ON FUNCTIONS OF SEVERAL VARIABLES

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THIS paper is mainly concerned with the problem, suggested by Weierstrass (*Ges. Werke*, Vol. II., p. 163), of showing that a function without finite essential singularities can be expressed as a quotient of two integral functions. In the *Acta Math.*, Vol. XXII., 1898, M. Poincaré had worked out in more detail, for the case of periodic functions, the suggestion considered in his earlier paper in *Acta Math.*, Vol. II., 1883, of expressing the real part of an integral function by the potential of the construct over which the function vanishes; this potential is a $(2p-2)$-fold integral, if p be the number of complex variables. In the *Trans. Camb. Phil. Soc.*, Vol. XVIII., 1899, p. 431, the present writer showed that the imaginary part of the function could be introduced concurrently with the real part, and the whole expressed as a $(2p-1)$-fold integral. So far as the form of the subject of integration only is concerned this integral is a particular case of one suggested by Kronecker in 1869 (*Werke*, Vol. I., p. 198), but it differs in that it is taken, not over a closed $(2p-1)$-fold, but over a $(2p-1)$-fold limited by a $(2p-2)$-fold defined by the vanishing of a function of the complex variables. In the recently published *Acta Math.*, Vol. XXVI., pp. 57–80, M. Poincaré has again given a solution of Weierstrass's problem, "qui tient pour aussi dire le milieu entre celle de M. Cousin et la mienne," depending upon an infinite series of $(2p-1)$-fold integrals. It would appear that this solution also can be deduced from the $(2p-1)$-fold integral used in the writer's previous paper, and the main object of the present paper is to explain this as simply as possible. Another object, however, is to attempt to put a point of view which appears to open a whole series of important questions.

CONTENTS.

1. Independent complex variables being denoted by $(\xi_1 \dots \xi_p)$ or $(\tau_1 \dots \tau_p)$, put $\xi = x_{2s-1}+ix_{2s}$, $\tau_s = t_{2s-1}+it_{2s}$, and speak of $(x_1 \dots x_n)$ or

$(t_1 \dots t_n)$ where $n = 2p$, as the coordinates of a point in n dimensions, calling the positive square root of $(x_1-t_1)^2+\dots+(x_n-t_n)^2$ the distance between these points. Imagine an analytical $(n-1)$-fold, expressed near any point of itself by a single power series $F(x_1 \dots x_n) = 0$, the function $F(t_1 \dots t_n)$ passing from negative to positive as (t) passes from the inside, so called, to the outside; let $(d_k x_1 \dots d_k x_n)$, for $k = 1 \dots (n-1)$, be independent sets of differentials on the surface, and dS_{n-1}, the so-called element of extent, be the positive square root of the determinant formed by multiplying into itself, row into row, the array of $(n-1)$ rows and n columns $(d_k x_1 \dots d_k x_n)$; and, taking $dt_1 \dots dt_n$ towards the outside from the point x of the $(n-1)$-fold so as to satisfy the $(n-1)$ equations $dt_1 d_k x_1+\dots+dt_n d_k x_n = 0$, let $l_r = h^{-1}\,\partial F/\partial t_r$, $l_1^2+\dots+l_n^2 = 1$, where h is real and positive. Further put

$$P(x, t) = -\frac{1}{n-2}\left[(x_1-t_1)^2+\dots+(x_n-t_n)^2\right]^{-\frac{1}{2}(n-2)}$$

or

$$= \tfrac{1}{2}\log\left[(x_1-t_1)^2+(x_2-t_2)^2\right],$$

the latter when $n = 2$, and

$$H = P(x, t)-P(x, 0)+\left(t\frac{\partial}{\partial x}\right)P(x, 0)-\dots+\frac{(-1)^{k-1}}{k!}\left(t\frac{\partial}{\partial x}\right)^k P(x, 0)$$

where k is a definite positive integer which may be zero and

$$\left(t\frac{\partial}{\partial x}\right) = t_1\frac{\partial}{\partial x_1}+\dots+t_n\frac{\partial}{\partial x_n}.$$

Then, taking (x) for a point of the $(n-1)$-fold, and (t) for any point of space, consider the integral

$$F = \frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

where $\varpi = 2\pi^p/(p-1)!$ is the total solid angle in $2p$ dimensions, and $f(\xi)$ is a function of $\xi_1 \dots \xi_p$ which therefore satisfies the p equations $(\partial/\partial x_{2s-1}+i\,\partial/\partial x_{2s})f(\xi) = 0$.

2. Notice, first, that for $n = 2$, putting $l_1 = dx_2/ds$, $l_2 = -dx_1/ds$, $dS_{n-1} = ds$, the integral is

$$\frac{1}{2\pi i}\int f(\xi)\,d\xi\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)$$

and

$$\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2} = \frac{x_1-t_1-i(x_2-t_2)}{(x_1-t_1)^2+(x_2-t_2)^2}-\frac{x_1-ix_2}{x_1^2+x_2^2}+\dots+\frac{(-1)^{k-1}}{k!}\left(t\frac{\partial}{\partial x}\right)^k\frac{x_1-ix_2}{x_1^2+x_2^2}$$

$$= \frac{1}{\xi-\tau}-\frac{1}{\xi}-\frac{\tau}{\xi^2}-\dots-\frac{\tau^k}{\xi^{k+1}};$$

so that, taken round a closed curve within which $f(\tau)$ is not singular, it gives

$$\epsilon_1 f(\tau)-\epsilon_2\left[f(0)+\frac{\tau}{1!}f'(0)+\frac{\tau^2}{2!}f''(0)+\ldots+\frac{\tau^k}{k!}f^{(k)}(0)\right]$$

where ϵ_1, ϵ_2 are both unity if τ and the origin be within the closed curve, but one or both of them otherwise zero; while, if we put $2\pi i$ in place of $f(\xi)$, and integrate from infinity along a curve not passing through the origin, up to ξ, the integral has the value

$$\log\left(1-\frac{\tau}{\xi}\right)+\frac{\tau}{\xi}+\frac{\tau^2}{2\xi^2}+\ldots+\frac{\tau^k}{k\xi^k},$$

which is the logarithm of Weierstrass's prime factor for an integral function having ξ for a simple zero.

3. Consider now values of n greater than 2. The integral F is unaltered by a slight deformation of the $(n-1)$-fold of integration, neither a singularity of $f(\tau)$, nor the point τ, nor the origin being passed over; for the condition this should be so is only

$$\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)f(\xi)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\left(\frac{\partial}{\partial x_3}+i\frac{\partial}{\partial x_4}\right)f(\xi)\left(\frac{\partial H}{\partial x_3}-i\frac{\partial H}{\partial x_4}\right)+\ldots=0,$$

namely,

$$f(\xi)\left[\frac{\partial^2 H}{\partial x_1^2}+\frac{\partial^2 H}{\partial x_2^2}+\ldots\right]=0;$$

further, when taken over a small closed $(n-1)$-fold given by

$$(x_1-t_1)^2+\ldots+(x_n-t_n)^2=\epsilon^2,$$

within which we suppose $f(\xi)$ is not singular and the origin is not found, the integral is equal to

$$\frac{1}{\varpi}\int f(\xi)\left\{\frac{x_1-t_1+i(x_2-t_2)}{\epsilon}\left[\frac{x_1-t_1-i(x_2-t_2)}{\epsilon^n}-\frac{x_1-ix_2}{(x_1^2+\ldots)^{\frac{1}{2}(n)}}+\ldots\right]\right.$$
$$\left.+\ldots\right\}\epsilon^{n-1}\,d\omega,$$

which, when $\epsilon=0$, gives $f(\tau)$; and, in fact, if $(y_1\ldots y_n)$ be any point, $\eta_1=y_1+iy_2$, &c., the integral

$$J=\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)P(x,y)+\ldots\right\}dS_{n-1}$$

taken over $(x_1-y_1)^2+\ldots+(x_n-y_n)^2=\epsilon^2$ is exactly equal to $f(\eta)$, independently of ϵ, provided $f(\xi)$ is throughout non-singular; and hence, as may be proved directly,

$$\left(\tau_1\frac{\partial}{\partial\eta_1}+\ldots+\tau_p\frac{\partial}{\partial\eta_p}\right)f(\eta)$$
$$=\left(t\frac{\partial}{\partial y}\right)J=\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial y}\right)P(x,y)+\ldots\right\}dS_{n-1}.$$

or, say, as
$$\frac{\partial P(x,y)}{\partial y_i} = -\frac{\partial P(x,y)}{\partial x_i},$$

$$\left(\tau\frac{\partial}{\partial\eta}\right)f(\eta) = -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial x}\right)P(x,y)+\ldots\right\}dS_{n-1},$$

and in general

$$\left(\tau\frac{\partial}{\partial\eta}\right)^m f(\eta) = \frac{(-1)^m}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial x}\right)^m P(x,y)+\ldots\right\}dS_{n-1}.$$

