Find t in powers of $-\psi\omega$ by common inversion, omitting ω after f for abbreviation, and writing f_n for $f^{(n)} : 2 . 3 \dots n$,

$$x = \omega - \frac{1}{f_{,}} . \psi\omega - \frac{f_{,,}}{f_{,}^{3}} . (\psi\omega)^2 - \frac{2f_{,,}^2 - f_{,} f_{,,,}}{f_{,}^{5}} . (\psi\omega)^3 - \frac{5f_{,,}^3 - 5f_{,} f_{,,} f_{,,,} + f_{,}^2 f_{,,}}{f_{,}^{7}} . (\psi\omega)^4 - \dots\dots ;$$

a representation of $f^{-1}\phi\omega$, or of $f^{-1}(f\omega - \psi\omega)$. Three terms will be more than sufficient.

Let $\phi x + \psi x$ be $ax + bx^2 + \dots + kx^m + (p + qx + \dots + sx^n)\, x^{n+1}$;
let $\alpha = \phi x + \psi x$ be the equation to be solved, and let $\alpha = \phi x$ give $x = \omega$. The root of $\alpha = \phi\omega$ being known, $\phi'\omega$ is known; call it l. Expanding the above in powers of ω, it is clear that the first three terms will give all short of ω^{3m+3}; Mr. Woolhouse, by an entirely different method, stops at ω^8 when $m = 2$. Taking this case, we find $\alpha = a\omega + b\omega^2$, $a + 2b\omega = \surd(a^2 + 4b\alpha) = l$. And, writing down no more than necessary for ω^8 inclusive,

$$x = \omega - \frac{c + e\omega + f\omega^2 + g\omega^3 + h\omega^4 + k\omega^5}{l + 3c\omega^2 + 4e\omega^3 + 5f\omega^4 + 6g\omega^5 + 7h\omega^6 + 8k\omega^7}\,\omega^3 - \frac{(b + 3c\omega + 6e\omega^2 + \dots + 28k\omega^6)\,(c + e\omega + f\omega^2)^2}{(l + 3c\omega^2 + \dots + 8k\omega^7)^3}\,\omega^6 - \dots$$

For l write 1; and, remembering that in this result al^{-1}, bl^{-1}, &c. must be written for a, b, &c., we have

$$x = \omega - c\omega^3 - e\omega^4 - (f - 3c^2)\,\omega^5 - (g - 7ce + bc^2)\,\omega^6 - (h - 8cf - 4e^2 + 12c^3 + 2bce)\,\omega^7 - (k - 9cg - 9ef + 45c^2e - 9bc^3 + 2bcf + be^2)\,\omega^8 - \dots$$

This result agrees entirely with that of Mr. Woolhouse.

The following Paper, read May 28th, 1868, could not be inserted in the account of the Proceedings of that day :—

On some Geometrical Constructions.

By H. J. Stephen Smith, F.R.S., Savilian Professor of Geometry in the University of Oxford.

Art. 1. A conic A is said to *circumscribe harmonically* a conic B, when A circumscribes a triangle which is self-conjugate with regard to B. Similarly, A is said to be *inscribed harmonically in* B, when A is inscribed in a triangle which is self-conjugate with regard to B. Though this mode of expression is not very accurate, it has the advantage of brevity, and it may serve to fix in the memory the well known theorems—

I. "If A circumscribe B harmonically, B is harmonically inscribed in A."

II. "If A circumscribe B harmonically, the conic corresponding to A in any correlative figure is harmonically inscribed in the conic corresponding to B."

Of these theorems we shall have to make frequent use, as also of the two following and their correlatives:—

III. "If A circumscribes B harmonically, A circumscribes an infinite number of triangles self-conjugate with regard to B; viz., if x_1 is any point of A, and x_2x_3 is the chord intercepted by A on the polar of x_1 with regard to B, $x_1x_2x_3$ is a self-conjugate triangle with regard to B. Or, which is the same thing, the *harmonic envelope of* A *and* B (*i. e.* the conic enveloped by lines cutting A and B harmonically) coincides with the polar reciprocal of A with regard to B."

IV. "If A circumscribe B harmonically, the centre of homology of any triangle inscribed in A, and of its polar triangle with regard to B, will lie on A."—(Dr. Salmon's *Conic Sections*, p. 326.)

The pairs of points, conjugate with regard to a conic A, which lie upon a line L, form a system in involution. Similarly, the pairs of lines, conjugate with respect to A, which intersect at a point P, form a pencil in involution. These involutions we shall term *the involutions of* A *upon the line* L, *and at the point* P, respectively. When we say that a conic is given, we shall understand that the polar system of the conic is given; *i. e.*, that the involution of the conic upon any line, and at any point in its plane, is given, or can be determined linearly. If any single element (point or tangent) of a given conic is given, we can determine linearly as many elements of it as we please. But if no single element of a given conic is given, the determination of any single element will require a quadratic construction, and the conic itself may be imaginary.

When there are two involutions I_1 and I_2 upon the same line L, the involution of which the double points are the extremities of the segment common to I_1 and I_2, is said to be *compounded of* I_1 *and* I_2. To obtain the involution compounded of two given involutions I_1 and I_2, let x be any point of L; x_1, x_2 the conjugates of x in I_1 and I_2 respectively, y_1 the conjugate of x_2 in I_1, y_2 the conjugate of x_1 in I_2; the harmonic conjugate of x with regard to y_1y_2 is also the conjugate of x in the involution compounded of I_1 and I_2. This construction may be demonstrated by projecting the involutions I_1 and I_2 upon a conic in the usual manner, and applying Pascal's theorem to the pentagon formed by the projections of the points x, x_1, y_2, y_1, x_2, x.

