ON THE FAILURE OF CONVERGENCE OF FOURIER'S SERIES

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IT is well known that, if $f(x)$ is a limited integrable function, defined for the interval $(-\pi, \pi)$, the corresponding Fourier's series, at any point x at which it converges, has for its limiting sum $\underset{\epsilon=0}{\mathrm{L}} \frac{1}{2}\{f(x+\epsilon)+f(x-\epsilon)\}$, provided this limit exists at the point. It is known that the nature of the function in an arbitrarily small neighbourhood of the point x determines whether the function converges at x or not; various sufficient conditions have been found that this convergence may take place. In the present communication a general theorem is obtained relating to the most general distribution of the points of the interval at which the series either converges to the value of the function, or can be made so to converge by bracketing the terms of the series in a suitable manner. In the latter case the series is an oscillating one: some preliminary remarks are accordingly made on the subject of oscillating series. An example, due to Schwarz, of a series which represents a continuous function, and fails to converge at a particular point, is considered in detail, and it is shown that the series is in reality an oscillating one, at the point, and that by introducing a suitable system of brackets, but without altering the order of the terms, the series can be made to converge to the value of the function. Finally, a function is constructed which fails to converge at points in every interval contained in the interval $(-\pi, \pi)$. A more complicated example of such a function has been given* by P. Du Bois Reymond.

On Oscillating Series.

1. Let us suppose that $u_1, u_2, \ldots, u_n, \ldots$ is an unending sequence of numbers, such that u_n has for each value of n a definite numerical value assigned by means of a prescribed law; let s_n denote the sum $u_1+u_2+\ldots+u_n$, and let us consider the aggregate $(s_1, s_2, \ldots, s_n, \ldots)$, the elements of which may be denoted in the usual manner by points on a

* Abhandlungen der bayerischen Akademie, Vol. XII.

straight line. This enumerable set of points will be denoted by G; the following cases may arise:—

(1) The set G may consist of points all of which lie between two fixed points A, B, and the derivative G' of the set G may consist of a single point s. In this case the series $u_1+u_2+\ldots+u_n+\ldots$ is convergent, and s is its limiting sum.

(2) The set G may be unlimited in one direction or in both directions, and the derivative G' may be non-existent. It may be said that G has the improper limiting point $+\infty$, or the improper limiting point $-\infty$; it may have both $+\infty$ and $-\infty$ as improper limiting points. In either case $|s_n|$ has no upper limit, and the series $u_1+u_2+\ldots+u_n+\ldots$ is said to be *divergent*. There are thus two species of divergent series: for example, the series $1/1+1/2+\ldots+1/n+\ldots$ is divergent and has the improper limiting point $+\infty$, whereas the series

$$1-2+3-4+\ldots+(2n-1)-2n+\ldots$$

has the two improper limiting points $+\infty$, $-\infty$.

(3) The set G may consist of points all lying between two fixed points A, B, and the derivative G' may consist of more than one point; in this case the series is said to be an *oscillating* series. The set G' may contain a finite or an infinite number of points, but is, in any case, in accordance with a well-known theorem, a closed set, and consequently has an upper limit U and a lower limit L. These limits U and L are called the *limits of indeterminacy* of the series Σu_n.

It is always possible to find a sequence $(s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$ of partial sums, where $n_1 < n_2 < n_3 \ldots$, which converges to the limit U, and another such sequence which converges to the limit L, or one which converges to any prescribed point of G'. It thus appears that, by means of a suitable system of brackets, the oscillating series Σu_n may be converted into a convergent series of which the sum is a prescribed point of G', the terms in each bracket being amalgamated. The set G' may be non-dense in the interval (L, U), or it may consist of all the points of that interval, or it may consist of a closed set of the most general type which is dense in some parts of the interval (L, U).

For example, the series $1-1+1-1+1-\ldots$ has the points 1, 0 for the upper and lower limits of indeterminacy, and G' consists of these two points.

Again, it is easy to construct a series which oscillates between the limits of indeterminacy 0, 1, and such that G' consists of the whole interval (0, 1).

