Moreover $m=0$ gives $\theta=\infty$, and $m=1$ gives $\theta=$ its value as defined by the equation $$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}+\frac{c^2}{h^2+\theta}=1,$$
so that, reversing the sign, the limits are ∞, θ; or, finally, writing under the integral sign ϕ in place of θ, the formula is

Resolved Attraction ÷ Mass of Ellipsoid
$$=\tfrac{3}{2}a\int_{\theta}^{\infty}\frac{d\phi}{(f^2+\phi)\sqrt{(f^2+\phi)(g^2+\phi)(h^2+\phi)}},$$
which is a known formula.

On the Solution of Linear Differential Equations in Series.

By J. Hammond.

[*Read January 14th*, 1875.]

By Leibnitz's theorem,
$$D^m\{\phi(x)y\}=\left\{\phi(x)D^m+m\phi'(x)D^{m-1}+\frac{m(m-1)}{\lfloor 2}\phi''(x)D^{m-2}+\dots\right\}y$$
$$=\phi(x+d)D^my,$$
where $dD^m=mD^{m-1}$, $d^2D^m=m(m-1)D^{m-2}$, ...,
and d operates on D only.

Thus the equation
$$\{\phi_0(x)D^n+\phi_1(x)D^{n-1}+\dots+\phi_n(x)\}y=\phi(x)\ \dots\dots\dots\dots(1),$$
when differentiated m times, gives
$$\{[\phi_0(x+d)D^m]D^n+[\phi_1(x+d)D^m]D^{n-1}+\dots+\phi_n(x+d)D^m\}y=\phi^m(x).$$

Now suppose $$y=y_0+y_1x+y_2\frac{x^2}{\lfloor 2}+\dots\dots;$$
then $y_0, y_1, \dots y_{n-1}$ are arbitrary constants, and $y_n, y_{n+1}, \dots$ are found from the equation
$$\{D^n\phi_0(d)+D^{n-1}\phi_1(d)+\dots+\phi_n(d)\}D^my_0=\phi^m(0)\ \dots\dots\dots(2)$$
by putting $m=0, 1, 2, \dots$, and $D^\kappa y_0=y_\kappa$.

Now $$\phi(d)=\phi+d\phi'+\frac{d^2}{\lfloor 2}\phi''+\dots\dots,$$
where $\phi, \phi', \phi''\dots$ are written instead of $\phi(0), \phi'(0), \phi''(0)\dots$ for shortness.

Thus (2) becomes
$$m_0y_{m+n}+m_1y_{m+n-1}+\dots+m_{m+n}y_0=\phi^m\dots\dots\dots\dots\dots\dots(3).$$

The general coefficient in (3) is

$$m_\kappa = \phi_\kappa + m\phi'_{\kappa-1} + \frac{m(m-1)}{\underline{|2}}\phi''_{\kappa-2} + \dots\dots,$$

$\phi_\kappa, \phi'_\kappa, \phi''_\kappa \dots$ being all zero for all values of κ not included among $\kappa = 0, 1, 2, \dots n$.

Thus, if $$p\phi_\kappa = \phi'_{\kappa-1},\quad p^2\phi_\kappa = \phi''_{\kappa-2}\dots\dots,$$

$$m_\kappa = (1+p)^m\phi_\kappa,$$

and (3) becomes

$$\phi_0 y_{m+n} + (1+p)^m(\phi_1 y_{m+n-1} + \phi_2 y_{m+n-2} + \dots + \phi_{m+n} y_0) = \phi^m \dots\dots (4),$$

p operating on ϕ only.

Now write (m, κ) for $(1+p)^m\phi_\kappa$;

and A_m for $(1+p)^m(\phi_{m+1}y_{n-1} + \phi_{m+2}y_{n-2} + \dots\dots + \phi_{m+n}y_0) - \phi^m$;

therefore, from (4),

$$A_m + (m, m)y_n + (m, m-1)y_{n+1} + \dots\dots + (m, 1)y_{m+n-1} + \phi_0 y_{m+n} = 0.$$

