On Clifford's Theory of Graphs. By A. Buchheim, M.A.

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In the present paper I attempt to reconstruct Clifford's theory of Graphs, from the lithographed volume of Mathematical Fragments, and from the letter to Prof. Sylvester, published in the first volume of the *American Journal of Mathematics.*

The first published account of a theory of graphs is contained in Prof. Sylvester's paper "On an Application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics" (*American Journal*, Vol. I., p. 64). In this paper Prof. Sylvester showed how any concomitant of a binary quantic could be graphically represented by a figure entirely analogous to the graphic formulæ used by chemists. In a letter to Prof. Sylvester, printed in the same volume of the *Journal*, Clifford showed that Sylvester's qualitative representation could be made quantitative, inasmuch as the graph could be interpreted as a direction to perform certain multiplications which would result in the form in question. It is this quantitative theory of graphs that I have attempted to explain in this paper. As regards the contents of the paper, I remark that I have given certain preliminary explanations of the elementary processes employed, and have then investigated the theory of the cubic and quartic. The parts of Clifford's Fragments that I have not considered are:—(1) The theory of systems of quantics, where the necessity of distinguishing between different forms makes the use of graphs troublesome, and where very little seems to be gained by using them. (2) The theory of the quintic, where Clifford has treated an unsymmetric graph as if it were symmetrical, and where the correct theory would involve more trouble than it seems worth. (3) A few fragmentary notes, some of which I was unable to understand.

It must be distinctly understood that, excepting a few corrections and the last section (on form-systems), this paper contains nothing that is not explicitly or implicitly contained in Clifford's Fragments,* and that my only object has been to make Clifford's theory more accessible, in the hope that it may be taken up by others, so that it may appear whether the method is likely to lead to new results. I must own that, owing to its essential

* An alteration which I have made in Clifford's method is pointed out and justified below.

identity with the symbolic methods employed by Cayley and Clebsch, it does not seem likely to furnish anything that could not be found quite as easily by the older methods; at the same time, there can, I think, be very little doubt that the representation of a concomitant by a graph throws considerable light on the genesis of a form system, and on Gordan's proof of the existence of a finite form system as presented by Clebsch.

I am not quite sure that I have presented the theory from Clifford's point of view; it is not quite clear what he conceived to be the function of the polar variables in a form, and what relation he supposed the polar form to bear to the ultimate form in which all the variables are scalars. The only passage bearing on this point (*Math. Papers*, p. 256, l. 24) is by no means conclusive, but I imagine that Clifford did not regard the polar form as a blank form to be filled up by multiplication by a polar variable, which is the point of view from which the form is considered in this paper.

I. *Fundamental Operations and Notation.*

If we use a well-known symbolic notation, we can write

$$a_{11}x_1y_1+a_{12}x_1y_2+a_{21}x_2y_1+a_{22}x_2y_2=(a_1x_1+a_2x_2)(a_1y_1+a_2y_2)=a_xa_y,$$

if we stipulate that $a_ia_k=a_{ik}$; and then, if we assume that the a's are to obey the commutative law, we have

$$a_{12}=a_1a_2=a_2a_1=a_{21},$$

that is to say, if we get a lineo-linear form by a symbolic multiplication of two linear forms, the resulting form must be *symmetrical*,* if the coefficients of the linear forms obey the commutative law.

In the same way, if we multiply together any number of linear forms, we get a form linear in the same number of variables, and it is easy to see that, if the coefficients of the linear forms obey the commutative law, the resulting multiple linear form will be symmetrical; that is to say, that all terms with the same number of 1's and the same number of 2's in the subscript indices will have the same coefficient; thus, for a triply linear form, we get a set of terms

$$x_1y_1z_2,\quad x_1y_2z_1,\quad x_2y_1z_1,$$

and, since each of these terms has two 1's and one 2 in the subscript indices, they will all have the same coefficient, $a_1^2a_2$; moreover, we have

$$a_xa_y=a_ya_x,$$

$$a_xa_ya_z=a_xa_za_y=\&c.$$

* As regards interchanges of x_1 with y_1 and x_2 with y_2.

Now, suppose we multiply two linear forms together without making any stipulation as to commutative multiplication, we shall obviously get an unsymmetrical lineo-linear form; for we get

$$(a_1x_1+a_2x_2)(a_1y_1+a_2y_2) = a_1a_1x_1y_1+a_1a_2x_1y_2+a_2a_1x_2y_1+a_2a_2x_2y_2,$$

and, if we do not stipulate that

$$a_1a_2 = a_2a_1,$$

this form is obviously unsymmetrical. In the same way, if we multiply n linear forms together, we shall get an n-tuply linear form, and it is obvious that it will consist of n^2 terms, no two of which will have the same coefficient.

If we suppose all the pairs of variables to become identical, we get a binary quantic of the n^{th} order, in which the coefficient of $x_1^r x_2^s$ will be the sum of all the products of a_1, a_2 containing a_1 r times as a factor, and a_2 s times, and we can still write

$$a_x^n = (a_1x_1+a_2x_2)^n,$$

provided we remember that we have made no stipulation as to the way the a's combine in multiplication. Most of what precedes is, of course, well known, but it was necessary to show how the symbolic notation of Clebsch and Aronhold could be applied to unsymmetrical forms. We have now to see how a linear form can itself be written as a product. I call to mind that, if we use Grassmann's methods, we replace a set of variables $x_1, x_2 \dots x_n$ by a single complex variable

$$x = x_1e_1+x_2e_2+\dots+x_ne_n,$$

where $e_1 \dots e_n$ are "units" supposed to obey the polar law of multiplication, so that we have

$$e_ie_j = -e\ e_i,$$

$$e_i^2 = 0,^*$$

and then we have for any sets

$$xy = -yx,$$

$$x^2 = 0.$$

The product of all the units is a scalar, and is assumed to be unity.

* It should be noticed that the second part of this polar law is not a consequence of the first, and that consequently the late Prof. Smith's objection to Clifford's system ($e_i^2 = -1$, $e_ie_j = -e_je_i$) does not seem to be valid. The same objection would apply to quaternions.

If we take any unit e_1, the product of the remaining units is the *conjugate* of e_1, and is denoted by Ke_1. We have

$$e_1 Ke_1 = 1,$$

and this is taken as the definition of K, viz., we have

$$e_i \,.\, Ke_i = 1,$$

$$e_j \,.\, Ke_i = 0.$$

Ke_i is obviously the product of the remaining units $e_1 \dots e_{i-1}, e_{i+1} \dots e_n$ taken in such an order as to satisfy the first of the above equations. In this paper I use instead of Ke_i another quantity ϵ_i defined by the equation

$$\epsilon_i e_i = 1,$$

$$\epsilon_i e_j = 0\,;$$

we have, obviously, $\epsilon_i = (-)^{n-1} Ke_i$.

And $\epsilon_i (x_1 e_1 + x_2 e_2 + \dots + x_n e_n) = x_i$.

Now consider the linear form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n.$$

Let $x = x_1 e_1 + \dots + x_n e_n$,

then $x_i = \epsilon_i x$,

and
$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = a_1 \epsilon_1 x + a_2 \epsilon_2 x + \dots + a_n \epsilon_n x$$
$$= (a_1 \epsilon_1 + a_2 \epsilon_2 + \dots + a_n \epsilon_n)\, x,$$

and we see that any linear form can be written as the product of two factors, one factor containing the variables, and the other containing the coefficients. Now a lineo-linear form was written above as the product of two linear forms, and we see now that it can be written as the product of four factors, two involving the coefficients, and two involving the variables. It is therefore necessary to see how the factors combine in multiplication.

