On certain Definite ϑ-Function Integrals. By L. J. ROGERS.

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1. If c_m is such a function of $\sqrt{-1}$ that c_{-m} is the same function of $-\sqrt{-1}$, then the series $\sum_{m=-\infty}^{\infty} c_m e^{mui}$ is real.

Let $$\vartheta(u) \sum_{m=-\infty}^{\infty} c_{2m} e^{2mui} = \sum_{r=-\infty}^{\infty} \kappa_{2r} e^{2rui} \quad \text{..................(1)}$$

Then κ_{2r} is the coefficient of e^{2rui} in the product

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} \times c_{2m} e^{2mui}$$

$$= \sum_{r=-\infty}^{\infty} \left\{ e^{2rui} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} c_{2m} \right\};$$

therefore $$\kappa_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r)} c_{2m} \quad \text{..................(2).}$$

Suppose $$c_{2m} = \frac{2q^m e^{vi}}{1+q^{2m} e^{2vi}},$$

so that $$c_{-2m} = \frac{2q^m e^{-vi}}{1+q^{2m} e^{-2vi}};$$

then the condition for the series being real is satisfied.

From (2), moreover, we have

$$\kappa_{2r} q^{-r^2} - \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r)} (1+q^{2m} e^{2vi}) c_{2m}$$

$$= 2e^{vi} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r+1)}$$

$$= 0 \quad \text{identically.}$$

Hence $$\kappa_{2r} q^{-r^2} = \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = \kappa_0 e^{2rvi}.$$

Moreover, $$\kappa_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} \frac{2q^m e^{vi}}{1+q^{2m} e^{2vi}},$$

which, by a theorem of Jacobi's,

$$= \frac{\vartheta_1'(0)}{\vartheta_2(v)}.$$

Hence
$$\vartheta(u) \sum_{m=-\infty}^{\infty} \frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}} e^{2mui}$$

$$= \frac{\vartheta_1'(0)}{\vartheta_2(v)} \{1+2q\cos 2(u+v)+2q^4\cos 4(u+v)+\ldots\}$$

$$= \frac{\vartheta_1'(0)\,\vartheta_3(u+v)}{\vartheta_2(v)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

2. We may derive a remarkable algebraic fact from the foregoing result.

Suppose the values of c_{2m} and κ_{2m} in (1) quite general, and let b_{2m} be such a function of u that

$$\vartheta(v) \sum_{m=-\infty}^{\infty} b_{2m} e^{2mvi} = \vartheta_3(u+v).$$

Then, as in § 1, (2),

$$q^{r^2} e^{2rui} = \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} b_{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But b_{2m} is the coefficient of e^{2mvi} in

$$\frac{\vartheta_2(u)}{\vartheta_1'(0)} \sum_{m=-\infty}^{\infty} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}},$$

by interchanging u and v in § 1, (3).

Hence
$$\frac{\vartheta_2(u)}{\vartheta_1'(0)} \sum_{m=-\infty}^{\infty} \kappa_{2m} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}} = \sum_{m=-\infty}^{\infty} \kappa_{2m} b_{2m}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} c_{2r} b_{2m}, \quad \text{by § 1, (1)},$$

$$= \sum_{r=-\infty}^{\infty} c_{2r} q^{r^2} e^{2rui}, \quad \text{by (1)}.$$

We see then that, *if c_{2m} and κ_{2m} are so related that*

$$\vartheta(u) \sum_{m=-\infty}^{\infty} c_{2m} e^{2mui} = \sum_{m=-\infty}^{\infty} \kappa_{2m} e^{2mui}$$

then these coefficients are also so related that

$$\vartheta_2(u) \sum_{m=-\infty}^{\infty} \kappa_{2m} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}} = \vartheta_1'(0) \sum_{m=-\infty}^{\infty} c_{2m} q^{m^2} e^{2mui} \quad \ldots\ldots\ldots(2).$$

