On the Motion of a Particle on the Surface of an Ellipsoid.

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1. The cases of motion of a particle constrained to move on the surface of an ellipsoid discussed in this paper are those in which the velocity, expressed as a function of the primary axes of the confocal hyperboloids determining the position of the particle on the surface, assumes a certain form.

The following theorem in the Calculus of Variations, which can be easily established, enables us to determine the motion and path of the particle in the cases considered.

If $\int \sqrt{\Phi(x)+\Psi(y)}\,\sqrt{dx^2+dy^2}$ be a minimum, then

$$\frac{dx}{\sqrt{\Phi(x)-C}} \pm \frac{dy}{\sqrt{\Psi(y)+C}} = 0$$

is a first integral of the differential equation afforded by the Calculus of Variations, C being an arbitrary constant; and hence, more generally,

if $$\int \sqrt{\Phi(x)+\Psi(y)}\,\sqrt{\phi(x)\,dx^2+\psi(y)\,dy^2}$$

be a minimum, then the first equation becomes

$$\frac{\sqrt{\phi(x)}\,.\,dx}{\sqrt{\Phi(x)-C}} \pm \frac{\sqrt{\psi(y)}\,.\,dy}{\sqrt{\Psi(y)+C}} = 0.$$

2. Let us now express ds the element of the arc of a curve on the surface of the ellipsoid in terms of μ and ν, the primary axes of the confocal surfaces through the point considered (see Salmon's *Geometry of Three Dimensions*, Art. 424).

Then $$ds^2 = d\sigma^2 + d\sigma'^2,$$

where $$d\sigma = \sqrt{\frac{(\mu^2-\nu^2)(a^2-\mu^2)}{(\mu^2-h^2)(k^2-\mu^2)}}\,d\mu \quad\ldots\ldots\ldots\ldots(1),$$

$$d\sigma' = \sqrt{\frac{(\mu^2-\nu^2)(a^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}}\,d\nu.$$

The Principle of Least Action gives us $\int v\,ds$ a minimum, or

$$\int v\sqrt{\mu^2-\nu^2}\sqrt{\frac{(a^2-\mu^2)\,d\mu^2}{(\mu^2-h^2)(k^2-\mu^2)} + \frac{(a^2-\nu^2)\,d\nu^2}{(h^2-\nu^2)(k^2-\nu^2)}}$$

a minimum for the motion of the particle on the surface; hence we see that, if $v\sqrt{\mu^2-\nu^2} \equiv \sqrt{\Phi(\mu)+\Psi(\nu)}$,

the dynamical problem is reduced to the mathematical problem already discussed. If then

$$v^2 = \frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2},$$

$\Phi(\mu)$ and $\Psi(\nu)$ involving the coordinates μ' and ν' of the initial position, and β the initial velocity of the particle, the differential equation of the path will be

$$\sqrt{\frac{(a^2-\mu^2)\,.\,d\mu}{(\mu^2-h^2)(k^2-\mu^2)\{\Phi(\mu)-C\}}} \pm \frac{\sqrt{a^2-\nu^2}\,.\,d\nu}{\sqrt{(h^2-\nu)(k^2-\nu)\{\Psi(\nu)+C\}}} = 0 \ldots (2),$$

or multiplying by $\sqrt{\mu^2-\nu^2}$, and referring to (1),

$$\frac{d\sigma}{\sqrt{\Phi(\mu)-C}} \pm \frac{d\sigma'}{\sqrt{\Psi(\nu)+C}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

3. The lines of curvature are obviously included in the above differential equation, hence if v^2 is of the form

$$\frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2},$$

the particle can, by a proper determination of the initial circumstances, be made to describe a line of curvature; if then it be required to make the particle, under the action of a force which causes the potential, and consequently v^2, to assume the above form, describe the line of curvature $\mu = \mu'$, it must obviously be projected in the direction of a tangent to this curve, and it remains to find the proper initial velocity β.

Let N be the resolved part of the force along a line perpendicular to the direction of motion and the normal to the surface, ρ the radius of curvature of the orbit, and ϕ the angle the osculating plane makes with the tangent plane to the surface; we have then the following equation from mechanical considerations :—

$$N + \frac{v^2}{\rho}\cos\phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If, now, the path is the line of curvature $\mu = \mu'$, $\frac{\cos\phi}{\rho} = \frac{1}{R'}$ when R' is the radius of curvature of the section of the hyperboloid of one sheet by the tangent plane to the ellipsoid. Now, $R' = \frac{\mu^2-\nu^2}{p'}$, p' being the perpendicular from the centre on the tangent plane to the hyperboloid of one sheet, hence (1) becomes

$$N + \frac{v^2}{\mu^2 - \nu^2} p' = 0 \quad \text{.........................} (2).$$