All the constructions which we shall employ in this paper are linear, except when the contrary is expressly stated. We shall, for the most part, leave the correlative of each proposition to be supplied by the reader.

ART. 2. *Problem* 1.—"To determine the conic σ, which passes through three given points a, b, c, and circumscribes harmonically two given conics S_1 and S_2."

Solution.—Let $a_1 b_1 c_1$, $a_2 b_2 c_2$ be the polar triangles of abc with regard to S_1 and S_2 respectively; and let aa_1, bb_1, cc_1 meet in x_1; aa_2, bb_2, cc_2 in x_2; then x_1, x_2 are points of σ (Theorem IV., art. 1), which is thus completely determined by the five points a, b, c, x_1, x_2.

Problem 2.—"To determine the conic σ, which passes through two given points a, b, and circumscribes harmonically three given conics S_1, S_2, S_3."

Solution.—Let c_1, c_2, c_3 be the poles of ab with regard to S_1, S_2, S_3 respectively. Through a draw any line aP, not passing through any one of the points c_1, c_2, c_3; let p_1, p_2, p_3 be the poles of aP with regard S_1, S_2, S_3; and let P be the unknown point in which σ meets aP for the second time. Considering the triangle abP, inscribed in σ, with regard to each of the conics S_1, S_2, S_3 in succession, we see that the three intersections (Pc_1, bp_1), (Pc_2, bp_2), (Pc_3, bp_3), as well as a, b, P, lie upon σ. We have therefore the anharmonic equation

$$P.[a, c_1, c_2, c_3] = b.[a, p_1, p_2, p_3],$$

which implies that the conic passing through a, c_1, c_2, c_3, and satisfying the anharmonic equation

$$[a, c_1, c_2, c_3] = b.[a, p_1, p_2, p_3],$$

also passes through P. Thus P is determined linearly, and with it σ, on which we now have six points. (The actual construction of P is as follows:—Let aP cut c_2c_3 in a'; determine on c_2c_3 a point c', satisfying the anharmonic equation

$$[a', c', c_2, c_3] = b.[a, p_1, p_2, p_3];$$

the point P is the intersection of aP and $c'c_1$.)

Problem 3.—"To determine the conic σ, which passes through a given point a, and circumscribes harmonically four given conics S_1, S_2, S_3, S_4."

Solution.—Let aP, aQ be any two straight lines passing through a, but not conjugate with regard to any one of the given conics. Let $p_1p_2p_3p_4$, $q_1q_2q_3q_4$ be the poles of aP, aQ respectively with regard to the conics S_1, S_2, S_3, S_4; and let P, Q be the unknown points in which aP, aQ meet σ for the second time. Considering the triangle aPQ, inscribed in σ, with regard to the conics S_1, S_2, S_3, S_4 in succession, we obtain the anharmonic equation

$$P . [a, q_1, q_2, q_3, q_4] = Q . [a, p_1, p_2, p_3, p_4],$$

which suffices for the linear determination of P and Q, by a kind of double position. Assume any point x on aP as the true position of P, and determine on aQ the corresponding position y of Q, first by the equation

$$y_3 . [a, p_1, p_2, p_3] = x . [a, q_1, q_2, q_3],$$

and then by the equation

$$y_4 . [a, p_1, p_2, p_4] = x . [a, q_1, q_2, q_4].$$

The two positions of y thus obtained will form, when x varies, two homographic ranges y_3 and y_4; of the double points, one is at the intersection of aQ and p_1p_2, the other can be determined linearly, and is the true position of Q. The details of the construction are as follows:—Denote by q the intersection (aP, q_1q_2), and by p the intersection (aQ, p_1p_2); let x be a point varying its position on aP, and let xq_3, xq_4 intersect q_1q_2 in $q'_3q'_4$. On the line p_1p_2 determine the points p'_3, p'_4, which satisfy the equation

$$[p'_3, p'_4, p, p_1, p_2] = [q'_3, q'_4, q, q_1, q_2],$$

and let $p_3p'_3$, $p_4p'_4$ cut aQ in y_3, y_4 respectively. The points y_3, y_4 will form two homographic ranges on aQ; for each of these ranges is homographic with the range x. One of the double points of the two ranges is at p; for if we imagine x to coincide with q, q'_3 and q'_4 will coincide with one another, and with q; whence p'_3 and p'_4, and with them y_3 and y_4, will coincide with one another, and with p. The other double point can therefore be obtained linearly; it will be the point Q, and the corresponding position of the point x will be the point P. We shall thus have seven points on σ; viz., the three points a, P, Q, and the four intersections (Pq_r, Qp_r), where $r = 1, 2, 3, 4$.

Problem 4.—"To determine the conic σ, which circumscribes harmonically five given conics S_1, S_2, S_3, S_4, S_5." Only the polar system of σ can be determined linearly, and σ itself may be imaginary.