By changing e^{vi} into $q^{\frac{1}{2}}e^{vi}$ in § 1, (3), so that $\vartheta_2(v)$ becomes $q^{-\frac{1}{4}}e^{-vi}\vartheta_3(v)$ and $\vartheta_3(u+v)$ becomes $q^{-\frac{1}{4}}e^{-(u+v)i}\vartheta_2(u+v)$, we get

$$\vartheta(u)\sum_{m=-\infty}^{\infty}\frac{2q^{m+\frac{1}{2}}e^{vi}}{1+q^{2m+1}e^{2vi}}=\frac{\vartheta_1'(0)\,\vartheta_2(u+v)}{\vartheta_3(v)}\quad\text{...........(3)},$$

whence, in the same manner as above, we find that, if

$$\vartheta(u)\sum_{m=-\infty}^{\infty}c_{2m+1}e^{(2m+1)ui}=\sum_{m=-\infty}^{\infty}\kappa_{2m+1}e^{(2m+1)ui},$$

then
$$\vartheta_3(u)\sum_{m=-\infty}^{\infty}\kappa_{2m+1}\frac{2q^{m+\frac{1}{2}}e^{ui}}{1+q^{2m+1}e^{2ui}}=\vartheta_1'(0)\sum_{m=-\infty}^{\infty}c_{2m+1}q^{(m+\frac{1}{2})^2}e^{(2m+1)ui}\quad\text{.................(4)}.$$

Moreover, since
$$\int_0^{\pi}e^{2mti}\,dt=0,$$

when
$$m\neq 0,$$

we see that

$$\left.\begin{aligned}\sum_{m=-\infty}^{\infty}c_{2m}q^{m^2}e^{2mui}&=\frac{1}{\pi}\int_0^{\pi}\sum_{m=-\infty}^{\infty}c_{2m}e^{2m(u+t)i}\,\vartheta_3(t)\,dt\\ \text{and}\quad\sum_{m=-\infty}^{\infty}c_{2m+1}q^{(m+\frac{1}{2})^2}e^{(2m+1)ui}&=\frac{1}{\pi}\int_0^{\pi}\sum_{m=-\infty}^{\infty}c_{2m+1}e^{(2m+1)(u+t)i}\,\vartheta_2(t)\,dt\end{aligned}\right\}\quad\text{..............(5)}.$$

3. In § 2, (2), let
$$c_{2m}=\frac{2q^{m}e^{vi}}{1+q^{2m}e^{2vi}},$$

so that, by § 1, (3),
$$\kappa_{2m}=\frac{\vartheta_1'(0)}{\vartheta_3(v)}q^{m^2}e^{2mvi}.$$

We see then that

$$\vartheta_3(u)\sum_{m=-\infty}^{\infty}\frac{2q^{m}e^{ui}}{1+q^{2m}e^{2ui}}q^{m^2}e^{2mvi}=\vartheta_3(v)\sum_{m=-\infty}^{\infty}\frac{2q^{m}e^{vi}}{1+q^{2m}e^{2vi}}q^{m^2}e^{2mui}\ \text{...(1)},$$

so that either side is symmetrical in u and v.

Similarly, from § 2, (2) and (4), we get that

$$\vartheta_3(u)\sum_{m=-\infty}^{\infty}\frac{2q^{m+\frac{1}{2}}e^{ui}}{1+q^{2m+1}e^{2ui}}q^{(m+\frac{1}{2})^2}e^{(2m+1)vi}\quad\text{.................(2)}$$

is symmetrical in u and v.

Either side of (1) will be denoted by $M_3(u, v)$, while (2) will be written $M_2(u, v)$, so that

$$M_3(u, v)=M_3(v, u)\quad\text{and}\quad M_2(u, v)=M_2(v, u).$$

Moreover, it is easy to see that

$$M_2(u, -v) = M_2(-u, v) = M_2(v, -u),$$

and
$$M_3(u, -v) = M_3(v, -u).$$

It will be convenient to write $\Lambda_2(u, v)$, $\Lambda_3(u, v)$ for $M_2(u, -v)$, $M_3(u, -v)$, so that the Λ-functions are also symmetrical in u and v. By adding and subtracting, we easily establish the symmetry in u and v of the four following expressions