Let the linear forms be

$$a_x = (a_1 \epsilon_1 + a_2 \epsilon_2)(x_1 e_1 + x_2 e_2),$$

$$a_y = (a_1 \epsilon_1' + a_2 \epsilon_2')(y_1 e_1' + y_2 e_2').$$

Now, we know that $\epsilon_1 \epsilon_2 = -\epsilon_2 \epsilon_1$,

and, as regards $\epsilon_1 \epsilon_1'$, $\epsilon_2 \epsilon_2'$, $\epsilon_2 \epsilon_1'$, $\epsilon_1 \epsilon_2'$, we stipulate that these products shall follow the commutative law; so that we have the following convention: *If we have any number of pairs* (ϵ_1, ϵ_2; ϵ_1', ϵ_2'; ...) *the elements of each pair combine according to the polar law, but combine with the elements of every other pair according to the commutative law.**

We have found that we can write

$$a_x = (a_1\epsilon_1 + a_2\epsilon_2)\,x.$$

Now, in all that follows, we shall only have to consider the coefficients of the forms we have to deal with, and we can therefore confine our attention to the first of the two factors giving a_x, that is to say, instead of working with the linear form $a_1x_1 + a_2x_2$, we can work with the form $a_1\epsilon_1 + a_2\epsilon_2$; or, in other words, we can consider all linear forms, and therefore forms of any order, as involving *polar* variables, instead of *scalars*.

I shall now change the notation, and shall use any letters to denote the two polar variables in a binary linear form; thus, for instance, I write as before,

$$a_u = a_1u_1 + a_2u_2;$$

but it must be remembered that u_1, u_2 are, not scalars, but *polars*,† and that we must multiply a_u by another polar, if we are to get a linear form involving scalars.

II. *The Fundamental Theorem.*

Now, suppose we take two linear forms

$$a_u = a_1u_1 + a_2u_2,$$

$$b_u = b_1u_1 + b_2u_2,$$

and multiply them together, we get, since

$$u_1^2 = u_2^2 = 0, \quad u_1u_2 = -u_2u_1 = 1,$$

$$a_ub_u = a_1b_2 - a_2b_1 = (ab).$$

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* This is not Clifford's convention; he makes the elements of different pairs combine according to the polar law; but, if we do this, we get into endless difficulties with the signs, and, as a matter of fact, several of Clifford's signs are wrong; with the convention in the text the signs of all forms can be determined without difficulty.

† There seems no obvious reason why *polar* should not be used as a noun, and it would simplify matters considerably.

Now, suppose we take two quadric* forms

$$a_{uv} = a_{11}u_1v_1 + a_{12}u_1v_2 + a_{21}u_2v_1 + a_{22}u_2v_2,$$

$$b_{uv} = b_{11}u_1v_1 + b_{12}u_1v_2 + b_{21}u_2v_1 + b_{22}u_2v_2.$$

If we write $$a_u = a_1u_1 + a_2u_2,$$

we have $$a_{uv} = a_u a_v,$$

$$b_{uv} = b_u b_v,$$

and, therefore,

$$a_{uv}b_{uv} = a_u a_v b_u b_v = a_u b_u \,.\, a_v b_v = (ab)(ab) = (ab)^2,$$

where we must remember that

$$(a_1b_2 - a_2b_1)^2 = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11}.$$

In the same way we should get

$$a_{uvw}b_{uvw} = (ab)^3.$$

And obviously $$a_u a_z \,.\, b_u b_z = (ab)\, a_z b_x,$$

$$a_u a_v a_w \,.\, b_u b_x b_y = (ab)\, a_v a_w \,.\, b_x b_y,$$

and $$a_u a_v a_w \,.\, b_u b_v b_y = (ab)^2\, a_w b_y.$$

In working with unsymmetrical forms, we must be careful to keep the variables in their right places, since $a_u a_v$ and $a_v a_u$ are not identical; thus

$$a_z a_u \,.\, b_x b_u = a_z b_x\, (a_u b_u) = a_z b_x\, (ab),$$

$$a_z a_u \,.\, b_u b_x = a_z\, (ab)\, b_x.$$

We have found that

$$a_u a_v a_w a_{w'} \ldots b_u b_v b_z b_{z'} \ldots = (ab)^2\, a_w a_{w'} \ldots b_z b_{z'} \ldots.$$

Now, if the forms a, b are symmetrical, the right-hand side is obviously the $2\,(n-2)$-tuply linear form answering to $(ab)^2 a_x^{n-2} b_x^{n-2}$; that is to say, if we multiply together two multiply linear forms having two pairs $(u_1, u_2;\ v_1, v_2)$ of polars in common, we get the second alliance (*Ueberschiebung*) of the quantics answering to the forms. And in the same way we see, generally, that *if we multiply*

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* I follow Clifford in classifying forms according to the order of the ultimate form obtained by introducing scalars, and making the set of variables identical: thus, an n-tuply linear form is an n-thic form.

*together two multiply-linear forms, having r pairs of polars in common, we get the r^{th} alliance of the quantics answering to the forms.**

III. *Graphs.*

We have now to consider the graphical representation of forms; suppose we have an n-tuply linear form, this is represented in the same way as an n-valent atom in chemistry; viz., by a small circle with n rays or bonds proceeding from it, each ray answering to a pair of polars; if we multiply together two forms having r pairs in common, we connect their representative atoms by r bonds; thus, if we take two cubics $a_x a_y a_z$, $b_x b_y b_w$, we represent their second alliance $(ab)^2 a_z b_w$ in the way shown in Fig. (1), where one atom is black to distinguish it from the other.

It is clear, without any formal proof, that what precedes can be extended to any number of forms, so that, if we take any concomitant written in its symbolic form, we can write down the corresponding graph. Thus, the discriminant of a cubic is $(ab)^3 (cd)^2 (ac) (bd)$; it is the result of the following multiplication,†

$$a_x a_y a_z \,.\, b_x b_y b_w \,.\, c_u c_v c_z \,.\, d_u d_v d_w \,,$$

and we get the graph in Fig. (2), and it is obvious that we could have got it by putting down four "atoms" answering to the four forms a, b, c, d, and joining two atoms (a, b) by a bond for every time (ab) occurs as a factor. In the same way we see that $(ab)^2 (bc)^2 (ca)^2$, the cubic invariant of a quartic, is represented by Fig. (3), and that $(ab)^2 (bc) a_x c_x^2$ is represented by Fig. (4).

It must not be forgotten that, in the first instance, a graph does not represent the quantic, but a certain polarised blank form of the quantic; thus, in Fig. (5), the graph does not represent the cubic a_x^3, but the product $a_u a_v a_w$, which gives, in the first instance, $a_x a_y a_z$, and then a_x^3 when we make the three pairs of variables identical.

IV. *Links and their Properties.*

The determinant $x_1 y_2 - x_2 y_1$ will be denoted by (xy) and represented by the graph Fig. (6); such determinants will be called *links.*

If we square (xy), we get

$$(x_1 y_2 - x_2 y_1)^2 = x_1^2 y_2^2 - x_1 x_2 y_2 y_1 - x_2 x_1 y_1 y_2 + x_2^2 y_1^2 = 2x_1 x_2 y_1 y_2 = 2.$$

* "*Quære*, whether this beautiful use of the method of polar multiplication is not, in its ultimate essence, identical with Professor Cayley's original method of hyper-determinants."—Prof. Sylvester, *American Journal*, I., 128.

† In what follows, the original quantic we work with will always be supposed symmetrical.

We also find

$$(xy)(zy) = (x_1 y_2 - x_2 y_1)(z_1 y_2 - z_2 y_1) = x_1 z_2 - x_2 z_1 = (xz),$$

and therefore $$(yx)(yz) = (xz).$$

If we multiply a_x by (yx), we get

$$(a_1 x_1 + a_2 x_2)(y_1 x_2 - y_2 x_1) = a_1 y_1 + a_2 y_2,$$

or $$a_x (yx) = a_y.$$

We have $a_x b_y = a_1 b_1 x_1 y_1 + a_1 b_2 x_1 y_2 + a_2 b_1 x_2 y_1 + a_2 b_2 x_2 y_2.$

Therefore $a_x b_y (xy) = a_1 b_2 - a_2 b_1 = (ab) = a_x b_x,$

and therefore in any product of this kind, when we multiply by (xy), we need only change a factor b_y into b_x.*

V. *Quadrics, Skew and Symmetrical.*

Consider the quadric

$$a_x a_y = a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{21} x_2 y_1 + a_{22} x_2 y_2;$$

if we multiply this by $x_1 y_2 - x_2 y_1$, we get

$$a_{12} - a_{21},$$

and therefore $$a_x a_y (xy)$$

vanishes if, and only if, $a_x a_y$ is symmetrical, and, since there is nothing to prevent the coefficients a_{ik} from involving polars, we see that any form a containing x, y is symmetrical with respect to these two variables, if, and only if, $a\,(xy)$ vanishes. Now, if a form is to be symmetrical with respect to all the variables involved, it is obviously necessary and sufficient that it should be symmetrical with respect to every pair of variables; that is to say, *the necessary and sufficient condition that a form should be symmetrical is that if we multiply the form by all the links formed by pairing its variables, each of the products must vanish.*

We have found the condition that a quadric may be symmetrical, that is, that we may have

$$a_x a_y = a_y a_x.$$

* It would not have been enough to say that $a_x b_y (xy) = a_x . b_y (xy) = a_x b_x = (ab)$, since we stipulated that x, y combined according to the commutative law.