Thus, our integral, denoted above (§ 1) by F, when taken over a closed $(n-1)$-fold within which $f(\tau)$ is not singular, has, as for $n=2$, the value $\epsilon_1 f(\tau)-\epsilon_2(1, \tau)_k$ where ϵ_1, ϵ_2 are both unity when (τ) and the origin are included but either or both otherwise zero, and $(1, \tau)_k$, arising as the value when $y=0$ of

$$f(\eta)+\frac{1}{1!}\left(\tau\frac{\partial}{\partial\eta}\right)f(\eta)+\frac{1}{2!}\left(\tau\frac{\partial}{\partial\eta}\right)^2 f(\eta)+\ldots+\frac{1}{k!}\left(\tau\frac{\partial}{\partial\eta}\right)^k f(\eta),$$

is the integral polynomial in $\tau_1 \ldots \tau_p$ of order k constituting the initial terms of the Taylor expansion of $f(\tau)$ about the origin.

4. It appears that the integral F also represents a function of $\tau_1 \ldots \tau_p$ when the $(n-1)$-fold over which it is taken is not closed, provided its boundary consists of an $(n-2)$-fold given by the vanishing of functions of the complex variables. We have $\partial H/\partial t_m = -\partial K/\partial x_m$ where K is obtained from H by changing k into $k-1$; hence

$$\frac{\partial F}{\partial t_{2r-1}}+i\frac{\partial F}{\partial t_{2r}}$$
$$= -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\ldots\right\}dS_{n-1}.$$

Now let
$$\zeta_m = \frac{1}{\varpi}\int\left\{(l_{m,1}+il_{m,2})f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\ldots\right\}dS_{n-2}$$

integrated over the $(n-2)$-fold which bounds the $(n-1)$-fold, where $l_{r,s}$ denote the $\frac{1}{2}n(n-1)$ direction cosines of the $(n-2)$-fold; this contour integral can, by the generalized Green-Stokes theorem, be expressed as an integral over the $(n-1)$-fold, given by

$$\zeta_m = \frac{1}{\varpi}\int l_m\left\{\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\ldots\right\}dS_{n-1}$$
$$-\frac{1}{\varpi}\int\left\{(l_1+il_2)\frac{\partial}{\partial x_m}f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\ldots\right\}dS_{n-1},$$

of which the first integral has an identically vanishing integrand; thus

$$\zeta_{2r-1}+i\zeta_{2r} = -\frac{1}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

$$= -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

$$= \frac{\partial F}{\partial t_{2r-1}}+i\frac{\partial F}{\partial t_{2r}}.$$

This is true whatever be the character of the bounding $(n-2)$-fold, provided $f(\xi)$ remain finite on the $(n-1)$-fold and $(n-2)$-fold, neither the origin or the point (t) be upon these, and provided any singularities which either construct may possess do not affect the validity of the transformation theorem; in case the bounding $(n-2)$-fold be given by the vanishing of functions of the complex variables (τ), we have, at any point in the neighbourhood of which one such function gives the $(n-2)$-fold $l_{2r-1,\,2s-1}+il_{2r-1,\,2s}+i(l_{2r,\,2s-1}+il_{2r,\,2s}) = 0$, and therefore $\zeta_{2r-1}+i\zeta_{2r} = 0$, giving $\partial F/\partial t_{2r-1}+i\partial F/\partial t_{2r} = 0$, which proves the theorem in question. The theorem therefore holds for an $(n-2)$-fold extending to infinity, for instance one given by the vanishing of a single integral function, provided the integrals are convergent. For more details we refer to the writer's paper, *Camb. Phil. Trans.*, Vol. XVIII., where the theorem was also used for the case of $f(\xi) = 1$.

5. Suppose now that $\Theta(\tau_1 \dots \tau_p)$ is an integral function of $\tau_1 \dots \tau_p$, capable, therefore, of expression about any finite point as a power series converging for all finite values of $\tau_1 \dots \tau_p$. The equation $\Theta(\tau) = 0$ defines an $(n-2)$-fold which we may denote by I; we suppose that, when (τ^0) is a point subject to no conditions but $\Theta(\tau^0) = 0$, the lowest terms when $\Theta(\tau)$ is arranged in ascending powers of the differences $\tau_1-\tau_1^0 \dots \tau_p-\tau_p^0$ are linear; then the intersection in the neighbourhood of (τ^0) of the $(n-2)$-fold I with any 2-fold given by such equations as

$$\tau_2-\tau_2^0 = m_2(\tau_1-\tau_1^0), \quad \dots, \quad \tau_p-\tau_p^0 = m_p(\tau_1-\tau_1^0)$$

consists only of the point (τ^0) counted once; and the increment of $\log\Theta(\tau)$, for a closed 1-fold path lying in this 2-fold, is $2\pi i$. Imagine now a very large closed $(n-1)$-fold S, afterwards to pass off entirely to infinity—it will enclose part of I, intersecting this in $(n-3)$-folds; and imagine that the interior of S is rendered simply connected by an $(n-1)$-fold diaphragm P bounded on one part by the $(n-2)$-fold I and on the other by the $(n-2)$-fold in which P intersects S; this diaphragm may be regarded as arising by the coincidence of the two $(n-1)$-fold sheets of an $(n-1)$-fold Π which encloses I and shuts it off from the interior of S, save only that, as

Π degenerates into the two sides of P, it gives rise also to a cylindrical $(n-1)$-fold Σ surrounding, and having every point of itself very near to, some point of the $(n-2)$-fold I. The portion of Π interior to S, and the whole of S other than the ultimately vanishing portion cut out from it by the $(n-2)$-fold in which Π intersects S, form together a closed $(n-1)$-fold in which $\log \Theta(\tau)$ is single valued and everywhere developable. Thus the integral $\frac{1}{\varpi}\int \log \Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ taken over this has a value $\log\Theta(\tau)-(1, \tau)_k$. Now let Π degenerate into the two sides of the diaphragm P, together with the cylindrical $(n-1)$-fold Σ surrounding I, and this itself degenerate into I. As the values of $\log\Theta(\xi)$ on the two sheets of Π differ by $2\pi i$, the integral contains a portion

$$\Phi = \frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

taken over one side of the diaphragm P. As regards the portion taken over Σ, it can be shown that, if the points of Σ be defined by equations

$$\tau_m-\tau_m^0 = \frac{1}{h}\left(\frac{\partial\Theta}{\partial\tau_m^0}\right)'\epsilon(\cos\theta+i\sin\theta)$$

where (τ^0) is a point of I, and $(\partial\Theta/\partial\tau_m^0)'$ is the conjugate complex of $\partial\Theta/\partial\tau_m^0$, and h the positive square root of the sum of the squares of the moduli of such quantities, then the element of extent of Σ has a form $dS_{n-1}=\epsilon\, d\theta\, dS_{n-2}$ where dS_{n-2} is a corresponding element of extent for I; it is clear that $\epsilon\log\Theta$ vanishes when ϵ vanishes; we shall assume, therefore, that when Σ degenerates into I the $(n-1)$-fold integral over Σ vanishes. We have then

$$\log\Theta(\tau) = (1, \tau)_k+\Phi+W$$

where $$W=\frac{1}{\varpi}\int\log\Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

taken over the $(n-1)$-fold S with the exclusion of a vanishing portion bounded by the $(n-2)$-fold which is the limit of the intersection of Π with S. We notice that Φ is not continuous across P.