$$\left.\begin{array}{l}
\vartheta_2(v)\left\{\dfrac{1}{\cos v}+\dfrac{4q(1+q^2)\cos v}{1+2q^2\cos 2v+q^4}\,q\cos 2u+\dfrac{4q^2(1+q^4)\cos v}{1+2q^4\cos 2v+q^8}\,q^4\cos 4u+\dots\right\} \\
\vartheta_2(v)\left\{\dfrac{4q(1-q^2)\sin v}{1+2q^2\cos 2v+q^4}\,q\sin 2u+\dfrac{4q^2(1-q^4)\sin v}{1+2q^4\cos 2v+q^8}\,q^4\sin 4u+\dots\right\} \\
\vartheta_3(v)\left\{\dfrac{4q^{\frac12}(1+q)\cos v}{1+2q\cos 2v+q^2}\,q^{\frac14}\cos u+\dfrac{4q^{\frac32}(1+q^3)\cos v}{1+2q^3\cos 2v+q^6}\,q\cos 3u+\dots\right\} \\
\vartheta_3(v)\left\{\dfrac{4q^{\frac12}(1-q)\sin v}{1+2q\cos 2v+q^2}\,q^{\frac14}\sin u+\dfrac{4q^{\frac32}(1-q^3)\cos v}{1+2q^3\cos 2v+q^6}\,q^{\frac94}\sin 3u+\dots\right\}
\end{array}\right\} \quad \dots\dots(3).$$

When $v = 0$, the first and third of these become, respectively,

$$\vartheta_2(0)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\frac{4q^{12}}{1+q^6}\cos 6u+\dots\right\}$$

and
$$\vartheta_3(0)\left\{\frac{4q^{\frac34}}{1+q}\cos u+\frac{4q^{\frac{15}{4}}}{1+q^3}\cos 3u+\dots\right\},$$

which will be called $\Lambda_3(u)$ and $\Lambda_2(u)$, respectively.

Expressing (1) and (2) in terms of definite integrals by the help of § 1, (3), and § 2, (3) and (5), we see that

$$\int_0^\pi \frac{\vartheta_3(u+v+t)\,\vartheta_3(t)}{\vartheta(u+t)}\,dt = \int_0^\pi \frac{\vartheta_3(u+v+t)\,\vartheta_3(t)}{\vartheta(v+t)}\,dt = \frac{M_3(u, v)}{\vartheta_1'(0)}$$

and
$$\int_0^\pi \frac{\vartheta_2(u+v+t)\,\vartheta_2(t)}{\vartheta(u+t)}\,dt = \int_0^\pi \frac{\vartheta_2(u+v+t)\,\vartheta_2(t)}{\vartheta(v+t)} = \frac{M_2(u, v)}{\vartheta_1'(0)}.$$

4. It may now be shown that $\Lambda_2(u, v)$ may be expressed in terms of $\Lambda_2(u+v)$, and $\Lambda_3(u, v)$ in terms of $\Lambda_3(u+v)$.

Suppose that

$$\vartheta_3(u)\sum_{m=-\infty}^{\infty} c_{2m}\,q^{m^2}e^{2mui} - \vartheta_2(u)\sum_{m=-\infty}^{\infty} c_{2m+1}\,q^{(m+\frac12)^2}e^{(2m+1)ui} = \sum_{r=-\infty}^{\infty} \kappa^{2r}e^{2rui}.$$

Then the left-hand side

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ c_{2m} q^{m^2+n^2} e^{2(m+n)ui} - c_{2m+1} q^{(m+\frac{1}{2})^2+(n-\frac{1}{2})^2} e^{2(m+n)ui} \right\},$$

so that, putting $m+n=r$, we see that

$$\kappa_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} \left\{ q^{2m(m-r)} c_{2m} - q^{(2m+1)(m-r+\frac{1}{2})} c_{2m+1} \right\}$$

$$= q^{r^2} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(m-2r)} c_m.$$

Hence, just as in § 1, when c_m has the same value,

$$\kappa_{2r} q^{-r^2} + \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = 0,$$

while $$\kappa_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m^2} c_m = \frac{\vartheta_1'(0, q^{\frac{1}{2}})}{\vartheta_2(v, q^{\frac{1}{2}})} = \frac{\vartheta_1'(0)\,\vartheta(0)}{\vartheta_2(v)\,\vartheta_3(v)},$$

so that $$\kappa_{2r} = \frac{\vartheta_1'(0)\,\vartheta(0)}{\vartheta_2(v)\,\vartheta_3(v)}\,\vartheta(u+v).$$

We have therefore the following linear relation connecting $M_2(u, v)$ and $M_3(u, v)$,

$$\vartheta_3(u)\,\vartheta_3(v)\,M_3(u, v) - \vartheta_2(u)\,\vartheta_2(v)\,M_2(u, v) = \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u+v) \quad \text{..............(1).}$$