Let $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{3}$, $s_3 = \frac{1}{4}$, $s_4 = \frac{2}{3}$, $s_5 = \frac{1}{5}$,

$s_6 = \frac{2}{4}$, $s_7 = \frac{1}{6}$, $s_8 = \frac{2}{5}$, $s_9 = \frac{3}{4}$, $s_{10} = \frac{1}{7}$, ...,

and generally

$$s_{m(m+1)+1} = \frac{1}{2m+2}, \quad s_{m(m+1)+2} = \frac{2}{2m+1}, \quad \dots, \quad s_{(m+1)^2} = \frac{m+1}{m+2},$$

$$s_{(m+1)^2+1} = \frac{1}{2m+3}, \quad s_{(m+1)^2+2} = \frac{2}{2m+2}, \quad \dots, \quad s_{(m+1)(m+2)} = \frac{m+1}{m+3}.$$

The set G consists of all the rational numbers between 0 and 1, and thus G' consists of the whole interval (0, 1).

It follows that the series

$$\frac{1}{2} - \frac{1}{2.3} - \frac{1}{3.4} + \frac{5}{3.4} - \frac{7}{3.5} + \dots$$

where

$$u_{m(m+1)+1} = \frac{1}{2m+2} - \frac{m}{m+2}, \qquad u_{m(m+1)+2} = \frac{2}{2m+1} - \frac{1}{2m+2},$$

$$u_{m(m+1)+3} = \frac{3}{2m} - \frac{2}{2m+1}, \qquad \dots, \qquad u_{(m+1)^2} = \frac{m+1}{m+2} - \frac{m}{m+3},$$

$$u_{(m+1)^2+1} = \frac{1}{2m+3} - \frac{m+1}{m+2}, \qquad u_{(m+1)^2+2} = \frac{2}{2m+2} - \frac{1}{2m+3}, \qquad \dots$$

can have its terms so bracketed that the sum of the resulting series is any prescribed number in the interval (0, 1).

(4) The derivative G' may exist, but it may be unlimited in one or in both directions: thus U may have the improper value $+\infty$, or L may have the improper value $-\infty$, or both these cases may arise simultaneously. In this case the series oscillates between limits one or both of which are indefinitely great. The series may be made to diverge by introducing a suitable system of brackets, or it may be made to converge to any point of G'.

For example, a series may be constructed which oscillates between infinite limits of indeterminacy, but which, by introducing a suitable system of brackets, may be made to converge to any prescribed number whatever.

If $x' = \dfrac{2x-1}{\sqrt{\{x(1-x)\}}}$, where the positive sign is ascribed to the radical, the points x of the interval (0, 1) have a (1, 1) correspondence with the points x' of the interval $(-\infty, \infty)$; it is clear that a set of points in the interval (0, 1) corresponds to a set in the interval $(-\infty, \infty)$, the relative order of pairs of points in the one interval being the same as that of the corresponding points in the other interval. Further, a limiting point of a set of points in the interval (0, 1) corresponds to a limiting point of the corresponding set of points in the interval $(-\infty, \infty)$. The rational points within the interval (0, 1) correspond to a set of points everywhere dense in the unlimited line $(-\infty, +\infty)$. We may apply this method to transform the series obtained in (3) which oscillates between the limits of indeterminacy 0, 1, and which can be made by introducing suitable brackets to converge to any number in the interval (0, 1).

We find

$$s_1' = 0, \quad s_2' = -\frac{1}{\sqrt{2}}, \quad s_3' = -\frac{2}{\sqrt{3}}, \quad s_4' = \frac{1}{\sqrt{2}}, \quad s_5' = -\frac{3}{2},$$

$$s_6' = 0, \quad s_7' = -\frac{4}{\sqrt{5}}, \quad s_8' = -\frac{1}{\sqrt{6}}, \quad s_9' = \frac{2}{\sqrt{3}}, \quad s_{10}' = -\frac{5}{\sqrt{6}}, \quad \ldots,$$

and generally

$$s'_{m(m+1)+1} = -\frac{2m}{\sqrt{(2m+1)}}, \qquad s'_{m(m+1)+2} = -\frac{2m-3}{\sqrt{\{2(2m-1)\}}}, \qquad \ldots,$$

$$s'_{(m+1)^2} = \frac{m}{\sqrt{(m+1)}}, \qquad s'_{(m+1)^2+1} = -\frac{2m+1}{\sqrt{(2m+2)}}, \qquad \ldots,$$

$$s'_{(m+1)(m+2)} = \frac{m-1}{\sqrt{\{2(m+1)\}}}.$$

Thus the series

$$-\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{2}}\right) - \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right) + \frac{3}{2}$$
$$- \frac{4}{\sqrt{5}} + \left(\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right) + \ldots$$

has the required character: it may be made to converge to any value whatever by suitably bracketing the terms together and amalgamating the terms in each bracket, without altering the order of the terms.