Now suppose the $(n-1)$-fold S to be taken at greater and greater distance; assume that the integer k can be chosen so that the $(n-1)$-fold integral Φ over one side of the diaphragm P remains convergent when taken over what ultimately becomes an infinite diaphragm P bounded only by the zero $(n-2)$-fold I; it is by no means asserted that this is so for any integral function Θ, but it appears that it is so for a very extensive class of functions, including those for which I is periodic. It follows then from the equation above that W does not become infinite or indeterminate for general positions of (τ); it is part of the purpose of the introduction of the $(n-1)$-fold Π to ensure that (τ) is not infinitely near to the $(n-2)$-fold I. But, in fact, the form of W shows that it cannot change from definite and finite to indefinite or infinite in consequence of any variation in the position of (τ) so long as this point remains in the interior of S; the indefinite approach of (τ) to the $(n-2)$-fold I introduces no infinite elements into W. We infer therefore, since by § 4 the integral Φ is a function of $\tau_1 \dots \tau_p$, that the limit of W when S passes off to infinity is an integral function of $\tau_1 \dots \tau_p$, possibly a constant or zero. This indeterminateness, too, in the expression for $\log \Theta(\tau)$ was to be anticipated; since the other integral Φ, over P, depends only on the position of the zero $(n-2)$-fold I, which is the same for any function Θe^{Ψ}, where Ψ is an integral function, as for Θ.

We have then the result following:—

An integral function having the zero $(n-2)$-fold I, supposed of multiplicity 1, is given by the exponential of the integral

$$\Phi = \frac{2\pi i}{\varpi} \int \left\{ (l_1 + i l_2) \left(\frac{\partial H}{\partial x_1} - i \frac{\partial H}{\partial x_2} \right) + \dots \right\} dS_{n-1},$$

taken over an infinite $(n-1)$-fold diaphragm P bounded by I, provided k can be chosen so that this is convergent.

We shall, however, find it desirable to bear in mind the more complete form

$$\log \Theta(\tau) = (1, \tau)_k + \frac{2\pi i}{\varpi} \int \left\{ (l_1 + i l_2) \left(\frac{\partial H}{\partial x_1} - i \frac{\partial H}{\partial x_2} \right) + \dots \right\} dS_{n-1} + W,$$

where W is the integral function arising as the limit of the $(n-1)$-fold integral previously put down, involving $\log \Theta(\xi)$ under the integral sign.

It has already been shown that this result is valid for the simple integral function of *one* variable $1-\tau/\xi$, the diaphragm P being then a curve coming from infinity and terminated at ξ; it is therefore also valid for any integral polynomial in the one variable τ, the diaphragm consisting then of several such curves, each terminated in one of the zeros of the polynomial; and therefore also valid for any integral function of τ of finite *genre*, the diaphragm consisting then of an infinite number of such curves. And, similarly, in the general investigation above we have not intended to assume that I consists of only one piece nor that P is simply connected; it may, moreover, be divided into unconnected portions by branches of I, in case this is, as would appear necessary for the case of periodic functions, a self-intersecting structure; it is believed, however, that the language employed can be understood in a sense suitable for all cases.

6. As a simple case where the assumption as to the convergence of the integral Φ over the $(n-1)$-fold diaphragm P is justified, we may take $p=2$, and $\Theta(\xi)=\xi_1-(a+ib)$; then the $(n-2)$-fold I is $x_1=a$, $x_2=b$, and the diaphragm P is given by $x_1=a$, $x_2<b$; also $l_1=1$, $l_2=l_3=l_4=0$; taking $k=0$, we find

$$\frac{2\pi i}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{b}\left\{\frac{a-t_1-i(x_2-t_2)}{[(a-t_1)^2+(x_2-t_2)^2+\ldots]^2}-\frac{a-ix_2}{[a^2+x_2^2+\ldots]^2}\right\}dx_2\,dx_3\,dx_4$$
$$=\log\left(1-\frac{\tau}{a+ib}\right).$$

In general, if the diaphragm P be such that for distant parts of space, at distance R from the origin, the extent of the diaphragm contained in n-fold extent V is at most of the order of magnitude of $R^{k-1}V$, the integral in question is convergent. For then, as the quantities $\partial H/\partial x_{2s-1}-i\,\partial H/\partial x_{2s}$ are ultimately of order $R^{-(n+k)}$, the portion of the integral over the part of the diaphragm for which $r<R<r_1$ is, in absolute value,

$$<\frac{\mu(r_1^n-r^n)r_1^{k-1}}{r^{n+k}},\qquad <\mu(1+\epsilon)^k\,\frac{(1+\epsilon)^n-1}{\epsilon}\left(\frac{1}{r}-\frac{1}{r_1}\right)$$

where μ is a fixed quantity and $r_1=r(1+\epsilon)$.

It is, however, to be remarked that, just as we have been able to infer the convergence of the integral W from the assumption of the convergence of the integral Φ, so the converse process may be possible. In absolute value W is of the order of

$$\frac{1}{\varpi}\int|\log\Theta(\xi)|\,\{R^{-(n+k)}\}\,R^{n-1}\,d\omega,$$

and vanishes if the limit of $R^{-(k+1)}\log|\Theta(\xi)|$ is zero as (x) passes to infinity in any direction not asymptotic to the zero $(n-2)$-fold I. For instance, if $\Theta(\xi)$ be an integral polynomial, this is so for $k=0$. Thus

Any integral polynomial $\Theta(\tau)$ can be represented in the form

$$\frac{\Theta(\tau)}{\Theta(0)}=\exp\left[\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\}dS_{n-1}\right]$$

where $H=P(x,t)-P(x,0)$ and the integral is over the infinite $(n-1)$-fold diaphragm limited by the $(n-2)$-fold on which $\Theta(\tau)=0$. It is supposed, as usual, that this $(n-2)$-fold is of multiplicity unity and does not contain the origin.

7. For the case of periodic functions consider first $p=1$. The diaphragm consists then, say, of straight lines directed from the origin,

one passing from each of the, say m, zeros in every parallelogram of periods, to infinity; the number of parallelograms in a circle of radius r, when r is large, is of the order r^2; the whole length of diaphragm in the annulus $r < R < r_1$ is then of order $m(r_1-r)r^2$; the ratio of this to the area of the annulus is

$$\frac{m}{\pi}\,\frac{(r_1-r)r^2}{r_1^2-r^2} = \frac{m}{\pi}\,r\,\frac{r}{r_1+r},$$

and is ultimately of order r; thus the condition of convergence is satisfied by taking $k = 2$.

For larger values of p, the $(n-2)$-fold I may be periodic in the sense that it is possible to divide n-fold space into period cells, the interior of any one of these being given by p equations

$$\tau_i = \lambda + 2\omega_{i,1}\lambda_1 + \ldots + 2\omega_{i,2p}\lambda_{2p} \quad (i = 1, \ldots, p),$$

where λ is a constant and $\lambda_1 \ldots \lambda_{2p}$ are real variables each between 0 and 1, in such a way that the portion of I in every cell is a repetition of that in any one cell. In that case the $(n-1)$-fold diaphragm P is presumably not periodic, there being in a distant cell, in addition to the portion bounded by the part of I actually contained therein, also portions continued from less distant cells and bounded by parts of I contained in these. It appears sufficient in order to show that the condition for the convergence of the integral is satisfied, also in this case by taking $k = 2$, to remark that the $(n-1)$-fold integral over the diaphragm P can be reduced to an $(n-2)$-fold integral over the bounding $(n-2)$-fold I; or, similarly, that a part of the $(n-1)$-fold integral over a distant portion of P can be reduced to a boundary $(n-2)$-fold integral; and that the condition for the convergence of the $(n-2)$-fold integral is that for an integral of the form $\int dS_{n-2}/R^{n+k-1}$; assuming that the extent of I included in any period cell is finite, the convergence of this last is reducible to that of the Eisenstein series

$$\sum_{m_1=-\infty}^{\infty} \ldots \sum_{m_n=-\infty}^{\infty} \left[\phi(m_1 \ldots m_n)\right]^{-\frac{1}{2}(n+k-1)},$$

where $\phi(m_1 \ldots m_n)$ is a definite quadratic form in the n integers $m_1 \ldots m_n$; this series converges if $n+k-1 > n$ or $k = 2$.