Changing v into $-v$, we have, moreover,

$$\vartheta_3(u)\,\vartheta_3(v)\,\Lambda_3(u, v) - \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_2(u, v) = \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u-v) \quad \text{..............(2).}$$

These equations may be immediately obtained by dividing the known equation

$$\vartheta_3(u+v+t)\,\vartheta_3(t)\,\vartheta_3(u)\,\vartheta_3(v) - \vartheta_2(u+v+t)\,\vartheta_2(t)\,\vartheta_2(u)\,\vartheta_2(v)$$
$$= \vartheta(0)\,\vartheta(u+v)\,\vartheta(u+t)\,\vartheta(v+t)$$

by $\vartheta(u+t)$ and integrating for t between limits π and 0.

5. If $\alpha = -i \log q$, so that $e^{(v+\alpha)i} = q^{\frac{1}{2}} e^{vi}$, we see that, by adding α to u and to v,

$$c_{2m} q^{m^2} e^{2mui} \text{ becomes } c_{2m+1} q^{m^2+m} e^{2mui} = e^{-ui} q^{-\frac{1}{4}} c_{2m+1} q^{(m+\frac{1}{2})^2} e^{(2m+1)ui}.$$

Hence $$M_2(u+\tfrac{1}{2}\alpha,\ v+\tfrac{1}{2}\alpha) = \vartheta_2(v+\tfrac{1}{2}\alpha)\, e^{-ui} q^{-\frac{1}{4}} \frac{M_3(u, v)}{\vartheta_3(v)}$$

$$= e^{-(u+v)i} q^{-\frac{1}{2}} M_3(u, v).$$

Thus, writing $v-\frac{1}{2}a$ for v, we have

$$M_3(u+\tfrac{1}{2}a, v) = e^{-(u+v)i} M_2(u-\tfrac{1}{2}a, v),$$

so that, changing v into $-v$,

$$\Lambda_3(u+\tfrac{1}{2}a, v) = e^{-(u-v)i} \Lambda_2(u-\tfrac{1}{2}a, v),$$

and replacing u by $u+\frac{1}{2}a$,

$$\Lambda_3(u+a, v) = e^{-(u-v)i} q^{-\frac{1}{4}} \Lambda_2(u, v),$$

so that
$$e^{2ui}\Lambda_3(u+a, v) = e^{(u+v)i} q^{\frac{1}{4}} \Lambda_2(u, v)$$
$$= e^{2vi}\Lambda_3(u, v+a), \text{ by symmetry.}$$

Similarly,
$$e^{4ui}\Lambda_3(u+2a, v) = e^{2(u+v)i}\Lambda_3(u+a, v+a)$$
$$= e^{4vi}\Lambda_3(u, v+2a), \text{ by symmetry.}$$

Proceeding in this way, we easily get

$$e^{2rui}\Lambda_3(u+ra, v) = e^{2rvi}\Lambda_3(u, v+ra) \quad\text{..................(1).}$$

We may notice, moreover, that $\Lambda_3(u)$ and $\Lambda_2(u)$ have a kind of imaginary period, for

$$\Lambda_3(u+\tfrac{1}{2}a)+\Lambda_3(u-\tfrac{1}{2}a)$$

$$= \vartheta_2(0)\left\{1+\frac{2q^2}{1+q^2}(q^{-1}+q)\cos 2u+\frac{2q^6}{1+q^4}(q^{-2}+q^2)\cos 4u+\dots\right\}$$

$$= \vartheta_2(0)\,\vartheta_3(u),$$

while
$$\Lambda_2(u+\tfrac{1}{2}a)+\Lambda_2(u-\tfrac{1}{2}a) = \vartheta_3(0)\,\vartheta_2(u).$$

6. Let us now expand

$$\vartheta_3(u+x)\sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2m(u-x)i}+\vartheta_2(u+x)\sum_{m=-\infty} q^{(m+\frac{1}{2})^2} c_{2m+1} e^{(2m+1)(u-x)i} \quad\text{..................(1)}$$

in the form
$$\Sigma A_{2r} e^{2rxi}.$$

The proposed expansion is

$$\sum_{n=-\infty}\sum_{m=-\infty}^{\infty} q^{m^2+n^2} c_{2m} e^{2(m+n)ui} e^{2(n-m)xi}$$
$$+\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2+(n+\frac{1}{2})^2} c_{2m+1} e^{2(m+n+1)ui} e^{2(n-m)xi},$$

so that, if $n = m+r$, it becomes

$$\sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{q^{r^2+2m(m+r)} c_{2m} e^{2(2m+r)ui} e^{2rxi} + q^{r^2+(2m+1)(m+r+\frac{1}{2})} c_{2m+1} e^{2(2m+1+r)} e^{2rxi}\right\},$$

and
$$A_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} q^{\frac{1}{2}m(m+2r)} c_m e^{2mui} e^{2rui}.$$