Solution.—(1°). If each of two conics A and B harmonically circumscribe a third S, every conic C, which passes through the intersections of A and B, also circumscribes S harmonically.* To prove this geometrically, let x be any one of the points of intersection of A and B, and let the polar of x with respect to S cut A, B, C in the points a_1a_2, b_1b_2, c_1c_2 respectively. Then a_1a_2 and b_1b_2 are pairs of conjugate points with respect to S; therefore also c_1c_2, which is in involution

* The theorem of M. Hesse, "If two pair of opposite vertices of a quadrilateral are conjugate pairs with regard to a conic, the third pair of opposite vertices is also a conjugate pair," is a particular case of the correlative theorem.

with a_1a_2 and b_1b_2, is a pair of conjugate points with respect to S; *i. e.*, C circumscribes a triangle xc_1c_2 which is self-conjugate with respect to S. From this it appears that the conics which harmonically circumscribe four given conics all pass through four fixed points; for, if A, B are two conics circumscribing S_1, S_2, S_3, S_4, the conic C, which passes through a given point c, and circumscribes harmonically those four conics, is no other than the conic of the system (A, B) which passes through the point c.

(2°). Let L be any line in the plane of the five given conics: to obtain the involution of σ upon L, we have first to determine the intersections of L by two of the conics which harmonically circumscribe the four conics S_2, S_3, S_4, S_5; and to do this, we have only, in Problem 3, to take successively for a two different points on L. Let I_1 represent the involution determined by the two pairs of intersections; similarly, let I_2 represent the involution determined on L by the system of conics which circumscribe harmonically the four conics S_1, S_3, S_4, S_5. The involution of σ upon L is the involution compounded of I_1 and I_2.

ART. 3. In the preceding problems, any number of the given conics may degenerate into pairs of points, but the solutions will remain applicable if we observe that the pole of a straight line L, with regard to a system of two points, is the harmonic conjugate, with regard to the two points, of the intersection of L by the line joining the two points. Since a conic circumscribing a conic, which has degenerated into a pair of points, is a conic with regard to which the two points are conjugate, our last problem includes that of M. de Jonquières ("Annales de Mathématiques, par MM. Terquem et Gerono, vol. xiv. p. 435), "To determine the conic which divides harmonically five given segments." Again, the two points of a degenerate conic may become coincident, in which case a conic circumscribing harmonically the degenerate conic, is simply a conic passing through the point which represents that conic. Thus the problem 4 includes the problems 1, 2, 3. We might have made the solution of these problems depend on the corresponding cases of M. de Jonquières' problem. For example, if x_1y_1, x_2y_2, x_3y_3 x_4y_4 are the polar chords of the point a with regard to the conics S_1, S_2, S_3, S_4, the conic passing through a, and dividing harmonically the four segments x_1y_1, x_2y_2, x_3y_3, x_4y_4, harmonically circumscribes the conics S_1, S_2, S_3, S_4. But the solutions which M. de Jonquières has given of the various cases of his problem are, perhaps, less direct than those which we have deduced from the theorem of Dr. Salmon (Theorem 4, Art. 1.)

We may add that it follows from the solution of the problem of M. de Jonquières, that the polar system of a conic is given, when five pair of conjugate points with regard to the conic are given (but these five pair must be independent; see Art. 4).

ART. 4. Some remarks, which may not be without interest, are suggested by the analysis corresponding to the problems 1—4.

Let $\alpha_i x^2+\beta_i y^2+\gamma_i z^2+2\alpha_i' yz+2\beta_i' xz+2\gamma_i' xy = 0$ ....(1)

be the equation in point coordinates of the conic σ_i, and let

$$\mathrm{A}_i\xi^2+\mathrm{B}_i\eta^2+\mathrm{C}_i\zeta^2+2\mathrm{A}_i'\eta\zeta+2\mathrm{B}_i'\xi\zeta+2\mathrm{C}_i'\xi\eta = 0 \text{ ....(2)}$$

be the equation in line coordinates of the conic S_i, the two sets of coordinates being connected by the relation

$$\xi x+\eta y+\zeta z = 0.$$

The equation which expresses that σ harmonically circumscribes S_i is

$$\mathrm{A}_i\alpha+\mathrm{B}_i\beta+\mathrm{C}_i\gamma+2\mathrm{A}_i'\alpha'+2\mathrm{B}_i'\beta'+2\mathrm{C}_i'\gamma' = 0 \text{ ....(3)};$$

and the problem 4 is the geometrical equivalent of the analytical problem, "To determine the ratios of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ from five linear and independent equations of the type (3)." In these equations the coefficients may have any values whatever; whereas, in the problem of M. de Jonquières, the five equations are subject to the condition, that in each of them the symmetrical determinant formed with the six coefficients must be equal to zero; and in the problem, "to determine the conic passing through five given points," there is the still further limitation, that the first minors of those determinants must also be equal to zero.

We shall denote the systems of conics represented by the equations

(i) ...... $\lambda_1\sigma_1+\lambda_2\sigma_1 = 0$,

(ii) ...... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3 = 0$,

(iii) ...... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3+\lambda_4\sigma_4 = 0$,

(iv) ...... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3+\lambda_4\sigma_4+\lambda_5\sigma_5 = 0$,

by the symbols (σ_1, σ_2), $(\sigma_1, \sigma_2, \sigma_3)$, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and $(\sigma_1, \sigma_3, \sigma_3, \sigma_4, \sigma_5)$; and we shall describe them as *systems of the orders* 1, 2, 3, 4 respectively. The coefficients λ are absolutely indeterminate, and it is understood that the conics σ are *independent*, *i.e.* that σ_3 does not belong to the system (σ_1, σ_2), nor σ_4 to the system $(\sigma_1, \sigma_2, \sigma_3)$, nor σ_5 to the system $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. If the equations (i), (ii), (iii), (iv), are in line-coordinates, we shall describe the corresponding systems of conics as *tangential systems of the orders* 1, 2, 3, and 4. In the enunciations and solutions of the problems 1—4, we have tacitly supposed that the data are such as to render them determinate; the necessary and sufficient condition for this determinateness is that the five conics S_1, S_2, S_3, S_4, S_5, (or the pairs of points, real, imaginary, or coincident, by which any, or all of them, are replaced) should be independent, or should form a tangential system of order 4. A system of order 1 is the "faisceau," a system of order 2 is the "réseau," of French geometers. A single conic may be regarded as forming a system of order 0.