The transformation $x' = \dfrac{2x-1}{\sqrt{\{x(1-x)\}}}$ is an example of an unlimited number of transformations by means of which sets of points in a finite interval may be made to correspond with sets of points in an unlimited straight line, the ordering of the sets being the same. Another simple transformation of this kind is $x' = \tan \pi x/2$, by which a set of points x in the interval $(-1, +1)$ is transformed into a set of points x' on an unlimited straight line.

The Points at which Fourier's Series does not converge.

2. Let $f(x)$, $\phi(x)$ denote two functions which are limited and integrable in the interval $(-\pi, \pi)$, and let a_s, b_s denote the Fourier's coefficients $\int_{-\pi}^{\pi} f(x)\cos sx\,dx$, $\int_{-\pi}^{\pi} f(x)\sin sx\,dx$, the corresponding coefficients for the function $\phi(x)$ being denoted by a'_s, b'_s. It has been well known for a long time that, provided the Fourier's series for the functions $f(x)$, $\phi(x)$ converge uniformly in the domain $(-\pi, \pi)$, the series $\tfrac{1}{2}a_0a'_0+\sum_1^n(a_sa'_s+b_sb'_s)$ converges, as n is indefinitely increased, to the sum $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\phi(x)\,dx$. The theorem has, however, more recently been established,* that the series converges to the same sum, quite independently of the mode of convergence of the Fourier's series, and in fact independently of any assumption that these series converge at all.

In the particular case in which the two functions $f(x)$, $\phi(x)$ are identical, the theorem takes the form that for any limited integrable function $f(x)$, the series $\tfrac{1}{2}a_0^2+\sum_1(a_s^2+b_s^2)$ is always convergent and has for its sum $\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2dx$, and that this is true independently of any assumption as to the convergence of the Fourier's series

$$\tfrac{1}{2}a_0+\sum_1^{\infty}(a_s\cos sx+b_s\sin sx).$$

This theorem will be here applied to an examination of the properties of the Fourier's series.

The function $f(x)$ being limited and integrable in the interval $(-\pi, \pi)$, let R_n denote $f(x)-\tfrac{1}{2}a_0-\sum_1^n(a_s\cos sx+b_s\sin sx)$; we have

$$\int_{-\pi}^{\pi} R_n^2\,dx = \int_{-\pi}^{\pi}\{f(x)\}^2dx-\pi\left[\tfrac{1}{2}a_0^2+\sum_1^n(a_s^2+b_s^2)\right].$$

The expression on the right-hand side is essentially positive, and converges to zero as n is indefinitely increased. Let ϵ, m be two fixed numbers which may be so chosen that $\epsilon^2/4m$ is as small as we please; then an integer N exists, dependent on m and ϵ, such that

$$\int_{-\pi}^{\pi} R_n^2\,dx < \frac{\epsilon^2}{4m}, \text{ provided } n \geqslant N.$$

* The first proof was given by De la Vallée Poussin; see *Annales de la Soc. Scient. de Bruxelles*; other proofs have been given by Hurwitz, *Math. Annalen*, Vol. LVII., and by Fischer, *Monatshefte für Math. u. Physik*, Vol. XV.

Let us take a fixed value of n which is $\geqslant N$; then, since $|R_n|$ is an integrable function, the interval $(-\pi, \pi)$ can be divided into a finite number of parts such that the sum of those parts in each of which the fluctuation of $|R_n|$ is $\geqslant \frac{1}{2}\epsilon$ is less than an arbitrarily chosen positive number η. In each of the other parts, the fluctuation of $|R_n|$ is $< \frac{1}{2}\epsilon$. Let these latter parts be denoted by δ_n; then $\int R_n^2 dx$, taken through all the parts δ_n, is $< \epsilon^2/4m$. If one of the intervals of δ_n contains a point at which $|R_n| \geqslant \epsilon$, then at every point in that interval $|R_n| \geqslant \frac{1}{2}\epsilon$; it is clear that the sum of those of the intervals δ_n for the whole of each of which $|R_n| \geqslant \frac{1}{2}\epsilon$ must be less than $1/m$. It has now been shown that there exists a finite set of intervals of which the sum is $> 2\pi - \eta - 1/m$, such that at every point in them the condition $|R_n| < \epsilon$ is satisfied; the set of intervals depends upon the particular value of n chosen.

The number of intervals in the set depends upon the value of η, and may increase indefinitely as η is indefinitely diminished; it follows that for each value of n which is $\geqslant N$ there exists a measurable set of points such that at each point $|R_n| < \epsilon$, the measure of the set being

$$\geqslant 2\pi - 1/m.$$

It has now been established that a sequence $G_N, G_{N+1}, G_{N+2}, \dots$ of measurable sets of points exists such that the measure of each set is $\geqslant 2\pi - 1/m$, and such that at any point of any one of them G_n the condition $|R_n| < \epsilon$ is satisfied.