Now, if we change q into q^2 in A_{2r}, we get

$$q^{2r^2} e^{2rui} \sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2mui} q^{2mr} = q^{2r^2} e^{2rui} \sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2m(u+x)i}$$

$$= q^{2r^2} e^{2rui} \frac{M_3(u+ra, v)}{\vartheta_2(v)}.$$

We see, then, that, if v were changed to $-v$, this coefficient after multiplying by $\vartheta_2(v)$ would be symmetrical in u and v, by § 5.

Hence (1) is symmetrical in u and v after multiplying by $\vartheta_2(v, q^{\frac{1}{2}})$, and changing v into $-v$.

But
$$\vartheta_2(v, q^{\frac{1}{2}})\, \vartheta_2(0, q^{\frac{1}{2}}) = 2\vartheta_2(v)\, \vartheta_3(v),$$

so that we finally get

$$\vartheta_3(u+x)\, \vartheta_3(v)\, \Lambda_3(u-x, v) + \vartheta_2(u+x)\, \vartheta_2(v)\, \Lambda_2(u-x, v)$$
$$= \vartheta_3(v+x)\, \vartheta_3(u)\, \Lambda_3(v-x, u) + \vartheta_2(v+x)\, \vartheta_2(u)\, \Lambda_2(v-x, u) \quad \ldots(2).$$

Eliminating $\Lambda_3(u-x, v)$ by § 4, we see that

$$\frac{\vartheta_3(u+x)}{\vartheta_3(u-x)} \left[\Lambda_2(u-x, v)\, \vartheta_2(u-x)\, \vartheta_2(v) + \vartheta_1'(0)\, \vartheta(0)\, \vartheta(u-v-x)\right]$$
$$+ \vartheta_2(u+x)\, \vartheta_2(v)\, \Lambda_2(u-x, v)$$

is symmetrical in u and v.

But $\vartheta_3(u+x)\, \vartheta_2(u-x) + \vartheta_2(u+x)\, \vartheta_3(u-x) = \vartheta_2(u, q^{\frac{1}{2}})\, \vartheta_2(x, q^{\frac{1}{2}})$,

so that

$$\vartheta_2(u, q^{\frac{1}{2}})\, \vartheta_2(x, q^{\frac{1}{2}})\, \vartheta_2(v, q^{\frac{1}{2}})\, \vartheta(0, q^{\frac{1}{2}}) \left\{ \frac{\Lambda_2(u-x, v)}{\vartheta_3(u-x)\, \vartheta_3(v)} - \frac{\Lambda_2(v-x, u)}{\vartheta_3(v-x)\, \vartheta_3(u)} \right\}$$

$$= 2\vartheta_1'(0)\, \vartheta(0) \left\{ \frac{\vartheta_3(v+x)\, \vartheta(u-v+x)}{\vartheta_3(v-x)} - \frac{\vartheta_3(u+x)\, \vartheta(u-v-x)}{\vartheta_3(u-x)} \right\}.$$

But

$$\vartheta_3(u+x)\, \vartheta_3(v-x)\, \vartheta(u-v-x)\, \vartheta(x) - \vartheta_3(v+x)\, \vartheta_3(u-x)\, \vartheta(u-x+x)\, \vartheta(x)$$
$$= \vartheta_1(v-u)\, \vartheta_1(2x)\, \vartheta_2(u)\, \vartheta_2(v),$$

and $\vartheta(0)\,\vartheta_1(2x)\,\vartheta_2(0)\,\vartheta_3(0) = 2\vartheta(x)\,\vartheta_1(x)\,\vartheta_2(x)\,\vartheta_3(x)$,

so that, finally, the above equation reduces to

$$\vartheta_3(v-x)\,\vartheta_3(u)\,\Lambda_2(u-x,\,v)-\vartheta_3(u-x)\,\vartheta_3(v)\,\Lambda_2(v-x,\,u)$$
$$= \vartheta_1'(0)\,\vartheta_1(u-v)\,\vartheta_1(x)\;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Changing u into $u+v$, and x into v, this becomes