It is evident that a system of order k is determined by $k+1$ independent conics, and that if $k+1$ independent conics of a system of order k harmonically circumscribe a given conic, every conic of the system harmonically circumscribes that conic. (The geometric proof for the case $k=1$, which has been given above, might easily be extended to the other cases). We have also the important theorem:

"All the conics of a given system of order k circumscribe harmonically all the conics of a certain tangential system of order $4-k$; and, conversely, all the conics which circumscribe harmonically the conics of a given tangential system of $4-k$, form a system of order k."

The tangential system of order $4-k$, which thus corresponds to a given system of order k, we shall call *the system contravariant to the given system.* The relations between the two systems may be inferred from the known properties of indeterminate systems of linear equations. Thus, given $k+1$ independent conics σ ($k \overline{>} 0$, $k \leqq 4$), we have a system of $k+1$ indeterminate equations of the type,

$$\alpha_i A + \beta_i B + \gamma_i C + 2\alpha'_i A'_i + 2\beta'_i B' + 2\gamma'_i C' = 0 \ldots\ldots (4),$$
$$[i = 1, 2, \ldots k+1,]$$

in which $\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i$ are the given coefficients, and A, B, C, A′, B′, C′ the indeterminates. The order of indeterminateness of the system is $5-k$; and if

$$A_j,\ B_j,\ C_j,\ A'_j,\ B'_j,\ C'_j, \ldots [j = 1, 2, \ldots 5-k,] \ldots\ldots (5)$$

represent $5-k$ independent solutions, every solution is included in the formulæ

$$\Sigma . \lambda_j A_j,\quad \Sigma . \lambda_j B_j,\quad \Sigma . \lambda_j C_j,\quad \Sigma . \lambda_j A'_j,\quad \Sigma . \lambda_j B'_j,\quad \Sigma . \lambda_j C'_j,$$

the $5-k$ coefficients λ being absolutely indeterminate. This establishes the first part of the theorem; the second is the correlative, as well as the converse, of the first. Again, considering the matrix of the system (4), and the matrix of the system (5) of independent solutions as two complementary matrices, we know (Phil. Trans., vol. 151, p. 301) that the determinants of the one are proportional to the complementary determinants of the other; so that, in particular, if any determinant of either matrix is zero, the complementary determinant of the other matrix is also zero. As, perhaps, no geometrical application has hitherto been given of this analytical property, we shall refer to a few of its consequences here, though the results are such as might be obtained by simple geometrical reasoning.

(a). Let $k = 4$; the system is determined by five independent conics, and the contravariant system is of order 0, *i.e.*, it is the single conic S harmonically inscribed in these five conics. The tangents of this conic are the conics of the given system, which degenerate into pairs of coincident straight lines; for if the line $\eta = 0$, $\zeta = 0$, be a tangent of S, we must have $A = 0$; and therefore the complementary determinant in the matrix of the given system is also zero; *i.e.*, $x^2 = 0$ is one of the conics contained in the given system. Similarly, we may show that those pairs of straight lines which are degenerate conics of the given system are no other than the pairs of straight lines which are harmonically conjugate with respect to S; viz., if the lines $(\xi = 0,\ \eta = 0)$, $(\xi = 0,\ \zeta = 0)$ are conjugate with respect to S, we must have $A' = 0$, a condition which implies that $yz = 0$ is one of the conics of the given system.

(*b*). Let $k = 3$; the given system is determined by four independent conics, the contravariant system (S_1, S_2) being a system of conics inscribed in the same quadrilateral. As in the former case, we may show that the only conics of the given system which resolve themselves into two straight lines are represented by the four common tangents of the system (S_1, S_2); and that the conics which degenerate into two straight lines are the pairs of lines harmonically conjugate with respect to the same system. Further, the three pairs of opposite vertices of the quadrilateral circumscribing the system (S_1, S_2) are conics of that system, and therefore are conics harmonically inscribed in the conics of the given system; *i.e.*, the three diagonals of that quadrilateral cut the given system in involutions of which the double points upon each diagonal are the opposite vertices of the quadrilateral on that diagonal. (Mathematical Questions from the *Educational Times*, Vol. IV., p. 110; M. Cremona, in Crelle's Journal, Vol. LXI., p. 110.)

(*c*). Let $k = 2$, so that the given system is a "réseau," and the contravariant system a tangential "réseau." As it is necessary to consider two contravariant "réseaux" in the theory of cubic curves, we shall place here the solution of two elementary problems relating to them.

Problem 5.—"Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, to determine the contravariant system."

We may determine as many elements as we please of a conic touching two given lines, and harmonically inscribed in $\sigma_1\sigma_2\sigma_3$ (Problem 2). This conic is one of the conics of the contravariant system.

Problem 6.—"To determine the conic of the system $(\sigma_1, \sigma_2, \sigma_3)$ which harmonically circumscribes two given conics."

Determine three independent conics of the contravariant system; the conic circumscribing them and the two given conics is the conic required; but if the three contravariant conics, and the two given conics, are not independent, when considered tangentially, the problem is indeterminate.