The following theorem* in the theory of sets of points will now be applied to the sequence $G_N, G_{N+1}, G_{N+2}, \dots$:—

If $P_1, P_2, \dots, P_n, \dots$ is a sequence of sets of points, each of which sets is a component of a closed set of finite content l, and if the interior measure of each of the sets is greater than a fixed number C, then there exists a set of points of interior measure $\geqslant C$, and of the power of the continuum, such that each point of the set belongs to an infinite number of the given sets.

In our case the sets $G_N, G_{N+1}, G_{N+2}, \dots$ are all measurable and their measures are all $> 2\pi - 1/m - \eta$, however small η may be; it follows that a set exists of interior measure $\geqslant 2\pi - 1/m - \eta$, each point of

* This theorem was stated and proved by W. H. Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, in his paper on "Open Sets and the Theory of Content." The theorem, so far as it relates to measurable sets, was stated without proof by Borel in the *Comptes Rendus* for December, 1903. The expression "measure" is employed in the present paper in accordance with the usage of Borel and Lebesgue, instead of the expression "content" employed by W. H. Young, because the latter has been used in another sense by Harnack, Cantor, and others. The word content is here restricted to the case of closed sets, in which case the "content" of Harnack and the "measure" of Borel are identical.

which belongs to an infinite number of the sets $G_N, G_{N+1}, \ldots$. Since η is arbitrarily small, it follows that the interior measure of the set of points thus found is $\geqslant 2\pi - 1/m$. For each point of this set $G(\epsilon)$ there are an infinite number of values of n such that $|R_n| < \epsilon$. Let us consider the set H_ϵ of points for each of which $|R_n| \geqslant \epsilon$, for all values of n except for a finite number of such values, and let a be the exterior measure of this set, and, if possible, let $a > 0$; for the complementary set which is of interior measure $2\pi - a$, there must be for each point an infinite number of values of n for which $|R_n| < \epsilon$. Let us choose m so large that $1/m < a$; then for each point of $G(\epsilon)$ which is of interior measure $2\pi - 1/m > 2\pi - a$ there are an infinite number of values of n for which $|R_n| < \epsilon$, and these cannot all be included in the set complementary to H_ϵ; it follows that it is impossible that $a > 0$; thus the set H_ϵ is measurable, and its measure is zero.

Since the set H_ϵ of those points for each of which $|R_n| < \epsilon$ at most for a finite number of values of n has zero measure, it follows that the complementary set K_ϵ, for each point of which $|R_n| < \epsilon$, for an infinite number of values of n, has the measure 2π. Let us now take a sequence of diminishing values of ϵ, say $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$, which converges to zero, and consider the sets $K_{\epsilon_1}, K_{\epsilon_2}, \ldots, K_{\epsilon_n}, \ldots$; each of these sets has the measure 2π: it follows, by applying again the theorem in sets of points already employed, that there exists a set of points of measure 2π each point of which belongs to an infinite number of the sets $K_{\epsilon_1}, K_{\epsilon_2}, \ldots, K_{\epsilon_n}, \ldots$. This set L is such that for any point P of it a sequence $\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}, \ldots$ of values of ϵ belonging to the sequence $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ exists such that $|R_n| < \epsilon_{p_1}$ for an infinite number of values of n, $|R_n| < \epsilon_{p_2}$ for an infinite number of values of n, and so on. The sequence $\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}, \ldots$ converges to the limit zero; thus for any point of the set L we have $\lim R_n = 0$, provided a properly chosen sequence of increasing values of n is taken.

For any point of the set L, either the Fourier's series converges to the value $f(x)$, or else it oscillates between finite or infinite limits of indeterminacy, but so that the functional value at the point is a limiting point of the partial sums of the series. In the latter case the series can be made to converge to the value $f(x)$, by bracketing the terms in a suitable manner, the terms in any one bracket being regarded as amalgamated; no change is made in the order of the terms.

If for any value of the variable x the Fourier's series does not converge to the value $f(x)$, but can be made to converge to that value by introducing a suitable system of brackets, the terms in each bracket *being amalgamated, but the order of the terms being unaltered, the series will be said to be* quasi-convergent *with* $f(x)$ *as its sum.*

The set complementary to L has zero measure, and contains (1) all those points for which the Fourier's series is divergent, (2) points at which the function $f(x)$ is discontinuous and the series converges to a value different from $f(x)$, and (3) points at which the series oscillates but is not quasi-convergent with $f(x)$ as sum.