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_2(u,\,v) = \vartheta_3(u)\,\vartheta_3(v)\,\Lambda_2(u+v)+\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)\;\ldots\ldots\ldots(4).$$

Similarly, by § 4, (2), we may eliminate the Λ_2-functions from (3), and obtain the relation

$$\vartheta_2(v-x)\,\vartheta_2(u)\,\Lambda_3(u-x,\,v)-\vartheta_2(u-x)\,\vartheta_2(v)\,\Lambda_3(v-x,\,u)$$
$$= \vartheta_1'(0)\,\vartheta_1(v-u)\,\vartheta_1(x)\;\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

or, with the same change as before,

$$\vartheta_2(u+v)\,\vartheta_2(0)\,\Lambda_3(u,\,v) = \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_3(u+v)-\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)\;\ldots\ldots\ldots(6).$$

We have, then, formulæ for expressing $\Lambda_2(u,\,v)$ and $\Lambda_3(u,\,v)$ in terms of the simple functions $\Lambda_2(u+v)$, $\Lambda_3(u+v)$ and ϑ-functions.

Again, by the equation

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3(u+v)-\vartheta_2(u+v)\,\vartheta_2(0)\,\Lambda_2(u+v)$$
$$= \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u+v),$$

which is easily obtained from § 4, (2), we may eliminate $\Lambda_3(u+v)$ from (6), and obtain

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3(u,\,v) = \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_2(u+v)+\vartheta_1'(0)\,\vartheta(u)\,\vartheta(v)\;\ldots\ldots\ldots(7).$$

Changing u, v into $u+\frac{\pi}{2}$, $v+\frac{\pi}{2}$, and noticing that

$$\Lambda_2(u+v+\pi) = -\Lambda_2(u+v),$$

we have, from (4) and (7),

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_2\left(u+\frac{\pi}{2},\,v+\frac{\pi}{2}\right)$$
$$= -\vartheta(u)\,\vartheta(v)\,\Lambda_2(u+v)+\vartheta_1'(0)\,\vartheta_2(u)\,\vartheta_2(v)\ldots\ldots(8),$$

and $\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$

$$= -\vartheta_1(u)\,\vartheta_1(v)\,\Lambda_2(u+v)+\vartheta_1'(0)\,\vartheta_3(u)\,\vartheta_3(v) \ldots\ldots (9).$$

Moreover, since

$$\vartheta_3^2(u)\,\vartheta_3^2(v)-\vartheta_2^2(u)\,\vartheta_2^2(v)+\vartheta_1^2(u)\,\vartheta_1^2(v)-\vartheta^2(u)\,\vartheta^2(v)=0,$$

we get, from (4), (7), (8), and (9),

$$\Lambda_3^2(u,v)-\Lambda_2^2(u,v)+\Lambda_2^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)-\Lambda_3^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=0.$$

Similarly, by changing v into $-v$, and seeing that

$$M_2\left(u+\frac{\pi}{2},\ v-\frac{\pi}{2}\right)=-M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right),$$

we have

$$M_3^2(u,v)-M_2^2(u,v)+M_2^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)-M_3^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=0.$$

The corresponding M-equations are, in fact,

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_2(u,v)=\vartheta_3(u)\,\vartheta_3(v)\,\Lambda_2(u-v)-\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_3(u,v)=\vartheta_2(u)\,\vartheta_2(v)\,\Lambda_3(u-v)+\vartheta_1'(0)\,\vartheta(u)\,\vartheta(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$= -\vartheta_1(u)\,\vartheta(v)\,\Lambda_3(u-v)+\vartheta_1'(0)\,\vartheta_2(u)\,\vartheta_2(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$= -\vartheta_1(u)\,\vartheta_1(v)\,\Lambda_2(u-v)-\vartheta_1'(0)\,\vartheta_3(u)\,\vartheta_3(v),$$

whence also

$$\Lambda_3(u,v)\,M_3(u,v)-\Lambda_2(u,v)\,M_2(u,v)$$

$$+\Lambda_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$-\Lambda_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)M_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=2\vartheta_1'(0)^2.$$