This is true for all positive integral values of m. Thus, putting $m = 0, 1, 2, \dots\dots$ in succession,

$$A_0 + \phi_0 y_n = 0,$$
$$A_1 + (1,1)\,y_n + \phi_0 y_{n+1} = 0,$$
$$A_2 + (2,2)\,y_n + (2,1)\,y_{n+1} + \phi_0 y_{n+2} = 0,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$A_m + (m,m)\,y_n + (m, m-1)\,y_{n+1} + \dots\dots + \phi_0 y_{m+n} = 0,$$

solving these equations,

$$\frac{y_{m+n}}{\begin{vmatrix} A_0, & \phi_0, & 0 & 0, & \dots\dots\dots & 0 \\ A_1, & (1,1), & \phi_0, & 0, & \dots\dots\dots & 0 \\ A_2, & (2,2), & (2,1), & \phi_0, & \dots\dots\dots & 0 \\ A_3, & (3,3), & (3,2), & (3,1), & \dots\dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_m, & (m,m), & (m,m-1), & (m,m-2), & \dots & (m,1) \end{vmatrix}} = \frac{(-1)^{m+1}}{\begin{vmatrix} \phi_0, & 0, & 0, & \dots\dots & 0 \\ (1,1), & \phi_0, & 0, & \dots\dots & 0 \\ (2,2), & (2,1), & \phi_0, & \dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (m,m), & (m,m-1), & (m,m-2), & \dots & \phi_0 \end{vmatrix}}$$

$$= \left(-\frac{1}{\phi_0}\right)^{m+1}.$$

Now $$A_0 = \phi_n y_0 + \phi_{n-1}y_1 + \dots\dots + \phi_1 y_{n-1} - \phi,$$
$$A_1 = (1, n+1)\,y_0 + (1, n)\,y_1 + \dots\dots + (1, 2)\,y_{n-1} - \phi',$$
$$A_2 = (2, n+2)\,y_0 + (2, n+1)\,y_1 + \dots\dots + (2, 3)\,y_{n-1} - \phi'',$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

and $$(m, \kappa) = \phi_\kappa + m\phi'_{\kappa-1} + \frac{m(m-1)}{\underline{|2}}\phi''_{\kappa-2} + \dots\dots$$

therefore $$(1, n+1) = \phi'_n,\quad (2, n+2) = \phi''_n,\quad \dots\dots$$

Thus, the expansion of y is

$$y=y_0\left\{1-\frac{1}{\phi_0}\cdot\phi_n\frac{x^n}{\underline{|n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_n & \phi_0\\ \phi_n' & (1,1)\end{vmatrix}\frac{x^{n+1}}{\underline{|n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_n & \phi_0 & 0\\ \phi_n' & (1,1) & \phi_0\\ \phi_n'' & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\underline{|n+2}}+\ldots\right\}$$

$$+y_1\left\{x-\frac{1}{\phi_0}\cdot\phi_{n-1}\frac{x^n}{\underline{|n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_{n-1} & \phi_0\\ (1,n) & (1,1)\end{vmatrix}\frac{x^{n+1}}{\underline{|n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_{n-1} & \phi_0 & 0\\ (1,n) & (1,1) & \phi_0\\ (2,n+1) & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\underline{|n+2}}+\ldots\right\}$$

$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$+y_{n-1}\left\{\frac{x^{n-1}}{\underline{|n-1}}-\frac{1}{\phi_0}\cdot\phi_1\frac{x^n}{\underline{|n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_1 & \phi_0\\ (1,2) & (1,1)\end{vmatrix}\frac{x^{n+1}}{\underline{|n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_1 & \phi_0 & 0\\ (1,2) & (1,1) & \phi_0\\ (2,3) & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\underline{|n+2}}+\ldots\right\}$$

$$+\frac{1}{\phi_0}\cdot\phi\frac{x^n}{\underline{|n}}-\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi & \phi_0\\ \phi' & (1,1)\end{vmatrix}\frac{x^{n+1}}{\underline{|n+1}}+\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi & \phi_0 & 0\\ \phi' & (1,1) & \phi_0\\ \phi'' & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\underline{|n+2}}-\ldots \quad\ldots\ldots (5).$$

When $\phi_0=0$, this expansion fails.

In this case, put $x=z+h$, where h is not a root of $\phi_0(h)=0$.

Then (1) becomes $\phi_0(z+h)\dfrac{d^ny}{dz^n}+\ldots\ldots=\phi(z+h)$,

and y can be expanded in powers of z.

The equation $\phi(x)y=\psi(x)$ .... (6),

when treated in the same way as (1), gives

$$\frac{\psi(x)}{\phi(x)}=\frac{\psi}{\phi}-\left(\frac{1}{\phi}\right)^2\begin{vmatrix}\psi & \phi\\ \psi' & \phi'\end{vmatrix}x+\left(\frac{1}{\phi}\right)^3\begin{vmatrix}\psi & \phi & 0\\ \psi' & \phi' & \phi\\ \psi'' & \phi'' & 2\phi'\end{vmatrix}\frac{x^2}{\underline{|2}}-\left(\frac{1}{\phi}\right)^4\begin{vmatrix}\psi & \phi & 0 & 0\\ \psi' & \phi' & \phi & 0\\ \psi'' & \phi'' & 2\phi' & \phi\\ \psi''' & \phi''' & 3\phi'' & 3\phi'\end{vmatrix}\frac{x^3}{\underline{|3}}+\ldots (7).$$