Two particular cases of the problem are of frequent occurrence:

"To determine the conic of the given system which passes through two given points;" and

"To determine the conic of the given system with regard to which a given point and line are pole and polar."

In the latter case (as indeed in the general case) only the polar system of the required conic can be obtained linearly, and the conic itself may be imaginary.

The "double points" of the given system $(\sigma_1, \sigma_2, \sigma_3)$ lie on a cubic curve, the Jacobian locus, or Hessian of the system. Each point of a pair of points self-conjugate with respect to the system is a double point of the system; and the three vertices of any triangle self-conjugate with respect to two conics of the system are three double points, of which the conjugates lie in the axis of homology of the triangle and of its polar triangle with regard to any third independent conic of the system. The contravariant system possesses the correlative properties; and its Jacobian envelope is the Pippian, or Cayleyan, of the given system. Every common chord of two conics of the given system is a tangent of the Cayleyan; for if L be a common chord of σ_2 and σ_3, and if $\omega_1 \omega_2$ be the double points of the involution determined on L by the system $(\sigma_1, \sigma_2, \sigma_3)$, $\omega_1 \omega_2$ is a conic of the contravariant system, so that L is a "double line" of that system, and consequently a tangent of the Cayleyan. Thus, the Cayleyan is the involution-envelope of the given system; and, reciprocally, the Hessian is the involution-locus of the contravariant system. These well known properties are introduced here to show the importance of considering explicitly the contravariant system, as the relation between the two systems of conics may be said to be the source of the contravariant relation of the Hessian and Cayleyan.

ART. 5. *Problem* 7.—" Given two conics σ_1 and σ_2, to determine the conic of the system (σ_1, σ_2) which harmonically circumscribes a given conic."

Solution.—Determine four independent conics harmonically inscribed in σ_1 and σ_2; the conic harmonically circumscribing these conics and the given conic, is the conic required. For the four auxiliary conics it will be convenient to take four pairs of points reciprocal with regard to the system (σ_1, σ_2).

Problem 8.—" Given two conics σ_1, σ_2, and two straight lines L_1, L_2; to determine the conics of the system (σ_1, σ_2) with regard to which L_1, L_2 are a pair of conjugate lines." The problem is quadratic.

Solution.—Let λ_1 be the conic reciprocal to L_1* with regard to the system (σ_1, σ_2), and let λ_1 cut L_2 in the points a, b. The conics required are the two conics A and B, with regard to which the poles of L_1 are a and b. When these points have been determined by a quadratic construction, the polar systems of the two conics will be known. But this quadratic determination we shall not require; and the fol-

* The point reciprocal to a given point P with regard to a system of conics (σ_1, σ_2) is the point in which the polars of P with regard to that system intersect. The conic reciprocal to a line is the locus of points reciprocal to the points of the line. Every such reciprocal conic passes through the three vertices of the *harmonic triangle* of the system; *i.e.*, of the triangle self-conjugate with regard to all the conics of the system.

lowing construction, which is linear, will suffice. Let X be any point on L_1, x the point reciprocal to X. The polars of X with regard to the conics (σ_1, σ_2) form a pencil of lines at x, of which the rays correspond anharmonically to the conics themselves; in this pencil xa, xb are the rays corresponding to A and B. And since the involution of λ_1 upon L_2 (of which a, b are the double points) may be obtained linearly, we can determine linearly a pencil in involution at x, of which the double rays are the rays corresponding to the conics A and B.

Problem 9.—" Given three conics σ_1, σ_2, S; to determine the couics of the system (σ_1, σ_2) which are harmonically inscribed in S." This problem depends upon the preceding, which is a particular case of it; it is of course quadratic.

Solution.—If any single element of S is given, let p, q be any two points of S, L_1 the line joining them, P, Q the points reciprocal to p, q with regard to the system (σ_1, σ_2). Let Σ be the conic reciprocal to P, Q; Σ will pass through p, q, and L_1 will be one of the chords of intersection of S and Σ; the opposite chord of intersection can then be determined linearly; let it be L_2; the two conics of the system (σ_1, σ_2) with regard to which L_1, L_2 are conjugate lines (Problem 8), are harmonically inscribed in S. For, if A be either of those conics, A is harmonically circumscribed by Σ, because Σ circumscribes the harmonic triangle of the system (σ_1, σ_2); but A is also harmonically circumscribed by the degenerate conic $L_1 L_2$; therefore A is harmonically circumscribed by S, which is a conic of the system $(\Sigma_1, L_1 L_2)$.

If only the polar system of S is given, let I be the involution of S upon any line L_1, and let λ_1 be the conic reciprocal to L_1 with regard to (σ_1, σ_2). To the involution I upon L_1 there will correspond an involution of points upon the reciprocal conic λ_1; let PQ be the polar line of the involution upon λ_1, and let Σ be the conic reciprocal to PQ; then, as before, L_1 will be one of the common chords of S and Σ, and the opposite common chord can be determined linearly.

Problem 10.—" Given three conics σ_1, σ_2, S, to find the fourth point common to the conics, which circumscribe the harmonic triangle of σ_1, σ_2, and which also harmonically circumscribe S." The harmonic triangle of (σ_1, σ_2) must not also be a self-conjugate triangle of S.