7. By putting $v = 0$ in various relations obtained above, we get the following relations between the simple series.

From § 4, (1),

$$\vartheta_3(u)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\dots\right\}$$

$$-\vartheta_2(u)\left\{\frac{4q^{\frac{3}{4}}}{1+q}\cos u+\frac{4q^{\frac{15}{4}}}{1+q^3}\cos 3u+\dots\right\}=\vartheta(0)^2\,\vartheta(u).$$

By changing q into $-q$,

$$\vartheta(u)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\dots\right\}$$

$$+\vartheta_2(u)\left\{\frac{4q^{\frac{3}{4}}}{1-q}\cos u-\frac{4q^{\frac{15}{4}}}{1-q^3}\cos 3u+\dots\right\}=\vartheta_3(0)^2\,\vartheta_3(u).$$

By eliminating $1+\frac{4q^2}{1+q^2}\cos 2u+\dots$ from these two equations,

$$\vartheta(u)\left\{\frac{4q^{\frac{3}{4}}}{1+q}\cos u+\frac{4q^{\frac{15}{4}}}{1+q^3}\cos 3u+\dots\right\}$$

$$+\vartheta_3(u)\left\{\frac{4q^{\frac{3}{4}}}{1-q}\cos u-\frac{4q^{\frac{15}{4}}}{1-q^3}\cos 3u+\dots\right\}=\vartheta_2(0)^2\,\vartheta_2(u).$$

Changing v into $-v$ in § 6, (6), and subtracting, we have, moreover,

$$\frac{2}{\vartheta_2(u)}\left\{\frac{4q(1-q^2)\sin v}{1+2q^2\cos 2v+q^4}\,q\sin 2u+\frac{4q^2(1-q^4)}{1+2q^4\cos 2v+q^8}\,q^4\sin 4u+\dots\right\}$$

$$=\frac{\Lambda_3(u+v)}{\vartheta_2(u+v)\,\vartheta_2(0)}-\frac{\Lambda_3(u-v)}{\vartheta_2(u-v)\,\vartheta_2(0)}$$

$$-\frac{\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)}{\vartheta_2(0)\,\vartheta_2(u)\,\vartheta_2(v)}\left\{\frac{1}{\vartheta_2(u+v)}+\frac{1}{\vartheta_2(u-v)}\right\}.$$

When $v=\frac{\pi}{2}$, this becomes

$$\vartheta_1(u)\left\{\frac{4q^2}{1-q^2}\sin 2u+\frac{4q^6}{1-q^4}\sin 4u+\dots\right\}$$

$$+\vartheta_2(u)\left\{1-\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u-\dots\right\}=\vartheta_1'(u).$$

When $v=0$, we get also

$$\vartheta_2(0)\left\{\frac{4q^3(1-q^2)}{(1+q^2)^2}\sin 2u+\frac{4q^6(1-q^4)}{(1+q^4)^2}\sin 4u+\dots\right\}$$

$$=\vartheta_2(u)\frac{d}{du}\frac{\Lambda_3(u)}{\vartheta_2(u)}-\frac{\vartheta_1'(0)^2\,\vartheta_1(u)}{\vartheta_3(0)\,\vartheta_2(u)}.$$

Similarly,

$$\vartheta_3(0)\left\{\frac{4q^{\frac{3}{2}}(1-q)}{(1+q)^2}\sin u+\frac{4q^{\frac{9}{2}}(1-q^3)}{(1+q^3)^2}\sin 3u+\dots\right\}$$

$$=\vartheta_3(u)\frac{d}{du}\frac{\Lambda_2(u)}{\vartheta_2(u)}+\frac{\vartheta_1'(0)^2\,\vartheta_1(u)}{\vartheta_3(0)\,\vartheta_3(u)}.$$

8. The series $\sum\limits_{m=-\infty}^{\infty}\frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}}\,q^{m^2}e^{2mui}$ is easily seen to satisfy a partial differential equation.