We have now to find the condition that it may be skew, that is, that we may have
$$a_x a_y = -a_y a_x.$$
This gives
$$a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 = -a_{11}x_1y_1 - a_{12}x_2y_1 - a_{21}x_1y_2 - a_{22}x_2y_2,$$
and therefore
$$a_{11} = a_{22} = 0,$$
$$a_{12} = -a_{21},$$
and the quadric reduces to
$$a_{12}(x_1y_2 - x_2y_1);$$
that is to say, if a quadric is skew, it is a multiple of the link of its variables; and, in the same way as before, we see that, if any form is skew as regards any pair of variables, it is a multiple of the link of this pair of variables.

It is always easy to determine the coefficient of the link; for, if we have
$$f(x, y \ldots) = A\,(xy),$$
we get, by multiplying by (xy),
$$f(x, x \ldots) = 2A,$$
and therefore
$$f(x, y \ldots) = \tfrac{1}{2}f(x, x \ldots)\,(xy).$$
Thus
$$a_z a_x \,.\, b_z b_y$$
is a skew function, if a, b refer to the same quadric, for, if we interchange x and y, we get
$$a_z a_y \,.\, b_z b_x.$$
Now, if we interchange a and b, the original form becomes
$$b_z b_x \,.\, a_z a_y,$$
and
$$b_x a_y = a_y b_x,$$
$$b_z a_z = -a_z b_z,$$
and therefore
$$a_z a_y \,.\, b_z b_x = -b_z b_x \,.\, a_z a_y = -a_z a_x \,.\, b_z b_y.$$
Therefore $a_z a_y \,.\, b_z b_x$ is a skew form, and the coefficient of (xy) is
$$\tfrac{1}{2}a_z a_y \,.\, b_z b_y = \tfrac{1}{2}(ab)^2.$$
In the same way, we see that the form in Fig. (7) is skew, and that twice the coefficient of (xy) is the graph in Fig. (8), that is to say, the quadric invariant (i) of the quartic.

VI. *Unsymmetrical Forms.*

Consider the graph Fig. (9). We see at once that its symbolical form is $(ab)^2 a_x^2 b_x^2$, and that it answers to the Hessian of a quartic; but, before we can identify the two, we must see whether the graph is symmetrical. Now, if we start with a symmetrical quartic, the graph is obviously unchanged if we write x for y, or u for v; to see whether it is unaltered when we interchange x and u, we must multiply $(ab)^2 a_x a_y . b_u b_v$ by (xu). Now we know that this comes to the same thing as identifying x and u, so that the product is

$$(ab)^3 a_y b_v,$$

and the graph for this is Fig. (7), and we know that this is

$$\frac{i}{2}(yv),$$

and does not vanish; we see, then, that Fig. (9) is not symmetrical with respect to x and u, and in the same way we see that it is not symmetrical with respect to y and v, or x and v, or y and u. Now, there is an essential and obvious difference between symmetrical and unsymmetrical graphs. Suppose we have two symmetrical quartic graphs, one having $xyzw$ as its variables, and the other having $stuv$; if we form the second alliance of these graphs, it is obviously a matter of indifference which pair of letters in the one we identify with a pair of letters in the other; if, however, the graphs are not both symmetrical, this is not the case, and we get different results according to the way we combine them. Thus, a sextic has an unsymmetrical quartic covariant represented by Fig. (10), where the broad bond denotes four bonds; if we join this to the sextic by the bonds u, v,* we get Fig. (11); if we join it by (x, u), we get Fig. (12). Now, these two covariants are not identical, for it can be shown that

$$(11) = (12) + \frac{A}{2} f,$$

where f is the sextic, and A is the invariant $(ab)^6$.

This distinction between symmetrical and unsymmetrical forms is of the greatest importance. If we work with unsymmetrical forms, we have to be extremely careful not to identify results got by combining them in different ways; four pages of Clifford's "Fragments" relate

* That is to say, if we multiply it by $a_u a_v$ ...; I shall use this abbreviated phrase throughout the rest of this paper.

to the form $\kappa f+\lambda g$, where f is a quintic, and g its covariant of degorder (3, 5); now, all the results obtained on these pages are vitiated by the fact that Clifford has treated unsymmetrical forms as if they were symmetrical. We must remember that every form is to be supposed unsymmetrical until it is proved to be symmetrical.

I proceed to show how we can always get a symmetrical form answering to a given unsymmetrical form.

VII. *Reduction of Graphs to Symmetrical Forms.*

Suppose we have any unsymmetrical form, and that we wish to make it symmetrical; the symmetrical form will obviously be obtained by taking the arithmetical mean of all the different values of the form. Thus, in the case of the quartic, we get an unsymmetrical quartic covariant $h = (ab)^2 a_x a_y . b_u b_v$: it is easy to see that, if a is a symmetrical form, we get six different values for h; for h is obviously unchanged if we interchange x, y, or u, v; so that the twenty-four possible values obtained by permuting xy, uv reduce to six. Let H be the symmetrical form of h, then we have

$$\begin{aligned} 6H = \; & (ab)^2 a_x a_y b_u b_v + (ab)^2 a_u a_y b_x b_v + (ab)^2 a_x a_v b_u b_y \\ & + (ab)^2 a_v a_y b_x b_u + (ab)^2 a_x a_u b_y b_v + (ab)^2 a_u a_v b_x b_y \\ = \; & 6\,(ab)^2 a_x a_y b_u b_v + \Sigma\left\{(ab)^2 a_u a_y b_x b_v - (ab)^2 a_x a_y b_u b_v\right\}, \end{aligned}$$

where the sign Σ denotes the sum of the terms obtained by subtracting the first term of $6H$ from each of the others.

$$\begin{aligned} \text{Now, } (ab)^2 a_u a_y b_x b_v - (ab)^2 a_x a_y b_u b_v &= (ab)^2 a_y b_v (a_u b_x - a_x b_u) \\ &= (ab)^3 a_y b_v (ux) \\ &= \tfrac{1}{2}(ab)^3 (ux)(a_y b_v - a_v b_y), \end{aligned}$$

if we interchange a, b, and take the semi-sum of the two expressions;

and this is $\qquad \frac{1}{2}(ab)^4 (ux)(yv)$.

In the same way, we get $\frac{1}{2}(ab)^4(ux)(yv)$ from the next term, and $\frac{1}{2}(ab)^4(xv)(uy)$ from each of the next two; the last difference

$$a_x a_y b_u b_v - a_u a_v b_x b_y$$

vanishes, since the second term reduces to the first if we interchange a and b.

We have, therefore,

$$6H = 6\,(ab)^2 a_x a_y b_u b_v - (ab)^4 (xu)(yv) - (ab)^4 (xv)(yu),$$

or $$H = (ab)^2 a_x a_y b_u b_v - \frac{(ab)^4}{6} \{(xu)(yv)+(xv)(yu)\}.*$$

In the same way, the quintic has an unsymmetrical quintic covariant

$$g = (ab)^4 (bc)\, a_x c_t c_u c_v c_w.$$

If c is symmetrical, the different values of g are got by interchanging x with t, u, v, w. We therefore have, if G is the symmetrical form of g,

$$5G = (ab)^4 (bc) \{a_x c_t c_u c_v c_w + a_t c_x c_u c_v c_w + a_u c_t c_x c_v c_w + a_v c_t c_u c_x c_w + a_w c_t c_u c_v c_x\}$$
$$= 5\,(ab)^4 (bc)\, a_x c_t c_u c_v c_w + \Sigma (ab)^4 (bc) \{a_t c_x c_u c_v c_w - a_x c_t c_u c_v c_w\},$$

where, as before, Σ denotes the sum of the four terms obtained by subtracting the first term of $5G$ from each of the others.