The following general theorem has now been established :—

If $f(x)$ *is a limited integrable function defined for the interval* $(-\pi, \pi)$, *there exists in this interval a set of points L of which the measure is equal to that of the whole interval, such that at each point of L the Fourier's series either converges to* $f(x)$, *or is quasi-convergent with* $f(x)$ *as sum. The complementary set is of zero measure, and contains all points at which the series is divergent, or converges to a value different from* $f(x)$, *or oscillates without being quasi-convergent.*

The set L is everywhere dense in the interval, and has the cardinal number of the continuum. The complementary set may or may not be everywhere dense.

In the case in which $f(x)$ is a continuous function, the set complementary to L consists of points at which the Fourier's series diverges or oscillates without being quasi-convergent.

The theorem leaves the possibility open that, even in the case of a continuous function, there may be no point in the interval at which the series is convergent.

A Continuous Function for which the Series does not converge.

3. It is well known that the question as to whether a Fourier's series converges at a particular point must be answered by an examination of the limit of an integral of Dirichlet's type

$$\int_0^a \phi(z) \frac{\sin(2m+1)z}{\sin z} dz \qquad (0 < a \leqslant \tfrac{1}{2}\pi),$$

when the positive integer m is indefinitely increased.

Let the product $1.3.5 \ldots (2\lambda+1)$ be denoted by $[2\lambda+1]$, and let the function $\phi(z)$ be defined for the interval $(0, a)$ in the following manner :—In the interval $(\pi/[\lambda-1], \pi/[\lambda])$ let $\phi(z) = c_\lambda \sin[\lambda]z$, where c_λ is a constant dependent upon the value of λ; let λ have all values $\lambda_1, \lambda_1+1, \lambda_1+2, \ldots$ where λ_1 is a fixed integer, and we may suppose a so chosen that $a = \pi/[\lambda_1-1]$; also let $\phi(0) = 0$. If the sequence $c_{\lambda_1}, c_{\lambda_1+1}, c_{\lambda_1+2}, \ldots$ be so chosen that it converges to the limit zero, the function $\phi(z)$ is continuous at the point $z = 0$, but has an indefinitely great number of oscillations in an arbitrarily small neighbourhood of that point. It has been shown by Schwarz* that, if c_λ is so chosen that $c_\lambda \log(2\lambda+1)$ becomes indefinitely great as λ is indefinitely increased, the integral may become indefinitely great as m is increased indefinitely, and thus that a Fourier's series exists which does not represent the given continuous function at a particular point.

It will here be shown that the series is in reality oscillatory, and that the point $z = 0$ is a point of quasi-convergence in the sense defined above. A simplified proof of Schwarz's result will first be given.

It is known that the integral $\int_0^\alpha \phi(z) \frac{\sin(2m+1)z}{\sin z} dz$ may, for the purpose of the consideration of the limiting values when m is indefinitely increased, be replaced by† $\int_0^\alpha \phi(z) \frac{\sin(2m+1)z}{z} dz$. Thus we may consider the latter integral.

Let $$2m+1 = 1, 3, 5 \dots 2\mu+1 \equiv [\mu];$$

then $$\int_0^\alpha \phi(z) \frac{\sin[\mu]z}{z} dz$$
$$= c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2[\mu]z}{z} dz + \sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin[\mu]z}{z} dz$$
$$+ \sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin[\mu]z}{z} dz;$$

the first integral on the right-hand side may be written in the form

$$\tfrac{1}{2}c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{1-\cos 2[\mu]z}{z} dz,$$

which is equivalent to

$$\tfrac{1}{2}c_\mu \log(2\mu+1) - \tfrac{1}{2}c_\mu \frac{[\mu]}{\pi} \int_{\pi/[\mu]}^{\beta} \cos 2[\mu]z\, dz,$$

where β is a number between $\pi/[\mu]$ and $\pi/[\mu-1]$.

Now let $c_\mu \log(2\mu+1)$ increase indefinitely with μ; this is consistent with c_μ having a zero limit, for we have only to take $c_\mu = \frac{1}{\{\log(2\mu+1)\}^s}$, where s is some fixed positive number less than unity.

Since $c_\mu \frac{[\mu]}{\pi} \int_{\pi/[\mu]}^{\beta} \cos 2[\mu]z\, dz$ is numerically not greater than c_μ/π,

* See the history of the theory of Fourier's series, by Sachs, *Schlömilch's Zeitschrift*, Supplement, Vol. XXV.