8. When $k = 2$ the second partial differential coefficient

$$-\frac{2\pi i}{\varpi}\,\frac{\partial^2}{\partial\tau_i\partial\tau_j}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1} - i\frac{\partial H}{\partial x_2}\right) + \ldots\right\} dS_{n-1}$$

is equal to $\quad -\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial H_0}{\partial x_1}-i\frac{\partial H_0}{\partial x_2}\right)+\ldots\right\}dS_{n-1}$

where $H_0 = P(x, t) - P(x, 0)$; if 2ω, a set of p complex quantities $(a_1+ia_2,\ a_3+ia_4,\ \ldots)$, be a period, it may be shown, by associating with any point (x) of the diaphragm of integration the equally arising point $x-a$; and with this the point $x-2a$, and so on, that the increment of the second partial differential coefficient, say $\wp_{ij}(\tau)$, which is given by

$$\wp_{ij}(\tau+2\omega)-\wp_{ij}(\tau) = -\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\ldots\right\}dS_{n-1},$$

where $K = P(x-a,\ t) - P(x,\ t)$, is zero. Thus the quotient, for an integral function Θ whose logarithm is represented by the $(n-1)$-fold integral over the diaphragm P, expressed by $\Theta(\tau+2\omega)/\Theta(\tau)$, is the exponential of a linear function of $\tau_1 \ldots \tau_p$. From this it follows, if (ξ) be a distant point obtained by addition of a general period from a finite point (τ), that the quotient $\Theta(\xi)/\Theta(\tau)$ is the exponential of a quadratic function of n integers $m_1 \ldots m_n$; and the condition previously remarked for the evanescence of the integral W, that $R^{-(k+1)}\log|\Theta(\xi)|$ should ultimately vanish, is clearly satisfied by $k = 2$; for R is the square root of a definite quadratic function of the n integers $m_1 \ldots m_n$.

The integral function Θ cannot be itself periodic; for, if we write $\Theta = U+iV$, the $(n-1)$-fold integral, taken over the perimeter of a period cell, $\int U\left(l_1\frac{\partial V}{\partial x_2}-l_2\frac{\partial V}{\partial x_1}+\ldots\right)dS_{n-1}$ would then be zero; it is, however, equal to $\int\left\{\frac{\partial}{\partial x_1}\left(U\frac{\partial V}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(U\frac{\partial V}{\partial x_1}\right)+\ldots\right\}dS_n$ taken through the cell, namely, to $\int\left\{\left(\frac{\partial U}{\partial x_1}\right)^2+\left(\frac{\partial U}{\partial x_2}\right)^2+\ldots\right\}dS_n$. On the other hand, it is not the case that every integral function whose zero $(n-2)$-fold is periodic is such that its second partial differential coefficients are periodic. We have shown that an integral function with such periodic differential coefficients can be found whose zero $(n-2)$-fold is an arbitrary given periodic $(n-2)$-fold; any other integral function having the same zero $(n-2)$-fold, with the same multiplicity, is obtainable from this by multiplication with a factor which is the exponential of an integral function. Integral functions whose second partial differential coefficients have, as here, sets of simultaneous periods are those considered by Frobenius under the name of "Jacobian functions"; it follows from his investigations, sketched in the present writer's *Abelian Functions*, pp. 579 *et seq.*, that the periods cannot be taken arbitrarily, and that the functions can be expressed by theta functions. This property then, as follows from the remarks to be next made, attaches to all single-valued analytic functions of p variables without finite essential singularities which have $2p$ sets of simultaneous periods—a theorem stated (on the authority of Hermite) to have been known to Riemann.

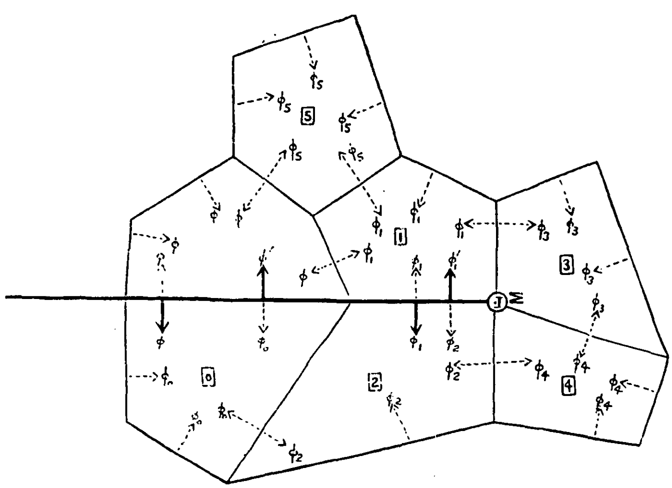
9. Suppose an analytic function is known to exist and to be single-valued and to be capable of expression about every finite point (τ^0) as the quotient of two power series converging in sufficiently near neighbourhood of (τ^0). If these series are both divisible by another power series vanishing at (τ^0), this factor may be supposed divided out (Weierstrass, *Ges. Werke*, t. II., p. 151); but further there exists a finite region about (τ^0), in the common region of convergence of the two series, such that, if the two series be developed about any point of the region, the resulting series have no common factor vanishing at this point (Weierstrass, *loc. cit.*, p. 154). Let a part of this region bounded by points all at the same distance from (τ^0) be called the proper region of (τ^0), and denoted by K_0, the expression of the function valid therein being ψ_0/ϕ_0, where ψ_0, ϕ_0 are power series in $\tau_1-\tau_1^0, \ldots, \tau_p-\tau_p^0$, having no common factor vanishing at any point of K_0. There may quite well be points of K_0 at which ψ_0 and ϕ_0 both vanish, these lying on the $(n-4)$-fold where the original function is not definite. Now let any finite region of space be for the moment called a suitable or unsuitable region according as it lies entirely within the proper region of some point within or upon the boundary of itself, or does not. Take any finite portion of space, however great, bounded by a closed $(n-1)$-fold: for definiteness we take the portion bounded by the $2n$ plane $(n-1)$-folds expressed by $x_s = -a_s$, $x_s = a_s$; let it be divided by planes into 2^{2n} similar rectangular cells each of extent 2^{-2n}-th of the original; these again divided into 2^{-2n} equal cells, and so on continually. After a finite number of steps, the original region must consist of sub-divisions each of which is a suitable region according to the definition above. For consider any indefinitely continued series of sub-divisions each of which is a sub-division of the preceding one of the series; the series will have a definite limiting point, say (τ^0), lying within or upon the boundary of every sub-division of the series. As the series is indefinitely continued, a stage can be assigned beyond which every sub-division is of less than an assigned extent, and therefore a stage can be assigned beyond which all the sub-divisions of the series lie entirely in the proper region of (τ^0), which by hypothesis is a spherical region about (τ^0) of assignable radius. Thus it is clear that in this series of sub-divisions we reach a suitable region after a finite number of steps; so that there exists no indefinitely continued series of wholly unsuitable regions, each contained in the preceding one of

the series. Thus the original cubical region, and similarly any finite portion of space, can be divided into a finite number of not-overlapping regions, each having the property of being entirely contained in the proper region of some point within or upon the boundary of itself.