Now, $$(ab)^4 (bc) (a_t c_x c_u c_v c_w - a_x c_t c_u c_v c_w)$$
$$= (ab)^4 (bc)\, c_u c_v c_w\, (a_t c_x - a_x c_t)$$
$$= (ab)^4 (bc)(ac)\, c_u c_v c_w\, (tx).$$

And therefore

$$G = (ab)^4 (bc)\, a_x c_t c_u c_v c_w - \tfrac{1}{5} \Sigma (ab)^4 (bc)(ac)\, c_u c_v c_w\, (xt),$$
$$= g\,(x, t, u, v, w) - \tfrac{1}{5} \Sigma j\, (u, v, w)(xt),$$

where $j\,(u, v, w)$ denotes the covariant (5, 3 ; 3),

$$(ab)^4 (bc)(ac)\, c_u c_v c_w.$$

As a last example, I take the covariant ϑ (2, 1 ; 3, 1 ; 3) of a cubic and quadric. If the cubic is a, and the quadric is α, we have

$$\vartheta\,(x, yz) = (\alpha a)\, \alpha_x a_y a_z.$$

Now, here the only changes which can affect the form of ϑ are the interchange, firstly of x and y, and secondly of x and z; and therefore we shall have, if Θ is the symmetrical form of ϑ,

$$3\Theta = 3\vartheta\,(x, yz) + \{\vartheta\,(y, xz) - \vartheta\,(x, yz)\} + \{\vartheta\,(z, xy) - \vartheta\,(x, yz)\}.$$

Now $$\vartheta\,(y, xz) - \vartheta\,(x, yz)$$

is a skew function of x, y, and therefore divides by (xy). We have

$$(\alpha a)\, \alpha_y a_x a_z - (\alpha a)\, \alpha_x a_y a_z = A\,(xy).$$

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* Clifford, p. iii.

Multiplying by (xy), we get

$$-2\,(\alpha a)^2 a_z = 2A,$$

or

$$A = -\,(\alpha a)^2 a_z.$$

We therefore have

$$\Theta = (\alpha a)\,\alpha_x a_y a_z - \tfrac{1}{3}\Sigma\,(\alpha a)^2 a_z\,(xy).$$

This result might, of course, have been found in the same way as the other two; I have adopted another method, partly for the sake of variety, and partly because this last method is generally more convenient when we are working with graphs.*

VIII. *Discriminations.*

It is, in general, easy to see when a graph vanishes, or is skew or symmetrical with respect to any pair of letters. Thus, consider the graph in Fig. (13); we can see at once that this vanishes, for it represents the covariant (5, 3; 1),†

$$(ab)^3\,(ac)^2\,(bc)^2\,c_x.$$

This covariant changes its sign if we interchange a and b, and therefore vanishes. In the same way, if we consider Fig. (14), we see that this is symmetrical with respect to (x, y); for it is

$$(ab)^2\,(ac)^2\,(bc)^2\,a_x b_y c_z,$$

and the interchange of x and y is identical with the interchange of a and b, and therefore leaves the graph unchanged. On the other hand, the graph in Fig. (15) is skew in (x, y); for it is

$$(ab)^3\,(ac)(bc)\,a_x b_y c_u c_v c_w,$$

and the interchange of x and y gives

$$(ab)^3\,(ac)(bc)\,a_y b_x c_u c_v c_w,$$

which is what the original covariant becomes when we interchange a and b and change the sign.

Clifford says (p. iii.) that the graph in Fig. (16) is a multiple of (xy); but we can see at once that it is not skew, but symmetrical, since it is

$$(ab)^2\,(ac)(bc)\,a_x b_y c_u c_v,$$

* The symmetrical forms of g, ϑ, on Clifford's pp. ix., iii., are not correct.

† Covariant of quintic of degorder (3, 1).

and the interchange of x and y leaves it unaltered. From this Clifford infers that it vanishes (which is not the case),* and this leads to some other incorrect results.

For triangular graphs we get the following rules :—

1. If two saturated† vertices of a triangle are joined by an odd number of bonds, and are joined to the third vertex by any the same number of bonds, the graph vanishes.

2. If two vertices, each having one free bond, are joined by an odd number of bonds, and are joined to the third vertex by any the same number of bonds, the graph is a multiple of the link of the free bonds at the two first mentioned vertices, the coefficient being one-half the graph obtained by joining these two vertices by an additional bond.

3. If two vertices, each having one free bond, are joined by an even number of bonds, and are joined to the third vertex by any the same number of bonds, the graph is symmetrical with respect to the free bonds at the two first mentioned vertices.

As a particular case of (1), we see that, if two saturated atoms are connected by an odd number of bonds, the resulting invariant vanishes.

If two atoms, having one free bond each, are connected by an odd number of bonds, the resulting quadric covariant is skew.

If two atoms, having one free bond each, are connected by an even number of bonds, the resulting quadric covariant is symmetrical.

IX. *Elementary Reductions.*

Suppose we have any form $f(xu)$ containing two letters, and as many besides as we please; then we have

$$f(x, u) = \tfrac{1}{2}\{f(x, u) + f(u, x)\} + \tfrac{1}{2}\{f(x, u) - f(u, x)\}.$$

Now, the first bracket is obviously a symmetrical function of x, u, and the second bracket is a skew function, and we see that every form containing two letters can be decomposed into two parts, one symmetrical and the other skew with respect to these two letters, and this decomposition is unique.‡

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* If the graph were a multiple of (xy), the coefficient would be $\tfrac{1}{2}(ab)^3(ac)(bc)c_u c_v$, which vanishes, and therefore the graph would vanish.

† In Fig. (13), the vertices at the base of the triangle are saturated, and the third angle has one free bond; in Fig. (16), the two vertices at the base have respectively the free bonds x, y.

‡ Let a be any quantity, E any distributive operator, then a can be decomposed uniquely into two parts, α, β, such that $E\alpha = \lambda\alpha$, $E\beta = \mu\beta$, λ, μ being unequal scalars, for we have $a = \alpha + \beta$, $Ea = \lambda\alpha + \mu\beta$, and α, β are determined uniquely.

In what follows $(A : xu, yv)$ denotes a function which is unaltered when we interchange x, u, and also when we interchange yv; $(A : xu)$ means a function which is unaltered when we interchange x, u*; $A\,(xu)$ denotes, as before, a multiple of the link (xu).

We have seen that we can write

$$f(x, y, u, v) = A\,(xu) + (A' : xu),$$

where, of course, A, A' involve yv; transforming these in the same way, we get

$$A = B_1\,(yv) + (B_2 : yv),$$

$$A' = B_1'\,(yv) + (B_2' : yv),$$

and therefore

$$f(x, y, u, v) = B_1\,(xu)(yv) + (B_2 : yv)(xu) + (B_3 : xu)(yv) + (B_4 : xu, yv).$$

If f contains more than four letters, we can, of course, go on in this way, and it is easy to see the form of the general result. The formula just given is enough for the purposes of the present paper.

Now, suppose that f is known to change sign when we interchange x, u, and also interchange y, v; we have

$$f = B_1\,(xu)(yv) + (B_2 : yv)(xu) + (B_3 : xu)(yv) + (B_4 : xu, yv),$$

$$-f = B_1\,(ux)(vy) + (B_2 : vy)(ux) + (B_3 : ux)(vy) + (B_4 : ux, vy),$$

and, remembering the meanings of $(B_2 : yv)$, &c., and that (xu), (yv) are skew, we get

$$0 = B_1\,(xu)\,(yv) + (B_4 : xu, yv),$$

and

$$f = (B_2 : yv)\,(xu) + (B_3 : xu)\,(yv) \quad \dots\dots\dots\dots (\alpha).$$

If f is unaltered when we interchange x, u, and also y, v, we shall find in the same way

$$f = B_1\,(xu)(yv) + (B_4 : xu, yv) \dots\dots\dots\dots (\beta).$$

Multiplying (α) by (yv), we get, if f was $f(x, y, u, v)$,

$$f(x, y, u, y) = 2\,(B_3 : xu).$$

Multiplying it by (xu), we get

$$f(x, y, x, v) = 2\,(B_2 : xu).$$

---

* This is not a good notation, but I have been unable to devise another that should look better, and at the same time guard against all risk of confusion.