This may be deduced from (5) by putting $n=0$,

$$\phi=\psi,\quad \phi'=\psi'\ldots\ldots$$
$$\phi_0=\phi,\quad \phi_0'=\phi'\ldots\ldots$$

and
$$(m,\kappa)=\frac{\underline{|m}}{\underline{|\kappa}\,\underline{|m-\kappa}}\phi^\kappa.$$

Again, putting $\psi(x)=1$ in (6) and (7),

$$\frac{1}{\phi(x)}=\frac{1}{\phi}-\left(\frac{1}{\phi}\right)^2\phi'.x+\left(\frac{1}{\phi}\right)^3\begin{vmatrix}\phi' & \phi\\ \phi'' & 2\phi'\end{vmatrix}\frac{x^2}{\underline{|2}}-\left(\frac{1}{\phi}\right)^4\begin{vmatrix}\phi' & \phi & 0\\ \phi'' & 2\phi' & \phi\\ \phi''' & 3\phi'' & 3\phi'\end{vmatrix}\frac{x^3}{\underline{|3}}+\ldots (8).$$

The coefficients of (7) suggest that

$$D^m\left\{\frac{\psi(x)}{\phi(x)}\right\}=\frac{(-1)^m}{\{\phi(x)\}^{m+1}}\begin{vmatrix}\psi(x) & \phi(x) & 0 & \ldots\\ \psi'(x) & \phi'(x) & \phi(x) & \ldots\\ \psi''(x) & \phi''(x) & 2\phi'(x) & \ldots\\ \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{vmatrix}\ldots\ldots\ldots\ldots\ldots (9),$$

there being $(m+1)$ rows in the determinant, and the last row being

$$\psi^m(x),\ \phi^m(x),\ m\phi^{m-1}(x),\ \frac{m(m-1)}{\lfloor 2}\phi^{m-2}(x), \ldots\ldots\ m\phi'(x).$$

Here (9) may be proved in exactly the same way, except that x is not made $=0$ in the course of the work.

There are some particular cases worth notice.

Case I.—Equation (4) reduces to $\phi_0 y_{m+n} + (1+p)^m y_{m+n-r} = 0$; therefore $(1+p)^m \phi_\kappa = 0$ when κ is not 0 or r.

Putting $m = 0, 1, 2, \ldots\ldots$ in succession,

$$\left.\begin{array}{lll} \phi_\kappa & = 0 & \\ \phi'_{\kappa-1} = -\phi_\kappa & & = 0 \\ \phi''_{\kappa-2} = \phi_\kappa & & = 0 \\ \ldots & \ldots\ \ldots & \ldots \\ \phi_0^\kappa & = (-1)^\kappa \phi_\kappa & = 0 \end{array}\right\} \text{ when } \kappa \text{ is not 0 or } r.$$

Thus

$$\phi_0(x) = \phi_0 + \frac{x^r}{\lfloor r}\phi_0^r = c\left(1 + a_0\frac{x^r}{\lfloor r}\right),$$

$$\phi_1(x) = \frac{x^{r-1}}{\lfloor r-1}\phi_1^{r-1} = ca_1\frac{x^{r-1}}{\lfloor r-1},$$

$$\phi_2(x) = \frac{x^{r-2}}{\lfloor r-2}\phi_2^{r-2} = ca_2\frac{x^{r-2}}{\lfloor r-2},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi_r(x) = \text{const.} \quad = ca_r,$$

$$\phi_{r+1}(x) = 0, \quad \phi_{r+2}(x) = 0 \ldots\ldots$$

Equation (1) then becomes, when $r < n$,

$$\left\{\left(1 + a_0\frac{x^r}{\lfloor r}\right)D^n + a_1\frac{x^{r-1}}{\lfloor r-1}D^{n-1} + \ldots\ldots + a_r D^{n-r}\right\} y = 0 \ \ldots\ldots\ (10);$$

and when $r > n$,

$$\left\{\left(1 + a_0\frac{x^r}{\lfloor r}\right)D^n + a_1\frac{x^{r-1}}{\lfloor r-1}D^{n-1} + \ldots\ldots + a_n\frac{x^{r-n}}{\lfloor r-n}\right\} y = 0 \ \ldots\ldots\ (11).$$

It is easily seen that the solution of (10) is obtained from that of (11) by solving the equation (11) when n is put $=r$, and then integrating the result $n-r$ times.