Solution.—Let P be any point in the plane of the conics, L the polar of P with regard to S; p the point reciprocal to P, and λ the conic reciprocal to L, with regard to the system (σ_1, σ_2). To the involution of S upon L, there corresponds an involution upon the conic λ; let q be the pole of this involution; the conic reciprocal to pq will circumscribe S harmonically, for it will pass through P, and will cut L in two points, which will be reciprocal to a pair of points of the involution upon λ, and which will therefore be a pair of points of the involution of S upon L. Let $p'q'$ be a second line, of which the reciprocal conic harmonically circumscribes S; let θ be the point of intersection of pq, $p'q'$; the point reciprocal to θ is the fourth point of intersection required.

This fourth point is evidently (Theorem IV., Art. 1) the centre of homology of the harmonic triangle of (σ_1, σ_2) and of its polar triangle with regard to S. We may obtain the axis of homology either by the correlative construction, or by observing that it is the polar, with regard to S, of the centre of homology.

The solution of the problem, "To determine the conic which circumscribes the harmonic triangle of (σ_1, σ_2), passes through a given point, and harmonically circumscribes a given conic," is explicitly contained in what precedes. To determine the conic which circumscribes the harmonic triangle of (σ_1, σ_2), and also harmonically circumscribes two given conics S_1, S_2, we should have to substitute successively S_1 and S_2 for S in the preceding solution, and to determine the two corresponding positions θ_1, θ_2 of the point θ; the conic reciprocal to $\theta_1\theta_2$ would then be the conic required. Though no single element of this conic is given, yet an *uneven* number of its points (the three vertices of the harmonic triangle) are given symmetrically. This explains why the conic is necessarily real, and why we can determine points on it linearly.

ART. 6.—The polar conics of a cubic curve form a system of conics of order 2. Conversely, every system of conics of order 2 may be regarded as the polar system of a certain cubic curve, which we shall call the *fundamental cubic* of the system. Let $\Delta_i = x_i \dfrac{d}{dx} + y_i \dfrac{d}{dy} + z_i \dfrac{d}{dz}$, $[i = 1, 2, 3]$; the analytical determination of the equation $C=0$ of the cubic curve, of which three given independent conics $\sigma_1, \sigma_2, \sigma_3$ are polar conics, requires the determination of the coordinates x_i, y_i, z_i of the poles of the three given conics. Nine equations, determining the ratios of these nine coordinates, are obtained by equating to zero the coefficients of x, y, z in the expressions

$$\Delta_2\sigma_3 - \Delta_3\sigma_2, \quad \Delta_3\sigma_1 - \Delta_1\sigma_3, \quad \Delta_1\sigma_2 - \Delta_2\sigma_1.$$

The matrix of these nine equations is skew-symmetrical; the determinant is therefore zero, and the equations can be satisfied by at least one system of ratios of the nine coordinates; and, except in special cases, by only one such system. When values have been assigned to the nine coordinates, the coefficients of C may be ascertained from the equations,

$$\Delta_1 C = \sigma_1, \quad \Delta_2 C = \sigma_2, \quad \Delta_3 C = \sigma_3.$$

There are thus two curves of the third order, and two curves of the third class, which we have to consider in connexion with a given system of conics of order 2:—(1), the fundamental cubic; (2), the Hessian; (3), the fundamental contravariant, *i.e.*, the curve of the third class related to the contravariant system of conics, precisely as the fundamental cubic is related to the given system; (4), the Cayleyan, which, as we have seen, is related to the contravariant system as the Hessian to the given system. If the equation of the fundamental cubic is

$$x^3+y^3+z^3+6m\,xyz=0,$$

the equation of the Hessian is

$$m^2(x^3+y^3+z^3)-(1+2m^3)\,xyz=0;$$

the equation of the fundamental contravariant in line coordinates is

$$m(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta=0,$$

and the equation of the Cayleyan in line coordinates is

$$-m(\xi^3+\eta^3+\zeta^3)+(-1+4m^3)\,\xi\eta\zeta=0.$$

The fundamental contravariant is mentioned by Professor Cayley (*Phil. Trans.*, vol. 147, p. 427) as a curve of the third class of which the Cayleyan is the Hessian envelope. It is also the curve designated as K_3 by M. Cremona (*Introduzione ad una Teoria Geometrica delle Curve Piane*, p. 117). It may be described as the evectant of $\frac{S^3}{T^2}$, S and T being the invariants of M. Aronhold. The harmonic relation between the polar conics of the fundamental cubic and the polar conics of the fundamental contravariant may be immediately verified by means of their equations.

Art. 7. *Problem* 11.—"Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ to determine the polar systems of the fundamental cubic and of the fundamental contravariant." It will be convenient to exclude the exceptional case in which the three given conics have a pole and polar in common.

Solution.—The conics of the system (σ_2, σ_3) are, of course, all conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, and their poles all lie on a right line L_1. Let abc be the harmonic triangle of (σ_2, σ_3), $a'b'c'$ the polar triangle of abc with regard to σ_1; A, B, C the pairs of common chords of σ_1 and σ_2, which intersect at a, b, c respectively. By a known property of cubic curves, the polar conic of one of two points, which are self-conjugate with regard to every conic of the polar system, is the degenerate conic of that system, which consists of two straight lines intersecting at the other of the two points. Thus the poles of the conics A, B, C are respectively the intersections $(bc, b'c')$, $(ac, a'c')$, $(ab, a'b')$. Hence L_1, the locus of the poles of the system (σ_2, σ_3), which includes the conics A, B, C, is the axis of homology of the triangles abc, $a'b'c'$, and can be determined linearly (Problem 10). Similarly, let L_2, L_3 be the loci of the poles of the conics (σ_3, σ_1), (σ_1, σ_2); the vertices P_1, P_2, P_3 of the triangle $L_1L_2L_3$ will be respectively the poles of the conics σ_1, σ_2, σ_3.