If f is $(f: xy, uv)$, B_2, B_3 are obviously the same forms. Thus, consider the form $$(ab)\, a_x a_y\, b_u b_v.$$

The substitution $(xu)(yv)$* is obviously equivalent to the interchange (ab), together with a change of sign, and we can therefore use (α).

This gives $$(ab)\, a_x a_y b_u b_v = B_2\,(xu) + B_3\,(yv).$$

To determine B_2 multiply by (yv), and we get

$$(ab)^2\, a_y b_v = 2B_2,$$

and in the same way $$(ab)^2\, a_x b_u = 2B_3;$$

and we have therefore

$$(ab)\, a_x a_y b_u b_v = \tfrac{1}{2}\,(ab)^2\, a_x b_u\,(yv) + \tfrac{1}{2}\,(ab)^2\, a_y b_v\,(xu).$$

This equation is represented graphically in Fig. (17).

X. *Form-Systems.*

We have now all the materials we require for the graphic construction of the form-system of a quantic.

I call to mind that forms are classified according to their degree and weight. If we write down the graph corresponding to a given form, the degree of the form is the number of atoms in the graph, and its weight is the total number of bonds connecting the atoms; thus Fig. (16) represents a form of order four, degree three, and weight four, appertaining to a quartic.†

Moreover, if we consider the reductions of graphs already given, we see that, if a graph of weight w reduces, as in Fig. (17), to a sum of links, the coefficients of the links are at least of weight $w+1$, and that, if we find the symmetric form answering to a given graph, of weight w, the two forms differ by a sum of products of links and forms of weight $w+1$, at least.

Now, the way we construct a form-system is as follows: Suppose we start with an n-thic. Joining this to itself by n bonds, we get the heaviest‡ form of the second degree; joining the quantic to itself by $n-1$, $n-2$ ... bonds, we get all the forms of the second degree

* I use here the ordinary notation for cyclic substitutions; the word *interchange* or *substitution* will always be used when this is the case, to prevent confusion.

† This rule is easily seen to be correct by considering the symbolic expression answering to the graph.

‡ It seems natural, and is certainly convenient, to describe a graph (or form) as *heavier* than a form of less weight.

arranged according to their weights, in descending order. Some of these forms (all the forms of odd weight) will reduce to sums of links, and these reductions must be effected, and unsymmetrical graphs must be made symmetrical. After finding in this way all the forms of the second degree, we get the forms of the third degree by joining the n-thic to these forms of the second degree, beginning with the heaviest, and in each case we form the combinations in the order of their weights, beginning with the heaviest.

We have now to see what happens when we join the quantic (f) by r bonds to a form reducible to a sum of links, or to an unsymmetric form. Suppose we have a form ϕ of any degree, and of weight w, and that this contains a term

$$\psi \,.\, (xu),^*$$

where ψ is of weight $w+1$; then ϕ gives a form of weight $w+r$, and, as regards the term just mentioned, three cases may present themselves:—

1. x, u may both be among the r bonds by which we join f to ϕ; then, since f is supposed symmetrical,

$$(xu) f(x, u, \ldots) = 0,$$

and the term contributes nothing.

2. Let one of the two letters, say x, be among the r bonds; the factor (xu) changes x to u, and we have to join f to ψ, by $r-1$ bonds, and the weight of the resulting term is

$$w+1+r-1 = w+r.$$

3. Let neither x nor u be among the r bonds; then we have to join x to ψ by r bonds, and we get the product of (xu) and a form of weight $w+r+1$.

In the third case, the form of weight $w+r+1$ will have been obtained before we got down to the forms of weight $w+r$, and need not be considered.

In the second case, the form of weight $w+r$ is got by joining f to a form of weight $w+1$, and all the combinations of this form with f will have been disposed of before we got down to forms of weight w.

We see, then, that if we arrange our forms in the order agreed

* It must be remembered that, since (Clebsch, p. 8) the coefficients and the variables are transformed by inverse substitutions, the weight of a link may be taken to be -1.

upon, we may neglect all forms reducing to sums of links, since everything that such forms could furnish will have been obtained before we get to them, and that we can neglect the complementary part* of any unsymmetric graph; or, in other words, that we can treat any graph as symmetrical, and join f to it in whatever way may happen to be most convenient.

It must be noticed that we are not at liberty to reject a graph because it contains, as part of itself, a graph reducible to a sum of links; this is simply because the reducible part is lighter than the graph from which the whole graph was derived. If, however, a graph contains, as part of itself, a graph reducing to a sum of products of links and invariants, it may obviously be rejected.

I proceed to apply these principles to the theory of quadric, cubic, and quartic forms.

SEC. 1. *Quadric.*

The forms of the second degree are given in Figs. (18, 19); the reduction in Fig. (19) is obvious, and we see that there can be no irreducible forms of the third degree.

SEC. 2. *Cubic.*

The forms of the second degree are given in Figs. (20—22); of these (20) vanishes, (21) is symmetrical [for, if we multiply it by (xy), we get (20)], (22) reduces by Fig. (17). We need, therefore, only consider (21), which is the Hessian.†

The forms of the third degree obtained from (21) are given in Figs. (23, 24); of these (23) vanishes, as I proceed to show. Written symbolically, the graph is

$$(ab)^2(bc)(ac)c_x;$$

we get three forms of this by interchange of a, b, c, and, taking one-third of the sum of these, we get

$$\begin{aligned}(ab)^2(bc)(ac)c_x &= \tfrac{1}{3}\{(ab)^2(bc)(ac)c_x + (bc)^2(ca)(ba)a_x \\ &\qquad + (ca)^2(ab)(cb)b_x\} \\ &= -\frac{(bc)(ca)(ab)}{3}\{(ab)c_x + (bc)a_x + (ca)b_x\} = 0.\ddagger\end{aligned}$$

* The complementary part of an unsymmetric graph is the sum of links that has to be added to make it symmetrical.

† Δ in Clebsch's notation; I use Clebsch's notation throughout for all the forms considered.

‡ I have given the above proof instead of Clifford's; Clifford proves that (23) vanishes, by reasoning of which I am unable to see the force.

Now that we have proved that (23) vanishes, we can see that (24) is symmetrical, for, if we multiply it by (xu), we get Fig. (23); (24) is therefore a symmetrical covariant of degorder (3, 3), and is therefore what Clebsch denotes by Q. The covariants of the fourth degree derived from Q are given in Figs. (25—27). Of these (25) is an invariant, the discriminant (R). The reduction in Fig. (26) is obvious; (27) is got by joining the Hessian to Fig. (22), and is found at once to be reducible, in the way shown in the figure; we see that the only irreducible form of the fourth degree is an invariant, and that there are, therefore, no irreducible forms of higher degrees, so that the form system of the cubic consists of the forms f, Δ, Q, R.