Denote any one of these regions by R_0; considering then the interface between two regions R_0, R_1 and the equality $\psi_0/\phi_0 = \psi_1/\phi_1$ between the two possible expressions of the original function existing on this interface as belonging to the proper regions of both the points (τ^0) and (τ^1), which we may, as above, associate respectively with the regions R_0 and R_1; and assuming that a power series in $(p-1)$ complex variables which vanishes at the origin and for all the points of a (limited) $2p-2$-fold continuous about the origin, with the possible exception of a $(2p-4)$-fold part of this for which nothing is known, necessarily has zero coefficients; we can infer that, in the region common to the proper regions K_0, K_1, the $(n-2)$-folds $\psi_0 = 0$, $\psi_1 = 0$ are the same, as also those expressed by $\phi_0 = 0$, $\phi_1 = 0$. We thus build up the idea of a zero $(n-2)$-fold I_1, and an infinity $(n-2)$-fold I_2, for the original function, whose common $(n-4)$-fold intersection consists of the points where the function is unassigned. Expressing the equations of these $(n-2)$-folds, as in the previous part of this paper, by integral functions Θ_1, Θ_2, the original function has a form $e^{\Theta_3}\Theta_1/\Theta_2$, where Θ_3 is an integral function, and Θ_1, Θ_2 have no common zero save where the function is unassigned.

10. Suppose now that, as in what precedes, the whole of any finite portion of n-fold space is divided into regions R, R_0, R_1, ..., separated by $(n-1)$-fold interfaces, the diaphragm P, limited by the zero $(n-2)$-fold J of an integral function Θ, forming part of the system of interfaces; and that each region R_s is within the domain of one of the component series ϕ_s by which Θ is expressed, while on the interface S_{rs} between R_r and R_s the ratio ϕ_r/ϕ_s is not zero or infinite.

The integral $\dfrac{2\pi i}{\varpi}\displaystyle\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ over the diaphragm P may be decomposed into a sum of parts, one part for each of the portions of P which forms a face of a region R; if such a portion of P be denoted by H, the corresponding part of the integral may be regarded as arising from the sum of two integrals each of the form $\dfrac{1}{\varpi}\displaystyle\int\log\phi\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$, and taken over H, but with opposite directions and signs for l_1, l_2, ... and values of $\log\phi$ differing by $2\pi i$; these two elementary integrals we shall represent diagrammatically by oppositely drawn arrows named ϕ and ϕ', the dash associated with the latter indicating that the corresponding values of $\log\phi$ is greater by $2\pi i$ than for the other integral; to these parts must be added, as in § 5, an ultimately vanishing integral taken over a cylindrical $(n-1)$-fold Σ enclosing and everywhere very near to I; we suppose, in addition to what has been said, that the division of space into regions R is so taken that I is very nearly an intersection of interfaces, but is shut off from the regions by portions of Σ which form part of the perimeter of these; in the diagram below the $(n-2)$-fold I is denoted by a dot and the cylindrical $(n-1)$-fold Σ by a small closed curve. We have seen that an integral $\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\}dS_{n-1}$ taken over a closed $(n-1)$-fold within which $f(\tau)$ is not singular is zero, unless (τ) or the origin is enclosed; in the former case it gives $f(\tau)$, in the latter an integral polynomial in $\tau_1 \ldots \tau_p$ of order k. With the exception of these possibilities, to be afterwards referred to, we may now make a further decomposition of the $(n-1)$-fold integral over P in a way perhaps best explained with the help of the diagram.



Here the arrows in dark line represent the original portions of our integral over the parts H of the diaphragm P which are contained in the system of interfaces. For each region R_s we supply an inwardly directed integral $\frac{1}{\varpi}\int \log\phi_s\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\}dS_{n-1}$ over its whole surface; this is zero, save that we must supply a correction $\log\phi(\tau)$ for the single region containing (τ), besides a correction, a polynomial in $\tau_1 \ldots \tau_p$, for the single region containing the origin. An inspection of the figure shows that the result is an integral over every interface R_{rs}, of the form $\frac{1}{\varpi}\int \log\left(\frac{\phi_r}{\phi_s}\right)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\} dS_{n-1}$, wherein, as has been said, $\log(\phi_r/\phi_s)$ is finite and developable over this interface; together with integrals over the ou side faces of the most external of the regions; these latter together form an integral of the character of that denoted in § 5 by W; we shall here denote this sum by $-W'$, so that W' is directed outwards over the whole outside $(n-1)$-fold S bounding the portion of space considered. We have thus the result—

The integral over the $(n-1)$-fold diaphragm P,

$$\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\} dS_{n-1},$$

is equal to $\log\phi(\tau)+\Sigma J_{rs}-W'+(1,\tau)_k$, where $\phi(\tau)$ is the series for the region R containing (τ), the symbol $(1,\tau)_k$ denotes an integral polynomial of order k, and J_{rs} denotes the integral

$$\frac{1}{\varpi}\int\log\left(\frac{\phi_r}{\phi_s}\right)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\} dS_{n-1}$$

taken over the interface separating the regions R_r and R_s, and the sum of these integrals is to be taken for all the interfaces.

To the two sides of this equation we may add the integral W of § 5,

$$\frac{1}{\varpi}\int\log\Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\} dS_{n-1},$$

taken over the whole outside $(n-1)$-fold S. It is manifest that the difference $W-W'$ is ultimately an integral function, the function $\log\Theta/\phi$ being finite and not zero over S; thus, if our diaphragm integral is convergent, the sum ΣJ_{rs} is ultimately convergent, and the sum W' is ultimately an integral function. Thus in this case we have the further result—

An integral function having the given zero construct I is represented by $\log\phi(\tau)+\Sigma J_{rs}$.

Now putting, for abbreviation,

$$l_1\frac{\partial}{\partial x_1}+\dots+l_n\frac{\partial}{\partial x_n}=\frac{\partial}{\partial\nu},\qquad l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\dots+l_{n-1}\frac{\partial}{\partial x_n}-l_n\frac{\partial}{\partial x_{n-1}}=\frac{\partial}{\partial\sigma},$$

so that a function $U+iV$ of $x_1+ix_2, \dots, x_{n-1}+ix_n$ satisfies

$$\left(\frac{\partial}{\partial\nu}+i\frac{\partial}{\partial\sigma}\right)(U+iV)=0,$$

we have

$$\int (U+iV)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\} dS_{n-1}$$

$$=\int (U+iV)\left(\frac{\partial H}{\partial \nu}-i\frac{\partial H}{\partial \sigma}\right) dS_{n-1}$$

$$=\int \left[(U+iV)\frac{\partial H}{\partial \nu}-H\frac{\partial (U+iV)}{\partial \nu}\right] dS_{n-1}-i\int \frac{\partial}{\partial \sigma}[(U+iV)H]\, dS_{n-1},$$

and any one of the integrals J_{rs} can be written in the form

$$J_{rs}=K_{rs}-\frac{i}{\varpi}\int \frac{\partial}{\partial \sigma}\left(H\log\frac{\phi_r}{\phi_s}\right) dS_{n-1},$$

where
$$K_{rs}=\frac{1}{\varpi}\int \left[\frac{\partial H}{\partial \nu}\log\frac{\phi_r}{\phi_s}-H\frac{\partial}{\partial \nu}\left(\log\frac{\phi_r}{\phi_s}\right)\right] dS_{n-1},$$

taken over the interface R_{rs}. Consider the sum of the integrals

$$-\frac{i}{\varpi}\int \frac{\partial}{\partial \sigma}\left(H\log\frac{\phi_r}{\phi_s}\right) dS_{n-1}$$

or
$$-\frac{i}{\varpi}\int \left(l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\ldots\right)(H\log\phi_r)\, dS_{n-1}$$

$$-\frac{i}{\varpi}\int \left(l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\ldots\right)(H\log\phi_s)\, dS_{n-1},$$

where in the former $(l_1, l_2, \ldots)$ are directed inwards to the region R_r and in the latter inwards to the region R_s; an integral $\int \partial/\partial\sigma\,[Hf(\xi)]\, dS_{n-1}$ is easily seen to be unaltered by a deformation of the $(n-1)$-fold of integration provided no singularity of $f(\xi)$ is passed over, and to give zero when taken over a vanishing closed $(n-1)$-fold containing (τ) or the origin. Hence it is easily seen that the sum of these integrals is ultimately zero. Compounding them together by a process the reverse of that followed in obtaining the sum ΣJ_{rs}, there result three parts.