Let P be a given point in the plane, and let σ be the polar conic of P.

If p_1 be the polar line of P with regard to σ_1, p_1 is also the polar line of P_1 with regard to σ. Thus the polar system of σ is determined linearly (Problem 6). Again, if σ be given, and its pole P be required, let p_1 be the polar of P_1 with regard to σ; the pole of p_1 with regard to σ_1 is P. Thus, in the preceding construction, it will suffice to determine the two axes of homology L_2 and L_3, since, when the pole of one conic σ_1 is known, the pole of every other conic of the system is known also.

To obtain the polar system of the fundamental contravariant, we have only to determine three conics of the contravariant system (Problem 5), and to apply to them the correlative of the preceding construction.

ART. 8. *Problem* 12. — "Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, to determine the polar system of the Hessian."

The solution of this problem depends on the following propositions:—

(1.) "The locus Σ of the poles of those conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ which are harmonically inscribed in a given conic S, is a conic."

For consider any straight line L; let (λ_1, λ_2) be the system of conics (contained in the given system) of the poles of which L is the locus; then two points of the locus Σ, and only two, lie upon L; viz., the poles of the two conics which belong to the system (λ_1, λ_2), and are harmonically inscribed in S (Problem 9). Or, we may prove the same thing analytically; for if $(x_i y_i z_i)$ is the pole of σ_i, a polar conic of the fundamental cubic C, the coefficients of σ_2 are linear in $x_i y_i z_i$; and the condition which expresses that σ_i is harmonically inscribed in S, will be quadratic in $x_i y_i z_i$; *i.e.*, the locus of $x_i y_i z_i$ is a conic section.

(2.) "Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, and the conic S, to determine Σ."

Let K be any straight line, (κ_1, κ_2) the system of conics, contained in the system $(\sigma_1, \sigma_2, \sigma_3)$, of which the poles lie on K. The poles of the conics $(\sigma_1, \sigma_2, \sigma_3)$ correspond correlatively to the polars of a fixed point X with regard to the conics themselves; for these points and lines are poles and polars with regard to the polar conic of X. If, in the construction of Problem 9, we draw the arbitrary straight line L_1 through the point X, we shall obtain a pencil in involution at the point x reciprocal to X with regard to the system (κ_1, κ_2), of which the double lines are precisely the lines corresponding to the two conics of the system (κ_1, κ_2) which are harmonically inscribed in S. The correlative involution will be an involution of points on the line K, for K is the polar of x with regard to the polar conic of X; and this involution will be no other than the involution of Σ upon K, because its double points will be the poles of the two conics (κ_1, κ_2) which are harmonically inscribed in S.

(3.) *Theorem.*—" If σ be any conic of the system $(\sigma_1, \sigma_2, \sigma_3)$, and P its pole, the locus of the poles of the conics of the system, which are harmonically inscribed in σ, is Σ, the Hessian polar conic of P."

This theorem is easily verified analytically. For the same equation which expresses that xyz is a point of the polar conic of $x'y'z'$ with regard to the Hessian, also expresses that the polar conic of xyz with regard to the fundamental cubic is harmonically inscribed in the polar conic of $x'y'z'$ with regard to the same curve. But the theorem may also be inferred geometrically from a known property of curves of the third order. For the six points R, in which Σ cuts the Hessian, are the points conjugate (upon the Hessian) to the six points r in which σ cuts the Hessian. (M. Cremona, loc. cit. p. 108.) The six degenerate polar conics which are composed of pairs of straight lines intersecting at the points r, are to be considered as conics harmonically inscribed in σ; their poles are the six points R. Therefore the locus of the poles of the conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ which are harmonically inscribed in σ, passes through the six points R; *i.e.*, that locus coincides with the Hessian polar conic of R.

(4). *Theorem.*—"If σ be any conic of the system $(\sigma_1, \sigma_2, \sigma_3)$, and P its pole, the locus of the poles of those conics of the system which harmonically circumscribe σ, is the Hessian polar line of P."

This theorem may be established by the same analysis as the last. Or it may be deduced from it geometrically; for, if σ' is any conic of the system which harmonically circumscribes σ, and if P$'$ is the pole of σ', P lies on the Hessian polar conic of P$'$, because σ is harmonically inscribed in σ'; *i.e.*, P$'$ lies on the Hessian polar line of P$'$."

It is evident that these propositions determine linearly the Hessian polar system. The polar system of the Cayleyan may be obtained by a correlative construction.

Art. 9. If C be a cubic, and Γ its Hessian, the cubics $C+\lambda\Gamma$ are termed the syzygetic cubics of C. If P is any point, σ the polar conic of P, and Σ its Hessian polar conic; its polar conic with regard to the syzygetic $C+\lambda\Gamma$ is $\sigma+\lambda\Sigma$. Thus the polar conics of a fixed point correspond anharmonically to the syzygetic cubics. Let P$'$, σ', Σ' represent a second given point, and its two polar conics; the reciprocal point of P$'$, with regard to the system (σ, Σ), will be the same as the reciprocal point of P with regard to the system (σ', Σ'). Let this point be Ω, and let Ωx be the mixed derivative of P, P$'$, with regard to the syzygetic $C+\lambda\Gamma$; the lines Ωx will correspond anharmonically to the polar conics of P, or of P$'$, and therefore to the syzygetic cubics. If we suppose that the polar systems of C and Γ are both given, the polar system of the syzygetic $C+\lambda\Gamma$, corresponding to any given line Ωx, is also given. For $\sigma+\lambda\Sigma$, the polar conic of P with regard to $C+\lambda\Gamma$, is given, since it is the conic of the system (σ, Σ) with regard to which

P′ and Ωx are pole and polar. And, similarly, if Q be any point in the plane, the polar conic of Q with regard to $C+\lambda\Gamma$ is given; for it belongs to a given system of order 1, and the polar line of P with regard to it is known, being the same as the polar line of Q with regard to $\sigma+\lambda\Sigma$.