SEC. 3. *Quartic.*

The forms of the second degree are given in Figs. (28—31). (28) is the invariant i; the reduction in (29) is obvious; (30) is unsymmetrical, and its symmetrical form has already been found to be (32); this symmetrical form is the Hessian (H); (31) reduces in the way shown in the figure. I shall go through the calculation here, as Fig. (31) does not agree with Clifford's results. The symbolic form of (31) is

$$(ab)\, a_x a_y a_z b_u b_v b_w .$$

This form changes sign if we effect the substitution $(xu)(yv)(zw)$; and therefore, when we expand it in a series of links, we need only keep the skew terms; we have, therefore,

$$(ab)\, a_x a_y a_z b_u b_v b_w = A\,(xu)(yv)(zw) + \Sigma\,(B : yv,\, zw)(xu).$$

Multiplying by $(xu)(yv)(zw)$, we get

$$(ab)^4 = 8A.$$

Multiplying by (xu), we get

$$(ab)^2\, a_y a_z b_v b_w = 2A\,(yv)(zw) + 2\,(B : yv,\, zw),$$

and therefore

$$(B : yv,\, zw) = \tfrac{1}{2}\,(ab)^2\, a_y a_z b_v b_w - A\,(yv)(zw),$$

and we get similar values for the other B's; substituting and reducing, we get

$$(ab)\, a_x a_y a_z b_u b_v b_w = \tfrac{1}{2}\,\Sigma\,(ab)^2\, a_y a_z b_v b_w\,(xu) - \frac{i}{4}\,(xu)(yv)(zw).*$$

* Clifford only gets the first part of this; but it can easily be verified directly that the above result is correct.

The forms of the third degree got from H are given in Figs. (33—36). Of these (33) is the invariant j; the reduction in Fig. (34) is obvious. I shall follow Clifford in showing that (35) is reducible, but I shall not use graphs in reducing it, as it is easier to get the signs right if we use symbolic methods. We have

$$(ab)\,a_x a_y a_z\,b_u b_v b_w = \tfrac{1}{2}(ab)^2 a_y a_z\,b_v b_w\,(xu) + \tfrac{1}{2}(ab)^2 a_x a_z\,b_u b_w\,(yv)$$
$$+\tfrac{1}{2}(ab)^2 a_x a_y\,b_u b_v\,(zw) - \frac{i}{4}(xu)(yv)(zw).$$

Now, multiply this *into* $c_x c_y c_u c_t$; we get

$$(ab)(ac)^2(bc)\,a_z b_v b_w c_t = \tfrac{1}{2}(ab)^2(ac)(bc)\,a_z b_w c_v c_t$$
$$+\tfrac{1}{2}(ab)^2(ac)^2(bc)\,b_v c_t\,(zw)$$
$$= -\tfrac{1}{2}(ac)^2(ab)(bc)\,a_z b_v b_t c_w$$
$$+\tfrac{1}{2}(ab)^2(ac)^2(bc)\,b_v c_t\,(zw).$$

Now $$(ab)(ac)^2(bc)\,a_z b_v b_w c_t - (ab)(ac)(bc)\,a_z b_v b_t c_w$$
$$= (ab)(ac)^2(bc)\,a_z b_v\,(b_w c_t - b_t c_w)$$
$$= (ab)(ac)^2(bc)^2 a_z b_v\,(wt),$$

and therefore, if we compare our results, we get

$$-\tfrac{1}{2}(ac)^2(ab)(bc)\,a_z b_v b_t c_w + \tfrac{1}{2}(ab)^2(ac)^2(bc)\,b_v c_t\,(zw)$$
$$= (ac)^2(ab)(bc)\,a_z b_v b_t c_w + (ac)^2(bc)^2(ab)\,a_z b_v\,(wt);$$

and therefore

$$-\tfrac{3}{2}(ab)(ac)^2(bc)\,a_z b_v b_t c_w$$
$$= \tfrac{1}{2}(ab)^2(ac)^2(bc)\,b_v c_t\,(zw) - (ab)(ac)^2(bc)^2\,a_z b_v\,(wt).\text{*}$$

But $$(ab)^2(ac)^2(bc)\,b_v c_t = \tfrac{1}{2}(ab)^2(ac)^2(bc)^2\,(vt),$$

and therefore $$= \tfrac{1}{2}j\,(vt),$$

$$-\tfrac{3}{2}(ab)(ac)^2(bc)\,a_z b_v b_t c_w = \frac{j}{2}\left\{(vt)(zw) - \frac{(zv)(wt)}{2}\right\}$$
$$= \frac{j}{2}\left\{(vt)(zw) + \frac{(vw)(zt)-(zw)(vt)}{2}\right\}$$

* Clifford finds

$$\tfrac{1}{2}(ab)(ac)^2(bc)\,a_z b_v b_t c_w = (ab)(ac)^2(bc)^2 a_z b_v\,(tw) - \tfrac{1}{2}(ab)(ac)^2(bc)^2 b_t c_t\,(zw).$$

[since $(zv)(wt)+(vw)(zt)+(wz)(vt) = 0$]

$$= \frac{j}{4}\left\{(vt)(zw)+(zt)(vw)\right\},$$

or $$(ab)(ac)^2(bc)\,a_z b_v b_t c_w = \frac{j}{6}\left\{(vt)(wz)+(vw)(tz)\right\}.^*$$

Now, it was found before that the symmetrical form of the Hessian is

$$H = (ab)^2 a_x a_y b_u b_v - \frac{i}{6}\left\{(xu)(yv)+(xv)(yu)\right\}.$$

Multiplying this into $c_x c_u c_t c_w$, we get

$$(ab)^2(ac)(bc)\,a_y b_v c_t c_w + \frac{i}{6}\,c_y c_v c_t c_w.$$

Multiplying H into $c_x c_y c_t c_w$, we get

$$(ab)^2(ac)^2 b_u b_v c_t c_w - \frac{i}{3}\,c_u c_v c_t c_w.$$

Now, H and c are symmetrical forms, and therefore, allowing for the change of y into v, we get the same result whether we join them by x, u or by x, y; we have, therefore, if f denotes the quartic,

$$(ab)^2(ac)^2 b_u b_v c_t c_w - \frac{i}{3}f = (ab)^2(ac)(bc)\,a_y b_v c_t c_w + \frac{i}{6}f$$

$$= -\frac{j}{6}\left\{(uw)(tv)+(ut)(wv)\right\} + \frac{i}{6}f,$$

and therefore

$$(ab)^2(ac)^2 b_u b_v c_t c_w = \frac{if}{2} - \frac{j}{6}\left\{(uw)(tv)+(ut)(wv)\right\}.^\dagger$$

Fig. (36) answers to the covariant T of degorder (3, 6); it is obviously unsymmetrical, since, if we join y, z, we get (35), which does not vanish.

In getting the forms of the fourth degree we need only consider T.

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* As already remarked, Clifford makes $(ab)(ac)^2(bc)\,a_z b_v b_t c_w$ vanish identically; but this is obviously impossible, since $(vt)(wz)+(vw)(zt)$ does not vanish.

† Clifford, having made $(ab)^2(ac)(bc)\,a_y b_v c_t c_w$ vanish identically, gets

$$(ab)^2(ac)^2 b_u b_v c_t c_w = -\frac{if}{2};$$

but, apart from the mistake in sign, this equation is impossible, since the left-hand side is unsymmetrical; if we make $(ab)^2(ac)^2 b_u b_v c_t c_w$ symmetrical, the complementary part consists of the terms involving j in the equation in the text.

The forms are given in Figs. (37—40). Of these (37) vanishes, since it contains the vanishing graph $(ab)^3 (ac)(bc) c_x c_y$; (38) is reducible since it contains (29). The reduction in Fig. (39) is obvious, and the coefficient of (xy) is got by joining f to the reducible graph (35); (40) is got by joining (31) to itself, and therefore reduces.* We see that there are no irreducible forms of the fourth degree, and therefore none of higher degrees.

It is obvious that we might use this method in finding the form-system of any quantic; but it is also obvious that in the case of higher quantics the application of it would be exceedingly tedious, and accordingly Clifford has abandoned this method for the quintic (after finding the forms of the first, second, and third degrees), and has contented himself with taking the irreducible forms from Clebsch's *Theorie der Binären Formen.*

XI. *Theory of the Compound Form.*

The cubic has a covariant (Q) of the third order, and the quartic has a covariant (H) of the fourth order; if we take two parameters κ, λ, we can find the form-systems of the compound forms, $\kappa f+\lambda Q$, $\kappa f+\lambda H$, for the cubic and quartic respectively; the problem is, to express each form of the system in terms of κ, λ, and the form-system of f. Clebsch solves this problem by the introduction of a certain differential operator; Clifford has used a method of direct formation, which I proceed to explain; it should be mentioned that Clifford has only worked out the results for the cubic and quintic; but, as already explained, the results for the quintic are vitiated by an error. As regards the quartic, he has put down certain results of Clebsch's theory, in a way which shows that at the time he had either forgotten, or not yet noticed, that the graph for H is unsymmetrical.