(1) a sum of integrals $-\dfrac{i}{\varpi}\displaystyle\int \frac{\partial}{\partial\sigma}(H\log\phi)\, dS_{n-1}$ over the outside $(n-1)$-fold S bounding the space considered, ultimately zero because H is ultimately of order $R^{-(n+k)}$, and the part of S where $\phi=0$ and $\log\phi=\infty$ is excluded from integration by the cylindrical $(n-1)$-fold Σ;

(2) an integral $\dfrac{2\pi}{\varpi}\displaystyle\int \frac{\partial H}{\partial\sigma}\, dS_{n-1}$ over the diaphragm P;

(3) an outward integral $\dfrac{i}{\varpi}\displaystyle\int \frac{\partial}{\partial\sigma}(H\log\phi)\, dS_{n-1}$ over the cylindrical

$(n-1)$-fold Σ; this, in virtue of $i\partial\phi/\partial\sigma = -\partial\phi/\partial\nu$, is equal to

$$\frac{1}{\varpi}\int\left(i\log\phi\frac{\partial H}{\partial\sigma} - \frac{H}{\phi}\frac{\partial\phi}{\partial\nu}\right)dS_{n-1},$$

which, if we put $dS_{n-1} = \epsilon\, d\theta\, dS_{n-2}$ as in § 5, and allow ϵ to vanish, so that Σ degenerates into I, becomes, since $\phi = 0$ on I,

$$-\frac{1}{\varpi}\int d\theta\int H\frac{\phi}{\phi}dS_{n-2} = -\frac{2\pi}{\varpi}\int H\,dS_{n-2};$$

and it is the fact that the sum of this last integral, over the boundary I of the diaphragm P, and the integral in (2), over this diaphragm, is zero. We have thus shown, for cases when our original integral is convergent, that

An integral function having the given zero $(n-2)$-fold I may be obtained by adding, to the sum of integrals

$$K_{rs} = \frac{1}{\varpi}\int\left[\log\frac{\phi_r}{\phi_s}\frac{\partial H}{\partial\nu} - H\frac{\partial}{\partial\nu}\left(\log\frac{\phi_r}{\phi_s}\right)\right]dS_{n-1}$$

over the interfaces, the quantity $\log\phi(\tau)$ associated with the region containing (τ).

This is the result obtained, for the less general case of periodic functions, by M. Poincaré in his last paper, *Acta Math.*, Vol. XXVI., pp. 67, 73, 78; while the identity just remarked,

$$\frac{2\pi}{\varpi}\int\frac{\partial H}{\partial\sigma}dS_{n-1} = \frac{2\pi}{\varpi}\int H\,dS_{n-2},$$

occurring *Trans. Camb. Phil. Soc.*, Vol. XVIII., p. 431, establishes the identity of the real part of the function considered by M. Poincaré, *Acta Math.*, Vol. XXII., p. 168, with the real part of the function

$$\frac{2\pi i}{\varpi}\int\left(\frac{\partial H}{\partial\nu} - i\frac{\partial H}{\partial\sigma}\right)dS_{n-1},$$

which is fundamental in the present paper.

11. To some readers the constant use of the language of hyperspace, and in particular the frequent mention of $(n-2)$-folds, in this paper, may seem to augur badly for the theory of the functions involved; on the contrary, the writer believes that this point of view, already found with some explicitness in Kronecker's *Berlin Monatsber.* paper of 1869, is of the greatest importance for the development of the theory of functions of several variables. It is desired to add here some general remarks, including a view of Mittag-Leffler's theorem for functions of more than one variable, which may add a little to the elucidation of the ideas.

In the plane which we employ in describing the properties of functions of one complex variable τ, the *element* is the point; it is, in general, at a point or a set of discrete points that a function of the complex variable τ has an assigned value, and we speak of infinite values of τ as lying in the neighbourhood of a point $1/\tau = 0$; and the region of existence of a power series in the difference $\tau-\tau^0$ is necessarily a circle described about and entirely enclosing the point τ^0, there being no point outside the circle at which the series converges. Suppose, however, we have, for instance, a power series in the two differences $\tau_1-\tau_1^0$, $\tau_2-\tau_2^0$, say $\phi(\tau_1, \tau_2)$; we have the theorem that, if the series converges for values τ_1', τ_2' such that $|\tau_1'-\tau_1^0| = R_1$, $|\tau_2'-\tau_2^0| = R_2$, a region of convergence is given by the equations $|\tau_1-\tau_1^0| < R_1$, $|\tau_2-\tau_2^0| < R_2$; but in a region of convergence of this form the radii R_1, R_2 are not in general to be regarded as fixed or independent; it may happen, indeed, that by taking one of them sufficiently small the other may be taken arbitrarily great; in that case, though, putting $\tau_1 = t_1+it_2$, $\tau_2 = t_3+it_4$, and using the language of hyperspace, we may take a sphere

$$(t_1-t_1^0)^2+(t_2-t_2^0)^2+(t_3-t_3^0)^2+(t_4-t_4^0)^2 = R^2,$$

and choose R less than the less of R_1 and R_2, to have its greatest value so that the power series converges at every interior point, yet this sphere will not in general contain all the points at which the series converges; the points at which the series $\phi(\tau_1, \tau_2)$ has any value which it takes within this sphere lie, in fact, upon a continuum of two dimensions satisfying $\phi_1 d\tau_1+\phi_2 d\tau_2 = 0$, which generally intersects the sphere, in a locus of one dimension, and passes to indefinite distance.

As an example of a power series whose region of existence is easily seen to pass to infinity, we may take the series obtained from the series

$$\text{(B)} \quad 1+\tau_1 e^{-\tau_2}+\tau_1^2 e^{-2\tau_2}+\ldots$$

by arranging according to terms of increasing dimension, that is, the series

$$\text{(A)} \quad 1+\tau_1+(-\tau_1\tau_2+\tau_1^2)+\left(\frac{\tau_1\tau_2^2}{2}-2\tau_1^2\tau_2+\tau_1^3\right)+\ldots;$$

if $\tau_1 = r_1 e^{i\theta_1}$, $\tau_2 = r_2 e^{i\theta_2}$, the sum of the moduli of the terms of (B) is $1+r_1 e^{r_2}+r_1^2 e^{2r_2}+\ldots$, which converges, only when $r_1 < 1$, for $r_1 < e^{-r_2}$; for any values of τ_1, τ_2 satisfying this condition the series (B) can be arranged as a convergent power series (A). Conversely, if the series (A) converges for any pair of values $\tau_1 = \tau_1'$, $\tau_2 = \tau_2'$, it will converge absolutely for values $|\tau_1| < |\tau_1'|$, $|\tau_2| < |\tau_2'|$ and converge therefore to the same value when arranged in the form (B); but the series (B) converges only when $|\tau_1| < |e^{\tau_2}|$, or, if $\tau_2 = t_3+it_4$, only when $r_1 < e^{t_3}$;

its region of convergence, represented on the two planes of τ_1 and τ_2, consists therefore of the interior of a circle, centre the origin, of radius r_1, in the first plane, taken with the region to the right of the straight line parallel to the imaginary axis in the second plane at distance $\log r_1$ to the right of this axis; there can, therefore, be absolute convergence only when $\log r_1$ is negative or $r_1 < 1$, and then only when $r_2 < \log 1/r_1$ or $r_1 < e^{-r_2}$. Thus the region of convergence of the series (A) consists of the interior of the three-fold $r_1 e^{r_2} = 1$; within this r_1 cannot be as great as unity, but can be as near as may be desired by taking r_2 small enough; on the other hand, r_2 can be as great as may be desired by taking r_1 small enough. We may then describe the region as *spindle-shaped*. It contains in its interior an infinite number of bi-cylindrical regions of convergence, each bounded by portions of two three-folds $r_1 = R_1$, $r_2 = R_2$, for which $R_2 = \log(1/R_1)$; this pair of three-folds intersects in a two-fold lying on the boundary of the spindle-shaped region. The spherical region of convergence of greatest radius is given by

$$t_1^2+t_2^2+t_3^2+t_4^2 = R^2$$

where R is the real positive quantity between $\frac{1}{2}$ and 1 for which $R = e^{-R}$. The two-fold on which the function represented by the series has the value, unity, which it has for $\tau_1 = 0$, $\tau_2 = 0$, has for equation $\tau_1 = 0$; it clearly passes to infinity; it intersects the boundary of the spherical region of convergence in the locus of one dimension $t_1 = 0$, $t_2 = 0$, $t_3^2+t_4^2 = R^2$; it intersects the perimeter of a bi-cylindrical region of convergence $|\tau_1| < R_1$, $|\tau_2| < R_2$ in the one-fold $t_1 = 0$, $t_2 = 0$, $t_3^2+t_4^2 = R_2^2$; but it does not intersect the perimeter of the spindle-shaped region of convergence in any finite point. Finally, it has appeared that this latter region is by no means co-extensive with the region of existence of the function represented by the power series, which is $(1-\tau_1 e^{-\tau_2})^{-1}$.