† A rigid proof of this is given by Brodén, *Math. Annalen*, Vol. LII., p. 220.

we see that, with the supposition made as to c_μ, the expression

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2 [\mu] z}{z} \, dz$$

becomes indefinitely great as μ is increased indefinitely.

To evaluate
$$\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin [n] z \sin [\mu] z}{z} \, dz,$$

we see, by writing $\sin [n] z \sin [\mu] z$ as half the difference of two cosines, and applying the second mean value theorem to each integral, that the absolute value of the expression is less than

$$\sum_{n=\lambda_1}^{\mu-1} c_n \frac{[n]}{\pi} \left\{ \frac{1}{[\mu]-[n]} + \frac{1}{[\mu]+[n]} \right\}$$

or than
$$\sum_{n=\lambda_1}^{\mu-1} \frac{c_n}{\pi} \frac{[n]}{[\mu-1]} \left\{ \frac{1}{2\mu+1-\frac{[n]}{[\mu-1]}} + \frac{1}{2\mu+1+\frac{[n]}{[\mu-1]}} \right\},$$

which is less than
$$\frac{c_{\lambda_1}}{\pi} \sum \frac{[n]}{[\mu-1]} \frac{1}{\mu},$$

and this is
$$\frac{c_{\lambda_1}}{\pi\mu} \left\{ 1 + \frac{1}{2\mu-1} + \frac{1}{(2\mu-1)(2\mu-3)} + \dots \right\}.$$

Thus the absolute value of the expression is less than $2c_\lambda/\pi\mu$, and this becomes indefinitely small as μ is indefinitely increased; thus the limiting value of the expression is zero.

Finally, we have to consider the expression

$$\sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin [n] z \sin [\mu] z}{z} \, dz \, ;$$

since $\left| \frac{\sin [\mu] z}{z} \right| < [\mu]$, and $|\sin [n] z| \leqslant 1$, the absolute value of the expression is less than $c_{\mu+1} [\mu] \frac{\pi}{[\mu]}$, and this has the limit zero when μ is indefinitely increased.

Schwarz's theorem has now been established, that

$$\int_0^a \frac{\phi(z) \sin [\mu] z}{\sin z} \, dz$$

increases indefinitely as μ is indefinitely increased, where

$$[\mu] = 1.3.5 \dots (2\mu+1),$$

and $\phi(z)$ is defined by $\phi(0) = 0$, $\phi(z) = c_\lambda \sin [\lambda] z$ in the interval $(\pi/[\lambda], \pi/[\lambda-1])$ where $\lambda = \lambda_1, \lambda_1+1, \lambda_1+2, \dots$ and $a = \pi/[\lambda_1-1]$, provided c_λ has the value $1/\{\log(2\lambda+1)\}^s$, where $0 < s < 1$.

4. We proceed to consider the case in which $2m+1 = (2p+1)[\mu-1]$, where p is an integer which varies with μ in such a manner that it always lies between 0 and μ.

In this case, as before, we divide the integral $\int_0^a \phi(z) \frac{\sin(2m+1)z}{z} dz$ into the three parts

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \sin[\mu]z \frac{\sin(2p+1)[\mu-1]z}{z} dz,$$

$$+\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin(2p+1)[\mu-1]z}{z} dz,$$

$$+\sum_{n=\mu+1}^{\infty} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin(2p+1)[\mu-1]z}{z} dz.$$

The first part is equal to

$$\frac{c_\mu[\mu]}{2\pi} \int_{\pi/[\mu]}^{\beta} \Big[\cos\{[\mu-1](2\mu-2p)\} - \cos\{[\mu-1](2\mu+2p+2)\}\Big] dz$$

where β is a number between $\pi/[\mu]$ and $\pi/[\mu-1]$, and this expression is less in absolute value than

$$\frac{c_\mu[\mu]}{\pi} \left\{\frac{1}{[\mu-1](2\mu-2p)} + \frac{1}{[\mu-1](2\mu+2p+2)}\right\},$$

or than

$$\frac{c_\mu}{\pi} \left\{\frac{2\mu+1}{2\mu-2p} + \frac{2\mu+1}{2\mu+2p+2}\right\},$$

and this may be written

$$\frac{c_\mu}{\pi} \left\{\frac{1+\frac{1}{2\mu}}{1-\frac{p}{\mu}} + \frac{1+\frac{1}{2\mu}}{1+\frac{1}{\mu}+\frac{p}{\mu}}\right\}.$$

If, now, p increases with μ in such a manner that p/μ is always less than some fixed number which is less than unity, this expression diminishes indefinitely as μ is indefinitely increased; it would also be sufficient that $p/\mu = 1-\kappa/\{\log(2\mu+1)\}^{s'}$, where $s' < s$, $c_\mu = 1/\{\log(2\mu+1)\}^s$, and κ is finite.