In what follows, I denote the r^{th} alliance of f, ϕ by

$$(f, \phi)_r.$$

It must be remembered that

$$(f, \phi)_r = (-)^r (\phi, f)_r.$$

I denote the compound form ($\kappa f+\lambda H$ or $\kappa f+\lambda Q$) by F. If ψ is any form appertaining to f, the corresponding form for F is denoted by ψ_F.

* Another form of this, chosen by Clifford, reduces as shown in Fig. (41).

SEC. 1. *Cubic.*

Consider first the theory of the form

$$F = \kappa f + \lambda Q,$$

f being a cubic.

We have first of all to find Δ_F. We have

$$\Delta_F = (\kappa f + \lambda Q, \kappa f + \lambda Q)_2 = \kappa^2 (f, f)_2 + 2\kappa\lambda (f, Q)_2 + \lambda^2 (Q, Q)_2 ;$$

$(f, f)_2$ is, of course, Δ; $(f, Q)_2$ is given in (26). We have, if x, y are the free bonds, and R is the discriminant,

$$(f, Q)_2 = \frac{R}{2} (xy).$$

$(Q, Q)_2$ is given in (42). Now, it is obvious that, in order to join the two graphs in the way shown in the figure, one of them had to be turned round end for end; and it is easy to prove, by considering the symbolic form answering to the graph, that this operation multiplies a form by $(-)^w$, if w is the number of bonds joining the atoms of the graph, *i.e.*, the weight of the form. Therefore the graph in Fig. (42) represents [not $(QQ)_2$ but] $-(QQ)_2$; now, this graph is obviously got by joining Δ to (26) by one bond, and therefore it is $-1/2 . R . \Delta$,* and therefore we have

$$(QQ)_2 = \tfrac{1}{2} R\Delta,$$

and therefore

$$\Delta_F = \left(\kappa^2 + \frac{\lambda^2}{2} R\right) \Delta + \kappa\lambda R (xy) = \Theta\Delta + \kappa\lambda R (xy).$$

Now, the left-hand side is a quadric form; the right-hand side consists of a symmetric part, $\Theta\Delta$, and a skew part; and therefore, as we have to make all forms symmetrical, we must leave out the skew term and write

$$\Delta_F = \Theta\Delta.\dagger$$

To get Q_F, we have to join this to F by one bond. We get

$$Q_F = \Theta \{\Delta, \kappa f + \lambda Q\}_1 = \Theta \{\kappa (\Delta, f)_1 + \lambda (\Delta, Q)_1\} ;$$

---

* The graph is

$$\tfrac{1}{2} . R . (xz) \Delta_z \Delta_y = -\tfrac{1}{2} . R . \Delta_z (xz) \Delta_y = -\tfrac{1}{2} . R . \Delta_x \Delta_y .$$

† Clifford says,—"In any kind of multiplication $fQ = -Qf$, and therefore we have only to find the Hessian of Q." I venture to think that the reason why the term involving $\kappa\lambda$ disappears from the result is that stated in the text; the term does not necessarily disappear, but it is rejected when we make Δ_F symmetrical.

$(\Delta, f)_1$ is Q; (Δ, Q) is given in (43), and is obviously got by joining (26) to f by one bond. We have therefore

$$(\Delta, Q)_1 = -\frac{R}{2} f,$$

which gives
$$Q_F = \Theta \left\{ \kappa Q - \frac{R}{2} \lambda f \right\}.$$

R is the discriminant of Δ, and therefore

$$R_F = \Theta^2 R.$$

SEC. 2. *Quartic.*

In the theory of the quartic we consider the function

$$F = \kappa f + \lambda H.$$

We must remember that the graph of the Hessian is unsymmetrical, and that we have to use the symmetrical form (32). The Hessian of F in its ultimate* form is

$$H_F = (\kappa f + \lambda H, \kappa f + \lambda H)_2 = \kappa^2 (f, f)_2 + 2\kappa\lambda (f, H)_2 + \lambda^2 (H, H)_2.$$

The coefficient of κ^2 is, of course, H.

To find the coefficient of $2\kappa\lambda$, we take Fig. (32), and join it to f by u, v, and we get at once Fig. (44); but it was proved before that

$$(ab)^2 (ac)^2 b_x b_y c_t c_w = \frac{if}{2} - \frac{j}{2} \{(xw)(ty) + (xt)(wy)\} \quad \ldots\ldots(a),$$

and therefore, if we take the ultimate forms, we get, for Fig. (44),

$$\frac{if}{2} - \frac{if}{3} = \frac{if}{6}.$$

To find the coefficient of λ^2 we have to join H to itself by two bonds, so that we have to multiply

$$(ab)^2 a_x a_y b_u b_v - \frac{i}{6} \{(xu)(yv) + (xv)(yu)\}$$

by
$$(cd)^2 c_u c_v d_\alpha d_\beta - \frac{i}{6} \{(\alpha u)(\beta v) + (\alpha v)(\beta u)\}.$$

---

* The ultimate form of a quantic is what we get when we introduce scalars, and make all the sets identical; it is, in fact, the ordinary form of the quantic.

We get
$$(ab)^2(bc)^2(cd)^2 a_x a_y d_a d_\beta$$
$$+\frac{i^2}{18}\{(xa)(y\beta)+(x\beta)(ya)\}-\frac{2i}{3}(ad)^2 a_x a_y d_a d_\beta.$$

If we take equation (*a*) above, and multiply it by $d_t d_w d_a d_\beta$, we get
$$(ab)^2(ac)^2(cd)^2 b_x b_y d_a d_\beta=\frac{i}{2}(cd)^2 d_a d_\beta c_x c_y+\frac{j}{3}d_a d_\beta d_x d_y,$$

and therefore, if we take the ultimate forms, the coefficient of λ^2 is
$$\frac{iH}{2}+\frac{jf}{3}-\frac{2i}{3}H=\frac{jf}{3}-\frac{iH}{6}.$$

And therefore we get
$$H_F=\kappa^2H+\kappa\lambda\frac{if}{3}+\lambda^2\left(\frac{jf}{3}-\frac{iH}{6}\right)$$
$$=H\left(\kappa^2-\frac{i\lambda^2}{6}\right)+f\left(\kappa\lambda\frac{i}{3}+\frac{\lambda^2 j}{3}\right)$$
$$=\tfrac{1}{3}\left(H\frac{d\Omega}{d\lambda}-f\frac{d\Omega}{d\kappa}\right),$$