Now $\frac{1}{2}\frac{dv^2}{d\sigma}$ is evidently the force perpendicular to the μ line of curvature or N; therefore, differentiating the equation

$$v^2(\mu^2 - \nu^2) = \Phi(\mu) + \Psi(\nu),$$

we get
$$(\mu^2 - \nu^2)\frac{dv^2}{d\sigma} + v^2 \times 2\mu \, . \, \frac{d\mu}{d\sigma} = \Phi'(\mu)\frac{d\mu}{d\sigma};$$

or
$$\frac{1}{2}\frac{dv^2}{d\sigma} + \frac{v^2\mu\frac{d\mu}{d\sigma}}{\mu^2 - \nu^2} = \frac{1}{2} \, . \, \frac{\Phi'(\mu)\frac{d\mu}{d\sigma}}{\mu^2 - \nu^2}.$$

But $\frac{\mu d\mu}{d\sigma} = p'$, therefore

$$N + \frac{v^2 p'}{\mu^2 - \nu^2} = \frac{1}{2} \, . \, \frac{\Phi'(\mu)\frac{d\mu}{d\sigma}}{\mu^2 - \nu^2} \quad \text{.......................} (3),$$

and this will be identical with (2) if $\Phi'(\mu) = 0$. This equation will determine β, since $\Phi(\mu)$ involves μ', ν', and β. The following examples will make this more clear.

A particle constrained to move on the surface of an ellipsoid is acted on by an attractive force to the centre as Ar. The potential is then $\frac{1}{2}Ar^2$, and consequently $v^2 = A(r'^2 - r^2) + \beta^2$ or $A(\mu'^2 + \nu'^2 - \mu^2 - \nu^2) + \beta^2$, since $r^2 = \mu^2 + \nu^2 + a^2 - h^2 - k^2$ (Salmon's *Surfaces*, Art. 161);

hence
$$v^2 = \frac{\{A(\mu'^2 + \nu'^2) + \beta^2\}(\mu^2 - \nu^2) - A(\mu^4 - \nu^4)}{\mu^2 - \nu^2},$$

which gives
$$(\mu) \equiv \{A(\mu'^2 + \nu'^2) + \beta^2\}\mu^2 - A\mu^4,$$

and the condition $\Psi'(\mu) = 0$ determines β from the equation

$$A(\mu'^2 + \nu'^2) + \beta^2 - 2A\mu'^2 = 0,$$

or
$$\beta^2 = A(\mu'^2 - \nu'^2),$$

and therefore
$$v^2 = A(\mu'^2 - \nu^2).$$

A particle moves on the surface, and is acted on by a central force situated on the axis major at a point S distant $\frac{hk}{a}$ from the centre, the force varying directly as the distance and inversely as the cube of the tangent to the sphere having S for centre and touching the ellipsoid at the umbilics, to determine the initial velocity in order that it may describe the line of curvature $\mu = \mu'$. The radius of this sphere will be found to be $\frac{bc}{a}$, and the tangent from any point on the ellipsoid will be consequently equal in length to $\mu + \nu$. Now the potential is $\frac{A}{\mu + \nu}$, hence $$\tfrac{1}{2}v^2 = \frac{A}{\mu+\nu} - \frac{A}{\mu'+\nu'} + \tfrac{1}{2}\beta^2,$$

or $$v^2 = \frac{(\mu^2-\nu^2)\left(\beta^2 - \frac{2A}{\mu'+\nu'}\right) + 2A(\mu-\nu)}{\mu^2-\nu^2};$$

hence $$\Phi(\mu) = \left(\beta^2 - \frac{2A}{\mu'+\nu'}\right)\mu^2 + 2A\mu;$$

and the condition $\Phi(\mu') = 0$ gives

$$\beta^2 = A\left(\frac{\mu'-\nu'}{\mu'+\nu'}\right).$$

Substituting this value of β^2, we find, since $r = \mu'+\nu$,

$$\frac{v^2}{A} = \frac{2}{r} - \frac{1}{\mu'}.$$

A particle moves on the surface and is acted on by an attractive force situated at the centre, S, of the sphere having double contact at the umbilics whose abscissa is positive, and by a repulsive force situated at S', the centre of the sphere having double contact at the umbilics whose abscissa is negative, the law of force being, as in the preceding problem, to determine the initial velocity, in order that the path may be the line of curvature $\mu = \mu'$.

The principle of vis viva gives us

$$v^2 = \frac{2A}{\mu+\nu} - \frac{2B}{\mu-\nu} + C,$$

where A and B are the absolute forces respectively,

and $$C = \beta^2 - \frac{2A}{\mu'+\nu'} + \frac{2B}{\mu'-\nu'}.$$

The condition $\Phi'(\mu) = 0$ gives us

$$\beta^2 = \frac{A(\mu'-\nu')}{\mu'+\nu'} - \frac{B(\mu'+\nu')}{\mu'-\nu'}.$$

If now we make $\beta^2 = 0$, we get

$$\mu' = \left(\frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}-\sqrt{B}}\right)\nu' = a\nu' \text{ say.}$$

Hence we see that, if the particle be placed at rest at any point of the curve $\mu = a\nu'$, it will describe the line of curvature $\mu = \mu'$ which passes through the point, and will obviously oscillate perpetually.