Thus the hyperspace of n, $= 2p$, dimensions which we speak of to describe the properties of a function of p complex variables is one of which the elements are not the points, but $(n-2)$-folds (cf. §§ 2, 4); the closed perimeters separating off regions where the function has an assigned character or value are $(n-1)$-folds which are not most naturally spheres, but multi-cylindrical surfaces often passing to an indefinite distance, and the infinity of the space consists not of one point, but of points lying on one or more of the $(n-2)$-folds, p in number, expressed by $\tau_1^{-1} = 0, \ldots \tau_p^{-1} = 0$; in general a point of the space is a derived element obtained by the co-intersection of p $(n-2)$-folds; whereas in the case of one variable the whole space is expressed by two equations of the form $|\tau| < R$, $|\tau^{-1}| < R$, in the case of p variables there are 2^p regions necessary to include the whole space, first the finite region $|\tau_1| < R_1 \ldots |\tau_p| < R_p$, and then all the regions of the form

$$|\tau_1^{-1}| < R_1 \ldots |\tau_m^{-1}| < R_m, \quad |\tau_{m+1}| < R_{m+1} \ldots \quad |\tau_p| < R_p,$$

where $m = 1, 2, \ldots, p$.

12. If now we have a single-valued function $f(\tau)$, either rational in the variables $\tau_1 \ldots \tau_p$, or without essential singularity for finite values of these, whose infinity $(n-2)$-fold I does not pass through the origin, and a multi-cylindrical region $|\tau_1| < R_1 \ldots |\tau_p| < R_p$ be constructed whose $(n-1)$-fold boundary, given by the aggregate of p sets of equations

$$|\tau_1| < R_1 \ldots |\tau_i| = R_i \ldots |\tau_p| < R_p,$$

excludes the $(n-2)$-fold I, the function $f(\tau)$ can be expanded in this region as a power series in $\tau_1 \ldots \tau_p$; and this expansion may be valid in a region of greater extent including in its interior all such possible multi-cylindrical regions; let C denote an $(n-1)$-fold excluding I, and including the origin, within which the expansion of $f(\tau)$

$$f(\tau) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_p=0}^{\infty} A_{m_1 \ldots m_p} \tau_1^{m_1} \ldots \tau_p^{m_p}$$

is uniformly convergent, in the sense that, for any assigned small positive quantity ϵ, values of $\mu_1 \ldots \mu_p$ can be assigned such that

$$F(\tau) = f(\tau) - \sum_{m_1=0}^{\mu_1-1} \ldots \sum_{m_p=0}^{\mu_p-1} A_{m_1 \ldots m_p} \tau_1^{m_1} \ldots \tau_p^{m_p}$$

$$= \sum_{m_1=\mu_1}^{\infty} \ldots \sum_{m_p=\mu_p}^{\infty} A_{m_1 \ldots m_p} \tau_1^{m_1} \ldots \tau_p^{m_p}$$

is in absolute value less than ϵ for all points interior to C; so that the region may be a multi-cylindrical region, or an interior spherical region, or be more extended.

Now suppose we have an enumerable series of such functions, infinity $(n-2)$-folds and $(n-1)$-folds, $f_1(\tau), f_2(\tau), \ldots, I_1, I_2, \ldots, C_1, C_2, \ldots$ such that any one $(n-1)$-fold C_s excludes $I_s, I_{s+1}, \ldots$, but includes $C_{s-1}, C_{s-2}, \ldots$ and the origin, while any finite point of space is interior to only a finite number of the $(n-1)$-folds $C_1, C_2, \ldots$; and, taking a convergent series of real positive terms $\epsilon_1, \epsilon_2, \ldots$, subtract from the expansion of $f_s(\tau)$ in C_s the sufficient polynomial that the remainder

$$F_s(\tau) = f_s(\tau) - \sum_{m_1=0}^{\mu_1-1} \ldots \sum_{m_p=0}^{\mu_p-1} A_{s\,m_1 \ldots m_p} \tau_1^{m_1} \ldots \tau_p^{m_p}$$

may within C_s be less than ϵ_s in absolute value, the integers $\mu_1 \ldots \mu_p$ presumably depending on s. Taking, then, any point (τ^0) not upon any one of the $(n-2)$-folds $I_1, I_2, \ldots$, but exterior, say, to $C_1 \ldots C_{s-1}$ and interior to $C_s C_{s+1} \ldots$, the infinite series $F_s(\tau)+F_{s+1}(\tau)+\ldots$ whose terms are power series in $\tau_1 \ldots \tau_p$ absolutely less respectively than $\epsilon_s, \epsilon_{s+1}, \ldots$, is uniformly convergent about this point and its sum can be arranged as a power series in the differences $\tau_1-\tau_1^0 \ldots \tau_p-\tau_p^0$; about this point also each of the finite number of functions $F_1(\tau) \ldots F_{s-1}(\tau)$ is non-singular and capable of expression as a power series in these differences, though possibly incapable of expression as a power series in $\tau_1 \ldots \tau_p$. Thus

The series $F(\tau)=F_1(\tau)+F_2(\tau)+F_3(\tau)+\ldots$ represents a single-valued function developable about every finite point not upon any one of the $(n-2)$-folds $I_1, I_2, \ldots$; while, as precisely the same reasoning applies if one of the $(n-2)$-folds and the corresponding function be omitted from consideration, and the difference $F_s(\tau)-f_s(\tau)$ is an integral polynomial, it follows that the difference $F(\tau)-f_s(\tau)$ is a function whose region of existence excludes only the $(n-2)$-folds other than I_s.

It is, of course, in the specification of the behaviour of the function $F(\tau)$ in the neighbourhood of the $(n-2)$-folds I_s that the subsidiary functions $f_s(\tau)$ have their chief utility; yet we have assumed a knowledge of the expansion of $f_s(\tau)$ about the origin in order to form the functions $F_s(\tau)$; it is worth remarking that when the functions $f_s(\tau)$ are known only in the immediate neighbourhood of the $(n-2)$-folds I_s, essentially a similar final theorem can be obtained. For one variable, if $f_s(\tau)$ be given only in an annulus surrounding the point I_s, consider the function

$$\phi_s(\tau)=\frac{1}{2\pi i}f_s(\tau)\int\frac{d\xi}{\xi-\tau}-\frac{1}{2\pi i}\int\frac{f_s(\xi)\,d\xi}{\xi-\tau}$$

taken round a closed curve in the annulus; it exists, is single-valued and developable about every finite point outside the inner boundary of the annulus, requires a knowledge of $f_s(\tau)$ only in the part of the annulus interior to the closed curve of integration, and is such that everywhere within this closed curve the difference $\phi_s(\tau)-f_s(\tau)$, being equal to $-\frac{1}{2\pi i}\int\frac{f_s(\xi)\,d\xi}{\xi-\tau}$, is non-singular (not excluding I_s). If then we use the expansion of $\phi_s(\tau)$ about the origin, just as before we used the expansion of $f_s(\tau)$, we shall obtain essentially the same character for the function $F(\tau)$. So, for any number of variables, imagine a cylindrical $(n-1)$-fold surface Γ_s passing to infinity which encloses the $(n-2)$-fold I_s, and suppose that the function $f_s(\tau)$ is known only in an annular cylindrical space which encloses Γ_s; then by § 4 we can use the expansion of the function

$$\phi_s(\tau) = \frac{f_s(\tau)}{\varpi}\int\left(\frac{\partial H}{\partial\nu} - i\frac{\partial H}{\partial\sigma}\right)dS_{n-1} - \frac{1}{\varpi}\int f_s(\xi)\left(\frac{\partial H}{\partial\nu} - i\frac{\partial H}{\partial\sigma}\right)dS_{n-1},$$

which is easily seen to be continuous as (τ) passes over Γ_s, in place of the expansion of $f_s(\tau)$, about the origin. This gives another example of the utility of the theorem of § 4.