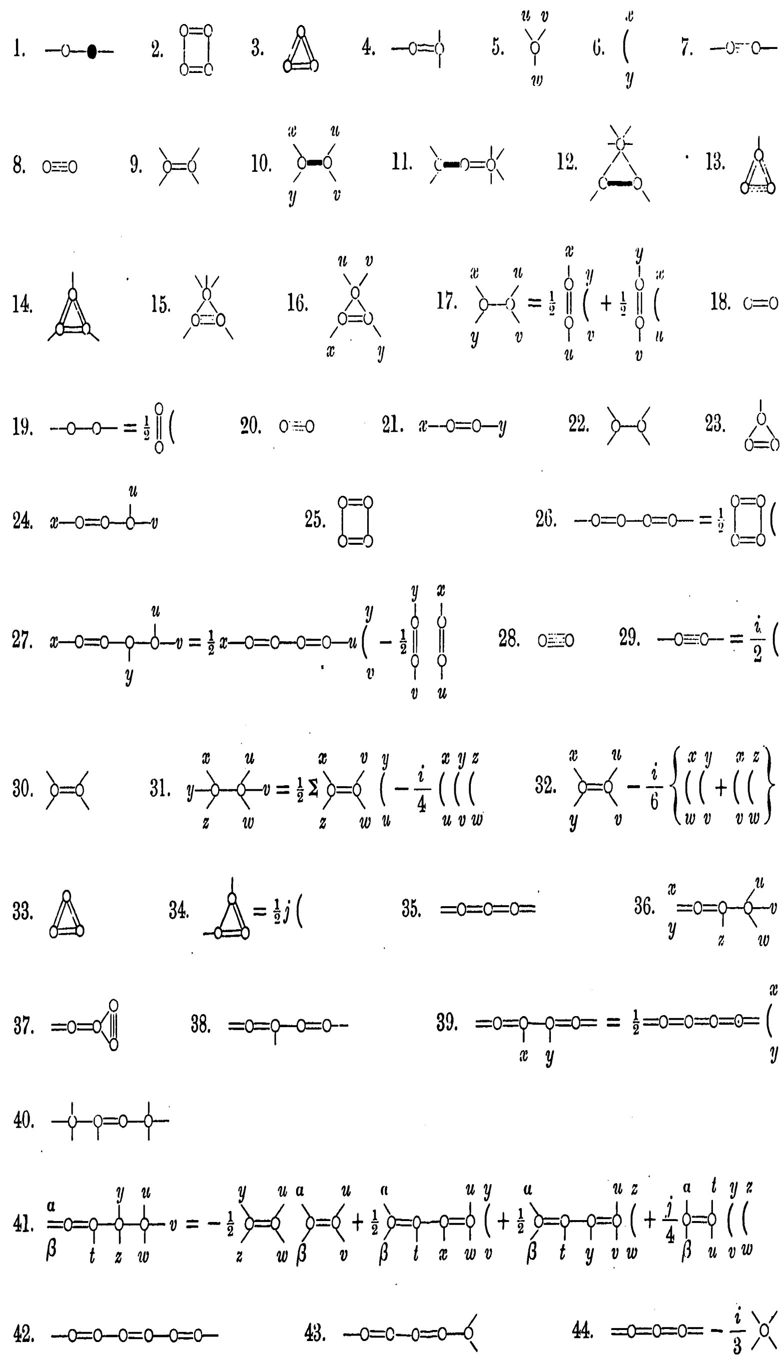
if
$$\Omega=\kappa^3-\frac{i}{2}\kappa\lambda^2-\frac{j}{3}\lambda^3.$$

To find T_F, we have to join this to F by one bond; we get
$$3T_F=\frac{d\Omega}{d\lambda}\{\kappa(f,H)_1+\lambda(H,H)_1\}-\frac{d\Omega}{d\kappa}\{\kappa(f,f)_1+\lambda(H,f)_1\}.$$

Now, if we take the ultimate forms, the first alliance of a quantic with itself vanishes, and we have also
$$(f,H)_1=-(H,f)_1,$$
and therefore
$$3T_F=(f,H)_1\left(\kappa\frac{d\Omega}{d\kappa}+\lambda\frac{d\Omega}{d\lambda}\right)=3T\Omega,$$

and therefore
$$T_F=\Omega\,.\,T.$$

To find i_F, we have to find
$$(\kappa f+\lambda H,\ \kappa f+\lambda H)_4=\kappa^2(f,f)_4+2\kappa\lambda(f,H)_4+\lambda^2(H,H)_4;$$


1. 2. 3. 4. 5. 6. 7.
8. 9. 10. 11. 12. 13.
14. 15. 16. 17. 18.
19. 20. 21. 22. 23.
24. 25. 26.
27. 28. 29.
30. 31. 32.
33. 34. 35. 36.
37. 38. 39.
40.
41.
42. 43. 44.

$(f, f)_4$ is i; $(H, f)_4$ is j, by definition; $(H, H)_4$ is

$$(ab)^2 (bc)^2 (cd)^2 (ad)^2 - \frac{i^2}{3};$$

but we easily find, by using equation (a) above, or by considering the graph, that

$$(ab)^2 (bc)^2 (cd)^2 (ad)^2 = \frac{i^2}{2},$$

and we therefere get

$$i_F = \kappa^2 i + 2\kappa\lambda j + \lambda^2 \frac{i^2}{6}.$$

We have $$3j_F = \left(H\frac{d\Omega}{d\kappa} - f\frac{d\Omega}{d\lambda},\ \kappa f + \lambda H\right)_4$$

$$= \frac{d\Omega}{d\kappa}\left\{\kappa\,(H, f)_4 + \lambda\,(H, H)_4\right\} - \frac{d\Omega}{d\lambda}\left\{\kappa\,(f, f)_4 + \lambda\,(f, H)_4\right\}$$

$$= \kappa j\frac{d\Omega}{d\kappa} + \frac{\lambda i^2}{6}\frac{d\Omega}{d\kappa} - \kappa i\frac{d\Omega}{d\lambda} - \lambda j\frac{d\Omega}{d\lambda},$$

and $$j_F = \frac{j}{3}\left(\kappa\frac{d\Omega}{d\kappa} - \lambda\frac{d\Omega}{d\lambda}\right) + \frac{i}{3}\left(\frac{\lambda i}{6}\frac{d\Omega}{d\kappa} - \kappa\frac{d\Omega}{d\lambda}\right)$$

$$= j\kappa^3 + \frac{i^2}{2}\kappa^2\lambda + \frac{ij}{2}\kappa\lambda^2 + \lambda^3\left(\frac{j^2}{3} - \frac{i^3}{36}\right).$$

XII. *Form-Systems.*

In this section I show how parts of Gordan's researches on form-systems, as given in Clebsch's *Binäre Formen,* can be simplified by the introduction of graphs.

1.

It will be remembered that the fundamental theorem in the theory of systems of quantics (*if two quantics have a finite form-system, then their joint system is derived from a finite form-system*) follows immediately from a lemma which can be expressed as follows:—If a power of a quantic is to be joined to any other quantic, the index of the power must not be greater than the order of the second quantic.* This is quite obvious if we consider the graphs of the two quantics. If the order of the second quantic ϕ is λ, and the index of the power of the first f is ρ, then, since the order of the alliance μ cannot be greater than λ, we are certain to satisfy all the conditions if we join

* This is not Clebsch's enunciation, but is equivalent to it.

μ f's to ϕ by one bond each, and then we are certain to have some f's left over; so that we get the product of a covariant and a power of f.

2.

In the theory of form-systems Clebsch speaks of parts of an alliance, and of the substitution of parts of an alliance for the alliance itself. This is a very simple matter if we consider graphs, for, if a graph is not symmetrical we have to make it symmetrical by adding links, and then, if we join two forms, we get, in the first instance, the graph got by joining their graphs, and then a series of terms obtained from the links. Moreover, if we join two graphs by a given number r of bonds, we can do so in various ways, since we can join any r bonds of the one to any r bonds of the other; the resulting graphs can only differ by terms derived from the complementary terms; and then it is obvious, from section (X.), that if we classify forms according to degree (in ascending order), and according to weight (in descending order), the graphs resulting from the union of two graphs by any given number of bonds can only differ by terms involving earlier forms, and that, therefore, in constructing a form-system, we can join two graphs in any way we please, provided we classify our forms in the way just described.

3.

The fundamental theorem (Clebsch's *Zerlegungssatz*) in the theory of form-systems seems much more obvious and natural if we regard it as a consequence of the following lemma:—*Every graph can be reduced to a sum of simple polygons*, where a simple polygon means an open or closed graph in which no atom is joined to more than two atoms.

For, assuming the truth of the lemma, it is obvious that in a simple polygon one of two things must happen; either all the vertices have free bonds proceeding from them, or some of the vertices are saturated; moreover, if a vertex containing an n-valent atom is saturated, it must be joined to one of the adjacent vertices by $n/2$ bonds at least; and, if the polygon was derived from an n-thic and has no saturated vertex, we can, by taking off one free bond from each vertex, get a graph derived from an $(n-1)$-thic, and we have the theorem: Every graph derived from an n-thic can be expressed as a sum of graphs, some of them derived from an $(n-1)$-thic, and the rest having one side at least containing at least $n/2$ bonds. This is the *Zerlegungssatz*.

As regards the proof of the lemma, we have only to start with the formula $(ab)(ac)(b_x c_y + b_y c_x) = (ab)^2 c_x c_y + (ac)^2 b_x b_y - (bc)^2 a_x a_y$, and then the lemma can be proved without any difficulty.