The curve $\mu = a\nu$ is the intersection of a sphere with the surface.

4. I now show how to find the reaction R of the surface for any cases of motion on the surface in which the forces have a potential.

Now $\frac{1}{2}\frac{dv^2}{d\sigma_0}$ is evidently the resolved part of the force perpendicular to the surface, $d\sigma_0$ being the element of the normal, and consequently

$$R = \tfrac{1}{2}\frac{dv^2}{d\sigma_0} + \frac{v^2}{\rho}\cos\theta,$$

θ being the angle the osculating plane makes with the normal. But $\frac{\cos\theta}{\rho} = \frac{1}{\rho_0}$, where ρ_0 is the radius of curvature of the normal section through the tangent line; and again $\rho_0 = \frac{\gamma^2}{p}$ (Salmon's *Geometry of Three Dimensions*, Art. 194), where γ is the semi-diameter parallel to the tangent line and p the perpendicular on the tangent plane from the centre; hence

$$R = \tfrac{1}{2}\frac{dv^2}{d\sigma_0} + \frac{v^2 p}{\gamma^2}.$$

But v^2 is a function of μ, ν, and a the primary axis of the surface, and if we consider for a moment a as variable where it enters through x, y, and z,

$$p\,d\sigma_0 = a\,da,$$

therefore finally

$$R = \tfrac{1}{2}\frac{p}{a}\cdot\frac{dv^2}{da} + \frac{pv^2}{\gamma^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

5. I now discuss some special cases of motion by assigning particular forms to $\Phi(\mu)$ and $\Psi(\nu)$ which render $\Phi(\mu)+\Psi(\nu)$ divisible by $\mu^2-\nu^2$.

In general, the differential equation of the path is

$$\frac{d\sigma}{\sqrt{\Phi(\mu)-C}} \pm \frac{d\sigma'}{\sqrt{\Psi(\nu)+C}} = 0.$$

Let us make

$$\Phi(\mu) - C = \frac{B}{\mu^2-h^2},$$

$$\Psi(\nu) + C = \frac{B}{h^2-\nu^2},$$

where B is a constant. The differential equation of the path then assumes the form

$$\sqrt{\frac{a^2-\mu^2}{k^2-\mu^2}}\,d\mu \pm \sqrt{\frac{a^2-\nu^2}{k^2-\nu^2}}\cdot d\nu = 0,$$

which is the differential equation of the orthogonal trajectory to an umbilicar geodesic. We have also

$$v^2 = \frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2} = \frac{B}{(\mu^2-h^2)(h^2-\nu^2)}.$$

Now

$$y^2 = \frac{(a^2-h^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k-h^2)},$$

hence $v^2 = \frac{A}{y^2}$, A being constant, and consequently the force is attractive and varies as $\frac{A}{y^3}$.

The principle of vis viva gives us

$$v^2 = \frac{A}{y^2} - \frac{A}{y'^2} + \beta^2,$$

therefore
$$\beta^2 = \frac{A}{y'^2}.$$

Hence:—*A particle constrained to move on the surface under the action of an attractive force varying as* $\frac{A}{y^3}$, *and projected perpendicularly to an umbilicar geodesic with the initial velocity* $\frac{\sqrt{A}}{y'}$, *will describe the curve traced on the surface by the extremity of an umbilicar geodesic of constant length.*

On the Points and Tangents common to Two Conics

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Taking the triangle *ABC* formed by three common tangents to two conics as that of reference in any system of point coordinates, let the equations to the conics be

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0 \quad \text{.......................(1)},$$

$$\sqrt{l'\alpha} + \sqrt{m'\beta} + \sqrt{n'\gamma} = 0 \quad \text{.......................(2)},$$

each radical being capable of the double sign; we may, without loss of generality, make the convention that the first term in each is to be taken positively.

For the points common to the two conics, we have

$$\frac{\sqrt{\alpha}}{\sqrt{m}\sqrt{n'} - \sqrt{n}\sqrt{m'}} = \frac{\sqrt{\beta}}{\sqrt{n}\sqrt{l'} - \sqrt{l}\sqrt{n'}} = \&c.,$$

or
$$\frac{\alpha}{mn' + nm' - 2\sqrt{m}\sqrt{n}\sqrt{m'}\sqrt{n'}} = \&c.,$$

or, say,
$$\frac{\alpha}{p - (p')} = \frac{\beta}{q - (q')} = \frac{\gamma}{r - (r')} \quad \text{.....................(3)},$$

where p', q', r' may take the double sign; but, since $q'r' = ll'p'$ always,