Comparing (11) with (1),

$$\phi_0 = 1,$$

$$\phi_0^r = a_0, \quad \phi_1^{r-1} = a_1, \quad \phi_2^{r-2} = a_2, \ \ldots\ldots$$

Also $(m, \kappa) = 0$ except when $\kappa = r$. And

$$(m, r) = \phi_r + m\phi'_{r-1} + \frac{m(m-1)}{\lfloor 2} \phi''_{r-2} + \&c.$$

$$= a_r + ma_{r-1} + \frac{m(m-1)}{\lfloor 2} a_{r-2} + \ldots\ldots + \frac{m(m-1)\ldots(m-r+1)}{\lfloor r} a_0.$$

And since r is to be taken not less than n, $a_r, a_{r-1}, \ldots\ldots a_{n+1}$ are all zero; therefore

$$(m, r) = \left\{ a_0 \frac{\lfloor m}{\lfloor r \lfloor m-r} + a_1 \frac{\lfloor m}{\lfloor r-1 \lfloor m-r+1} + \ldots + a_n \frac{\lfloor m}{\lfloor r-n \lfloor m-r+n} \right\}.$$

Thus, expanding y by means of the relation

$$y_{m+n} = -(m, r)\, y_{m+n-r},$$

$$y = y_0 \left\{ 1 - \frac{(r-n, r)}{\lfloor r} x^r + \frac{(2r-n, r)(r-n, r)}{\lfloor 2r} x^{2r} - \ldots\ldots \right\}$$

$$+ y_1 \left\{ x - \frac{(r-n+1, r)}{\lfloor r+1} x^{r+1} + \frac{(2r-n+1, r)(r-n+1, r)}{\lfloor 2r+1} x^{2r+1} - \ldots\ldots \right\}$$

$$+ y_2 \left\{ \frac{x^2}{\lfloor 2} - \frac{(r-n+2, r)}{\lfloor r+2} x^{r+2} + \frac{(2r-n+2, r)(r-n+2, r)}{\lfloor 2r+2} x^{2r+2} - \ldots\ldots \right\}$$

$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+ y_{n-1} \left\{ \frac{x^{n-1}}{\lfloor n-1} - \frac{(r-1, r)}{\lfloor n+r-1} x^{n+r-1} + \frac{(2r-1, r)(r-1, r)}{\lfloor n+2r-1} x^{n+2r-1} - \ldots\ldots \right\}.$$

Many well known expansions are particular cases of the solution of (10) and (11).

Thus

$\{(1+x)D-n\}y = 0$	gives the expansion of		$(1+x)^n$,
$\{(1+x)D^2+D\}y = 0$	,,	,,	$\log(1+x)$,
$\{(1+x^2)D^2+2xD\}y = 0$	,,	,,	$\tan^{-1}x$,
$\{(1-x^2)D^2-xD\}y = 0$	,,	,,	$\sin^{-1}x$,
$\{(1-x^2)D^2-xD+m^2\}y = 0$	,,	,,	$A\sin(m\sin^{-1}x) + B\cos(m\sin^{-1}x)$.

Case II.—The general coefficient of equation (4) is

$$(1+p)^m \phi_\kappa = a_\kappa F(m+n) F(m+n-1) \ldots\ldots F(m+n-\kappa+1);$$

therefore, putting $m=0$,

$$\phi_\kappa = a_\kappa F(n) F(n-1) \ldots\ldots F(n-\kappa+1).$$

And (4) reduces to

$$a_0 u_{m+n} + a_1 u_{m+n-1} + \ldots\ldots + a_n u_m = 0 \quad \ldots\ldots\ldots\ldots\ldots (12),$$

where $$u_{m+n} = \frac{y_{m+n}}{F(m+n)\,F(m+n-1)\,\dots}.$$

Here $$(1+p)^m = E_n^m, \text{ and } p = \Delta_n.$$

Now $$\phi_{\kappa-r}^r = p^r\phi_\kappa = \Delta_n^r\phi_\kappa;$$

therefore $$\phi_\kappa^r = E_\kappa^r\Delta_n^r\phi_\kappa.$$

Thus $$\phi_\kappa(x) = e^{xE_\kappa\Delta_n}\phi_\kappa \quad\dots\dots\dots\dots (13).$$