Next, we have $\sum_{n=\lambda_1}^{\mu-1} c_n \int_{\pi/[n]}^{\pi/[n-1]} \frac{\sin[n]z \sin(2p+1)[\mu-1]z}{z} dz$ is less in absolute value than

$$\sum_{n=\lambda_1}^{\mu-1} \frac{c_n[n]}{\pi} \left\{\frac{1}{(2p+1)[\mu-1]-[n]} + \frac{1}{(2p+1)[\mu-1]+[n]}\right\}$$

or than
$$\frac{c_{\lambda_1}}{\pi}\Sigma\frac{[n]}{[\mu-1]}\left\{\frac{1}{2p+1-\frac{[n]}{[\mu-1]}}+\frac{1}{2p+1+\frac{[n]}{[\mu-1]}}\right\},$$
and this is less than
$$\frac{c_{\lambda_1}}{p\pi}\left\{1+\frac{1}{2\mu-1}+\frac{1}{(2\mu-1)(2\mu-3)}+\dots\right\}$$
or than $2c_{\lambda_1}/p\pi$; thus the expression becomes indefinitely small when p is increased indefinitely.

That
$$\sum_{n=\mu+1}^{\infty} c_n\int_{\pi/[n]}^{\pi/[n-1]}\frac{\sin[n]z\sin(2p+1)[\mu-1]z}{z}\,dz$$
has the limit zero is seen from the fact that its absolute value is less than $c_{\mu+1}(2p+1)[\mu-1]\frac{\pi}{[\mu]}$, or than $\pi c_{\mu+1}\frac{2p+1}{2\mu+1}$.

We have now established that $\int_0^a\phi(z)\frac{\sin(2m+1)z}{z}\,dz$ has the limit zero if $2m+1$ increases indefinitely through a sequence of the form
$$[\mu_1-1](2p_1+1),\qquad[\mu_2-1](2p_2+1),\qquad[\mu_3-1](2p_3+1),\qquad\dots$$
where $\mu_1, \mu_2, \mu_3, \dots$ is an increasing sequence of integers, and $p_1, p_2, p_3, \dots$ are such that $p/\mu \leqslant 1-\kappa/\{\log(2\mu+1)\}^s$.

The limit of the same integral has been shown to be infinite if $2m+1$ increases indefinitely through a sequence of values $[\mu_1]$, $[\mu_2]$, $[\mu_3]$, ...

A Continuous Function for which the Series does not converge at a Dense Set of Points.

5. Let the continuous function $f(x)$ be defined for the interval $(-\pi, \pi)$ as follows:—If $-\pi\leqslant x\leqslant\xi$, where ξ is a fixed point in the interval, let $f(x)=0$; if $0\leqslant x-\xi<2a$, let $f(x)=\phi\left(\frac{x-\xi}{2}\right)$, where $\phi\left(\frac{x-\xi}{2}\right)$ is the function $\phi(z)$ which has been already discussed; in case $\xi+2a<\pi$, we take $f(x)=0$, for $\xi+2a<x\leqslant\pi$.

The limit of the sum of the first $2m+1$ terms of the Fourier's series for the function $f(x)$ is that of the expression
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x')\frac{\sin(2m+1)\frac{x'-x}{2}}{\sin\frac{x'-x}{2}}\,dx',$$

and at the point ξ the limit depends upon that of

$$\frac{1}{\pi}\int_0^a \phi(z)\frac{\sin(2m+1)z}{\sin z}\,dz.$$

It has been shown that this limit is zero, or indefinitely great, according to the nature of the sequence of values through which $2m+1$ increases indefinitely. It follows that, at the point ξ, the Fourier's series is not convergent, but is quasi-convergent to the value $f(\xi)\equiv 0$ of the function.

Let us now denote the function $f(x)$ by $\mathfrak{f}(x, \xi)$, and let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be an enumerable set of values of ξ, everywhere dense in the interval $(-\pi, \pi)$, and let us consider the function

$$F(x) = c_1\mathfrak{f}(x, \xi_1)+c_2\mathfrak{f}(x, \xi_2)+\ldots+c_n\mathfrak{f}(x, \xi_n)+\ldots,$$

where $c_1, c_2, \ldots, c_n, \ldots$ are constants so chosen that the series

$$c_1+c_2+\ldots+c_n+\ldots$$

is absolutely convergent.