Further, the reason for the use of the functions $F_s(\tau)$ in the place of the functions $f_s(\tau)$ is the need of a series converging uniformly about every finite point (τ^0) not upon the $(n-2)$-folds $I_1, I_2, \ldots$ and of definite construction; it may be possible, in place of choosing the integers $\mu_1 \ldots \mu_p$ so that $F_s(\tau)$ is absolutely less than ϵ_s for the whole interior of C_s, to choose them smaller than in that case in such a way that for the whole interior of any assigned C_r the series $F_r(\tau)+F_{r+1}(\tau)+\ldots$, and therefore the series $F(\tau)$, is uniformly convergent, which is sufficient for the theorem. An example is the case where $F(\tau) = \cot \pi\tau$; another example occurs below.

The series obtained permit of integration and differentiation term by term, as may be proved in a manner quite analogous to that given by Weierstrass for one variable, or by consideration of the integrals

$$F_r(\tau) = \int F_r(\xi)\left(\frac{\partial}{\partial\nu} - i\frac{\partial}{\partial\sigma}\right) P(x, t)\, dS_{n-1}$$

taken over a closed $(n-1)$-fold within which no one of the functions $F_1(\tau), F_2(\tau), \ldots$ is singular.

The theorem and proof are capable of extension to functions not altogether single-valued or of pseudo-rational character; in particular, analogous to the way in which Weierstrass's factor theorem for integral functions of one variable is derivable from Mittag-Leffler's theorem, there is a derivable factor theorem for functions of several variables; the expression in § 6 for the logarithm of a rational function gives rise to the necessary expansion of this logarithm about the origin.

13. To form an example of the theorem it seems natural in the first instance to consider the case when the $(n-2)$-folds $I_1, I_2, \ldots$ are given by linear equations. For two complex variables $\xi = x+iy$, $\eta = z+it$, the square of the distance from the origin to the nearest point of the two-fold $(a+ib)\xi+(c+id)\eta = 1$ is $(a^2+b^2+c^2+d^2)^{-1}$; and, in fact, the square of the modulus of the left side of this equation, that is, of

$$(a+ib)(x+iy) + (c+id)(z+it),$$

is $(ax-by+cz-dt)^2+(ay+bx+ct+dz)^2$, which is equal to

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2) - (az+bt-cx-dy)^2 - (at-bz-cy+dx)^2$$

and is less than unity when $x^2+y^2+z^2+t^2 < (a^2+b^2+c^2+d^2)^{-1}$; so that under this condition we have, if $u = a+ib$, $v = c+id$, the expansion

$$(1-u\xi-v\eta)^{-1} = 1+(u\xi+v\eta)+(u\xi+v\eta)^2+\ldots.$$

Now take an infinite series of functions

$$w_1(1-u_1\xi-v_1\eta)^{-1}, \qquad w_2(1-u_2\xi-v_2\eta)^{-1}, \qquad \ldots$$

where $w_1, w_2, \ldots$ are constants, so that the numbers $R_1 = [|u_1|^2+|v_1|^2]^{-\frac{1}{2}}$, $R_2 = [|u_2|^2+|v_2|]^{-\frac{1}{2}}$, ... constantly increase to infinity, but in such a way that only a finite number of them are less than any assigned real positive number; then, θ being a fixed real quantity just less than unity, the $(n-1)$-folds $|\xi|^2+|\eta|^2 = (\theta R_s)^2$ are such a series as those denoted in § 12 by $C_1, C_2, \ldots$; and, if ϵ be a real quantity between 0 and θ, $W_s = |w_s|$ and $\rho_s = |u_s\xi+v_s\eta|$, the sum of the moduli of the terms which follow the μ_s-th term of the expansion within C_s of $w_s(1-u_s\xi-v_s\eta)^{-1}$ in powers of $u_s\xi+v_s\eta$, namely $W_s\rho_s^{\mu_s}/(1-\rho_s)$, is, for $|\xi|^2+|\eta|^2 \leqslant (\epsilon R_s)^2$, and therefore $\rho_s \leqslant \epsilon$, less than or equal to $W_s\epsilon^{\mu_s}/(1-\epsilon)$, and the functions $F_s(\xi, \eta)$ may be defined by choosing the numbers μ_s so that $W_s\epsilon^{\mu_s}/(1-\epsilon) < \epsilon_s$ and taking

$$F_s(\xi, \eta) = \frac{w_s}{1-u_s\xi-v_s\eta} - \{1+u_s\xi+v_s\eta+\ldots+(u_s\xi+v_s\eta)^{\mu_s-1}\};$$

but we have remarked that it is sufficient if the numbers μ_s be such that the series $\sum_s W_s\rho_s^{\mu_s}/(1-\rho_s)$ be uniformly convergent within any assigned C_r; which is satisfied, since $\rho_s < \epsilon$, if the series $\sum_s W_s\rho_s^{\mu_s}$ be so uniformly convergent; and this again, since $\rho_s \leqslant (|u_s|^2+|v_s|^2)^{\frac{1}{2}}(x^2+y^2+z^2+t^2)^{\frac{1}{2}}$, provided the series of constants $\sum_s W_s[|u_s|^2+|v_s|^2]^{\frac{1}{2}\mu_s}$ be convergent.

The most obvious case is when the series $\sum_s W_s$ is convergent; then we have

$$F(\xi, \eta) = \sum_{s=1}^{\infty} \frac{w_s}{1-u_s\xi-v_s\eta},$$

the aggregate of numbers $[|u_s|^2+|v_s|^2]^{-\frac{1}{2}}$ satisfying the condition of having infinity as its sole point of condensation.

For another case we may take $w_s = 1$ and $u_s = m_s^{-1}$, $v_s = n_s^{-1}$, wherein $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ are both series of constantly increasing integers; then clearly $\sum_s \left(\frac{1}{m_s^2}+\frac{1}{n_s^2}\right)^{\frac{1}{2}\mu_s}$ converges for $\mu_s = 2$, and we have

$$F(\xi, \eta) = \sum_s \left\{\frac{1}{1-\frac{\xi}{m_s}-\frac{\eta}{n_s}} - \left(1+\frac{\xi}{m_s}+\frac{\eta}{n_s}\right)\right\} = \sum_s \frac{\left(\frac{\xi}{m_s}+\frac{\eta}{n_s}\right)^2}{1-\frac{\xi}{m_s}-\frac{\eta}{n_s}}.$$

14. Such considerations appear to the writer to have great interest as throwing light on the question, "How far do there exist Mittag-Leffler series* of simpler functions for multiperiodic functions without finite essential singularities analogous to the well known series for the elliptic function $\wp(u)$?" When we consider the periodicity of the infinity construct of such a function, it appears unlikely that such a series can be built under the hypothesis that the $(n-1)$-folds C_1, C_2, ... used in the demonstration of § 12 are spheres or multicylindrical surfaces of all finite radii; for that case, moreover, already considered by M. Appell, *Acta Math.*, Vol. II., 1883, p. 71, the proof is only a very obvious generalization of the case of one variable; we believe it to be of importance to plead for standing ground for the more general formulation.

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* [*September 1st*, 1903.—For triply periodic functions of two variables, such series are derivable at once from the known unsymmetrical forms given, for instance, in Part III. of the writer's note on hyperelliptic functions in *Proc. Camb. Phil. Soc.*, Vol. XII., Part III. (Easter Term, 1903). See also Painlevé, *Compt. Rend.*, April 14th, 1902.]