When $F(n)=1$, this is the case of linear differential equations with constant coefficients; and the general coefficient in the solution of

$$(a_0D^n + a_1D^{n-1} + \dots\dots + a_n)\,y = 0$$

is the general solution of the difference equation

$$a_0y_{m+n} + a_1y_{m+n-1} + \dots\dots + a_ny_m = 0.$$

When $F(n) = n+b-1$,

$$\phi_\kappa = a_\kappa(n+b-1)(n+b-2)\dots\dots(n+b-\kappa) = a_\kappa\frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)},$$

and $$E_\kappa^r\Delta_n^r\phi_\kappa = a_{\kappa+r}(\kappa+r)(\kappa+r-1)\dots\dots(\kappa+1)\frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)};$$

therefore, expanding (13) and putting $\kappa=0$,

$$\phi_0(x) = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n.$$

The general value of $\phi_\kappa(x)$, obtained by expanding (13), is

$$\phi_\kappa(x) = \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)}\left\{a_\kappa + a_{\kappa+1}(\kappa+1)x + a_{\kappa+2}\frac{(\kappa+2)(\kappa+1)}{\underline{|2}}x^2 + \dots\right\}$$

$$= \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)\,\underline{|\kappa}}\left\{a_\kappa\underline{|\kappa} + a_{\kappa+1}\underline{|\kappa+1}\,x + a_{\kappa+2}\frac{\underline{|\kappa+2}}{\underline{|2}}x^2 + \dots \right.$$

$$\left.\dots + a_n\frac{\underline{|n}}{\underline{|n-\kappa}}x^{n-\kappa}\right\}$$

$$= \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)\,\underline{|\kappa}}D^\kappa\phi_0(x)$$

Thus equation (1) becomes

$$\left\{\frac{\phi_0(x)}{\Gamma(n+b)}D^n + \frac{\phi_0'(x)}{\Gamma(n+b-1)}D^{n-1} + \frac{\phi_0''(x)}{\Gamma(n+b-2)\,\underline{|2}}D^{n-2} + \dots\right.$$

$$\left.\dots + \frac{\phi_0^n(x)}{\Gamma(b)\,\underline{|n}}\right\}y = 0 \quad\dots\dots\dots\dots (14),$$

where $\phi_0(x)$ is any rational integral function of x of the nth degree. And equation (12) becomes

$$\frac{a_0y_{m+n}}{\Gamma(m+n+b)} + \frac{a_1y_{m+n-1}}{\Gamma(m+n+b-1)} + \dots\dots + \frac{a_ny_m}{\Gamma(m+b)} = 0 \quad\dots (15).$$

Thus if a be a single root of the equation $\phi_0\left(\frac{1}{x}\right) = 0$, the corresponding solution of (15) is

$$y_m = \Gamma(m+b)\, A a^m,$$

and the corresponding expansion of y from (14)

$$y = A\left\{\Gamma(b) + \Gamma(b+1)\, ax + \Gamma(b+2)\frac{a^2x^2}{\lfloor 2} + \dots\dots\right\} \dots\dots (16).$$

The n arbitrary constants A, ... are not $y_0, y_1, \dots$, but are connected with them by linear equations.

When $b=1$, equation (14) reduces to

$$\left\{\phi_0(x)\,D^n + n\phi_0'(x)\,D^{n-1} + \frac{n(n-1)}{\lfloor 2}\phi_0''(x)\,D^{n-2} + \dots\dots\right\} y = 0,$$

or

$$D^n\{\phi_0(x)\,y\} = 0.$$

This gives the expansion for rational fractions; and in the same manner, when b is put $=0$, or any positive or negative integer, the expansion obtained is that of $D^{b-1}(f)$, where f stands for the rational fraction, and $D^{b-1}(f)$ for its $(b-1)^{\text{th}}$ differential coefficient; negative indices of course meaning integrations.

The Diagonal Scale Principle applied to Angular Measurement in the Circular Slide Rule. By JOHN R. CAMPBELL.

[*Abstract of Paper, read January* 14*th*, 1875.]

Before entering upon the construction of diagonal scales, having, in place of the usual equidistant parallel lines crossed by a straight diagonal, as many equidistant concentric arcs of circles crossed by a curved diagonal, it will be necessary for me briefly to describe the instrument, or rather the somewhat rough home-made model of one, in which I have introduced them.

It is simply a form of circular slide rule, combining in one arrangement both the ordinary principle of two logometric scales for multiplication and division, and that introduced by the late Dr. Roget (*vide* Phil. Trans., Nov. 17, 1814) for the finding of powers and roots.

Fig. 1 is a plan of the face; fig. 2, a section of the instrument by a vertical plane through the centre.

The face is a circular cardboard surface AA, 12 or 14 inches in diameter, but which might well be made of smaller dimensions. It consists of two parts,—an annular rim AB for the *fixed* scale, and a circular disc BCB corresponding to the *slide*, turning on an axis C in