Since the upper limits of each of the functions $|\mathfrak{f}(x, \xi)|$ has the same finite value, it follows that the series $c_1\mathfrak{f}(x, \xi_1)+c_2\mathfrak{f}(x, \xi_2)+\ldots$ is uniformly convergent in the interval $(-\pi, \pi)$, and thus the function $F(x)$ is continuous, and the expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x')\frac{\sin(2m+1)\frac{x'-x}{2}}{\sin\frac{x'-x}{2}}\,dx'$$

is equal to the sum

$$\frac{1}{2\pi}\sum_1^\infty c_n\int_{-\pi}^{\pi}\mathfrak{f}(x', \xi_n)\frac{\sin(2m+1)\frac{x'-x}{2}}{\frac{x'-x}{2}}\,dx',$$

which may be written in the form

$$c_1\chi_1(x, m)+c_2\chi_2(x, m)+\ldots+c_n\chi_n(x, m)+\ldots,$$

where $\underset{m=\infty}{\mathrm{L}}\chi_1(x, m)$ is zero, unless $x=\xi_1$, at which point the limit may be either 0 or ∞, according to the mode in which m is indefinitely increased; a similar statement holds as regards $\chi_2(x, m)$ at the point ξ_2, and generally $\underset{m=\infty}{\mathrm{L}}\chi_n(x, m)$ is zero, unless $x=\xi_n$, in which case the limit depends upon the mode in which m becomes infinite.

At the point ξ_n, the term $c_n\chi_n(x, m)$ has an infinite limit provided m is increased indefinitely in a proper manner, but it might happen that the limit of

$$c_{n+1}\chi_{n+1}(\xi_n, m)+c_{n+2}\chi_{n+2}(\xi_n, m)+\ldots$$

is also infinite, although each separate term has a zero limit; in that case the limit of the whole expression for the sum of the series might be finite or zero, in whatever manner m were made indefinitely great. If this happened for a particular set of values of the constants $c_1, c_2, \ldots, c_n, c_{n+1}, \ldots$, it would no longer happen if these constants were replaced by $c_1 e_1$, $c_2 e_1 e_2$, $c_3 e_1 e_2 e_3$, ..., $c_n e_1 e_2 \ldots e_n$, ..., where $e_1, e_2, e_3, \ldots$ is a sequence of descending positive numbers, provided they are properly chosen. For, if

$$c_n \chi_n(\xi_n, m) + \{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots\},$$

when m is indefinitely increased, were finite, being dependent on the form $\infty - \infty$, the expression

$$e_1 e_2 \ldots e_n c_n \chi_n(\xi_n, m) + e_1 e_2 \ldots e_{n+1}\{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2}\chi_{n-2}(\xi_n, m) + \ldots\}$$

would also be finite or zero, only in case

$$e_{n+1}\frac{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2}\chi_{n+2}(\xi_n, m) + \ldots}{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots}$$

had as its limit unity, when m is indefinitely increased. But this limit can be altered by changing e_{n+1}, without altering $e_{n+2}, e_{n+3}, \ldots$; and thus e_{n+1} can certainly be chosen so that this expression does not converge to unity when m is indefinitely increased.

It has thus been shown that, by choosing the numbers $e_1, e_2, \ldots$ properly, the limit of the ratio o

$$e_1 e_2 \ldots e_n e_{n+1}\{c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2} e_{n+2}\chi_{n+2}(\xi_n, m) + \ldots\}$$

to

$$e_1 e_2 \ldots e_n c_n \chi_n(x, m)$$

will be different from what it was when all the e's were equal to unity. It has therefore been shown that, by altering the numbers $c_1, c_2, c_3, \ldots$ in a suitable way, the infinite limit of $c_n \chi_n(\xi_n, m)$ when m is indefinitely increased will no longer be removed by an infinite limit of the sum

$$c_{n+1}\chi_{n+1}(\xi_n, m) + c_{n+2}\chi_{n+2}(\xi_n, m) + \ldots.$$

It has thus been shown that it is possible to choose the numbers $c_1, c_2, c_3, \ldots$ *in such a manner that the continuous function*

$$F(x) = \sum_1^\infty c_n \mathfrak{f}(x, \xi_n)$$

is such that the Fourier's series fails to converge to the value of the function at each point of the everywhere dense set of points $(\xi_1, \xi_2, \ldots \xi_n)$.