There are three cubics of which any given cubic is the Hessian, and these three cubics are syzygetic with the given cubic. We proceed to determine the polar systems of these three cubics.

Problem 13.—"To determine the polar systems of the cubics of which a given cubic C is the Hessian." By a given cubic, we understand a cubic of which the polar system is given. The problem is, of course, cubical.

Solution.—Determine the polar system of Γ, the Hessian of C. Let P, P′ be any two points of which the second lies on the polar conic of the first; and, as before, let σ, σ' be the polar conics; Σ, Σ' the Hessian polar conics, of P, P′; Ωx the mixed derivative of P, P′ with regard to $C+\lambda\Gamma$. Consider the lines Ωx as corresponding anharmonically to the conics $\sigma'+\lambda\Sigma'$, and $\sigma+\lambda\Sigma$. For any given conic $\sigma+\lambda\Sigma$, determine the pencil in involution having its centre at the point Ω, of which the double rays Ωy_1, Ωy_2 correspond to the two conics of the system (σ', Σ'), which are inscribed harmonically in $\sigma+\lambda\Sigma$; we may obtain this determination by taking for L, in the construction of Problem 9, a straight line passing through P which is the point reciprocal to Ω with regard to the system (σ', Σ'). We thus have, at the point Ω, a pencil of lines Ωx, and a pencil of pairs of lines Ωy_1, Ωy_2. These two pencils correspond anharmonically to one another; for to each ray Ωx there corresponds but one pair Ωy_1, Ωy_2; and to each ray Ωy there corresponds but one ray Ωx, because there is but one conic in the system (σ, Σ), which circumscribes harmonically a given conic in the system (σ', Σ'). The ray Ωx will in three different directions coincide with one of the corresponding rays Ωy. Let Ωx_1, Ωx_2, Ωx_3 be the three directions of coincidence, which are to be determined by the cubic construction of M. Chasles.* The three syzygetic cubics corresponding to the three rays Ωx_2, Ωx_2, Ωx_3, are the three cubics of which C is the Hessian; because, for each of those three syzygetics, the Hessian polar conic of P passes through P′, and therefore coincides with σ; so that the Hessian itself coincides with C.

The problem, "To determine the two cubics which have the same Hessian as a given cubic C," may be solved in the same manner, but is only quadratic. In fact, when one of the three directions of coincidence Ωx_1, Ωx_2, Ωx_3, is known *à priori*, a pencil in involution of which the other two are the double lines may be determined linearly.

* Comptes Rendus, vol. 41, p. 681.

ART. 10. *Problem* 14.—"Given the polar system of a cubic C, to determine its nine points of inflexion." The problem requires one biquadratic, and two cubic constructions.

Solution.—The syzygetic cubics comprise four triangles; of which one is real, and one consists of one real line and a pair of conjugate imaginary lines; the two others are two imaginary triangles. The first two of these triangles will serve to determine the three real, and the six imaginary points of inflexion. Each of the four triangles, considered as a syzygetic cubic, is characterized by the property that it is its own Hessian. Retaining the notation of the last article, let Ωx be the ray corresponding to any given syzygetic $C+\lambda\Gamma$; and similarly let Ωh correspond to the Hessian of $C+\lambda\Gamma$; Ωx_1, Ωx_2, to those two syzygetics, other than $C+\lambda\Gamma$, which have its Hessian for their Hessians. We shall thus have, at the point Ω, a pencil of lines Ωh, and a pencil of triplets of lines Ωx, Ωx_1, Ωx_2. And since to every Hessian only one triplet of fundamental cubics corresponds, and to every cubic only one Hessian, the rays of the one pencil will correspond anharmonically to the triplets of the other. The ray Ωh will in four different directions coincide with one of the rays of its corresponding triplet; and the syzygetic cubics corresponding to the four directions of coincidence are precisely the four syzygetic triangles. Let K be any conic passing through Ω, and let the point in which any ray Ωx meets K for the second time be designated by x; also let ω, ω' be two given points, of which the first is, and the second is not, a point of K. The rays Ωh, and the conics $(\omega, \omega', x, x_1, x_2)$, will correspond to one another anharmonically; and the locus of the intersections of the corresponding rays and conics will be a cubic curve, which will cut K in the two points Ω, ω, and in four other points h_1, h_2, h_3, h_4, which may be determined by a biquadratic construction indicated by M. Chasles.* The four rays Ωh_1, Ωh_2, Ωh_3, Ωh_4, (of which two, and only two, are real) are the four directions of coincidence. Let Ωh be either of the two real rays. The polar system of the syzygetic triangle corresponding to Ωh, is real, and its polar conics all pass through the three vertices of the triangle. Determine two of these polar conics which pass through one and the same point taken arbitrarily in the plane; the three remaining intersections of the two conics are to be obtained by a cubic construction, and are the three vertices of the syzygetic triangle.

* Comptes Rendus, vol. 41, p. 1193.