For the operation of $q\frac{d}{dq}$ on the general terms is equal to that of

$$-\frac{1}{2}\frac{d^2}{du\,dv}-\frac{1}{4}\frac{d^2}{du^2}.$$

Thus $$q\frac{d}{dq}\frac{M_3(u,v)}{\vartheta_2(v)}=-\frac{1}{2}\frac{d^2}{du\,dv}\frac{M_3(u,v)}{\vartheta_2(v)}-\frac{1}{4}\frac{d^2}{du^2}\frac{M_3(u,v)}{\vartheta_2(v)}.$$

If we write X_3 for $M_3(u,v)/\vartheta_2(u)\,\vartheta_2(v)$, this equation becomes

$$q\frac{dX_3}{dq}=-\frac{1}{2}\frac{d^2X_3}{du\,dv}-\frac{1}{4}\frac{d^2X_3}{du^2}-\frac{1}{2}\frac{\vartheta_2'(u)}{\vartheta_2(u)}\left(\frac{dX_3}{du}+\frac{dX_3}{dv}\right)\ \dots(1).$$

But, by § 6, (6),

$$X_3=\frac{\Lambda_3(u-v)}{\vartheta_2(u-v)\,\vartheta_2(0)}-\frac{\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)}{\vartheta_2(u-v)\,\vartheta_3(0)\,\vartheta_2(u)\,\vartheta_2(v)}.$$

Hence $\left(\frac{d}{du}+\frac{d}{dv}\right)X_3$ is purely elliptic, since the differential operator annihilates all functions of $u-v$, and by ordinary ϑ-function formulæ, including

$$\vartheta(u)\,\vartheta_3(u)\,\vartheta_1(v)\,\vartheta_2(v)+\vartheta(v)\,\vartheta_3(v)\,\vartheta_1(u)\,\vartheta_2(u)$$

$$=\vartheta_3(0)\,\vartheta(0)\,\vartheta_1(u+v)\,\vartheta_2(u-v),$$

we have $$\left(\frac{d}{du}+\frac{d}{dv}\right)X_3=\frac{\{\vartheta_1'(0)\}^2\,\vartheta_1(u+v)}{\vartheta_2^2(u)\,\vartheta_2^2(v)}\ \dots\dots\dots\dots(2).$$

This relation will help to reduce the unsymmetric equation (1) to the symmetrical form

$$4q\frac{dX_3}{dq}+\frac{d^2X_3}{du\,dv}=-\frac{\{\vartheta_1'(0)\}^2\,\vartheta_1'(u+v)}{\vartheta_2^2(u)\,\vartheta_2^2(v)} \quad\ldots\ldots\ldots\ldots\ldots(3).$$

Moreover, eliminating $\frac{d}{dv}$ by (2), we get

$$4q\frac{dX_3}{dq}-\frac{d^2X_3}{du^2}=-2\frac{\{\vartheta_1'(0)\}^2}{\vartheta_2(u)\,\vartheta_2(v)}\frac{d}{du}\frac{\vartheta_1(u+v)}{\vartheta_2(u)}.$$

When $v=0$, this becomes

$$\left(4q\frac{d}{dq}-\frac{d^2}{du^2}\right)\frac{\Lambda_2(u)}{\vartheta_2(0)\,\vartheta_2(u)}=-2\{\vartheta_1'(0)\}^2\frac{\vartheta(u)\,\vartheta_3(u)}{\vartheta_2^3(u)}.$$

In a similar manner, we obtain

$$\left(4q\frac{d}{dq}-\frac{d^2}{du^2}\right)\frac{\Lambda_3(u)}{\vartheta_3(0)\,\vartheta_3(u)}=-2\{\vartheta_1'(0)\}^2\frac{\vartheta(u)\,\vartheta_2(u)}{\vartheta_3^3(u)}.$$

The Electrical Distribution on a Conductor bounded by Two Spherical Surfaces cutting at any Angle. By H. M. Macdonald.

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In Maxwell's *Electricity and Magnetism*, Vol. I., §§ 165, 166, the problem of the distribution of electricity induced by an electrified point placed between them on two planes cutting at an angle which is a submultiple of two right angles, and the inverse problem of the conductor formed by two spherical surfaces cutting at such an angle (the angle referring to the dielectric), are solved by the method of point images. This method is inapplicable when the (dielectric) angle is not a submultiple of two right angles, as has been shown by W. D. Niven, *Proc. Lond. Math. Soc.*, Vol. XXI., p. 27. The only other case which has been hitherto solved is, as far as I know, that of the spherical bowl (Lord Kelvin, *Papers on Electrostatics and Magnetism*, p. 178). In the paper by W. D. Niven mentioned above