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Notes on a General Method of Solving Partial Differential Equations of the First Order with several Dependent Variables. By H. W. Lloyd Tanner, M.A.

[*Read March 11th*, 1880.]

1. The following notes relate to a method of solving equations with several dependent variables, which is a generalization of the process employed in the case where only one dependent variable occurs. The extension to the more general equations herein considered is easy, but it is perhaps worth stating since it leads to this curious conclusion; that the solution of such equations depends upon the solution of an auxiliary system similar to that which is required in the integration of equations with one dependent variable, but involving differential coefficients of the second or higher orders.

When there is one dependent variable, z, we determine p_i $\left(\text{or } \dfrac{dz}{dx_i}\right)$ by means of equations

$$F_1 = 0, \quad F_2 = 0, \; \dots \; F_n = 0,$$

each F being a function of the p's and the variables z, x. The form of the functions, F, is derived from certain equations obtained from the identities

$$\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}=0.$$

It so happens that the equations for the determination of F, may be expressed in a form, the theory of which is well known; but this form does not appear to be capable of extension to more general cases. When all the F's are determined, the p's are given by algebraical processes, and z is determined from an integrable equation

$$dz-\Sigma p_i\,dx_i=0.$$

This theory has been discussed at some length, "Proceedings of London Mathematical Society," Vol. ix., pp. 77, &c., and the F-equations there given are of the same general type as those hereinafter obtained.

2. In the case in which we have m dependent variables

$$z_1,\ z_2\ \dots\ z_m,$$

and n independent variables $x_1, x_2 \dots x_n$,
the most obvious analogue to the process just described, would be to determine

$$\frac{dz_i}{dx_j}$$

by means of mn equations $F=0$,
and to determine the forms of F by relations derived from the identities

$$\frac{d}{dx_k}\cdot\frac{dz_i}{dx_j}=\frac{d}{dx_j}\cdot\frac{dz_i}{dx_k}.$$

Each of these identities gives a relation between the F's; so that these mn functions are the solutions of $\frac{1}{2}mn(n-1)$ differential equations, which are of a very simple form. Represent, for convenience, $\frac{dz_i}{dx_j}$ by ij, and consider the Jacobian

$$\frac{d(F_1\dots F_{mn})}{d(11\dots ij\dots mn)},$$

in the numerator of which are all the F's, in the denominator all the differential coefficients ij. If from this we form another Jacobian by substituting x_k for ij, a third by substituting x_j for ik, and express that the two Jacobians thus derived are identically equal, we obtain one of the equations for the determination of the F's, viz., the one derived from the identity

$$\frac{d}{dx_k}\cdot\frac{dz_i}{dx_j}=\frac{d}{dx_j}\cdot\frac{dz_i}{dx_k}.$$

If we suppose that the F's may involve $z_1 \dots z_m$, we must replace $\frac{dF}{dx_i}$, say, by the sum

$$\frac{dF}{dx_i}+\Sigma_k(ki)\frac{dF}{dz_k},$$

and the effect upon the Jacobians is similar, viz., each is replaced by a sum of Jacobians. Judging from analogy, we infer that this increase in the number of terms does not increase the difficulty of solving the system for F.

If the equations for F could be solved, we should be able at once to solve any proposed system of equations in $\frac{dz_i}{dx_j}$. Amongst the F's we should include the proposed system, and then determine the other F's. From all of these we may, by algebraical transformations, obtain $\frac{dz_i}{dx_j}$ as explicit functions of $x_1, x_2 \ldots x_n, z_1 \ldots z_m$, and arbitrary parameters. These values make the system

$$dz_i - \frac{dz_i}{dx_1}\, dx_1 - \frac{dz_i}{dx_2}\, dx_2 - \ldots - \frac{dz_i}{dx_n}\, dx_n = 0,$$
$$i = 1, 2 \ldots m,$$

an integrable system, from which z_i is to be determined by well-known processes.

3. Another course is open to us, leading to an auxiliary system of the same general character, but with a different number of F's, and consisting of a different number of equations. In some cases one process will be more convenient, in some cases the other; and, besides the advantage of having a choice, there is the further result that we get two distinct auxiliary systems, which must be equivalent, and thus may get hints as to the possible transformations of such systems.

For the application of the second method, we suppose that the equations to be solved are presented in what I have elsewhere* called the homogeneous form. The differential coefficients only occur in Jacobians of the n^{th} order; viz., in the determinants of the matrix

$$\left\| \begin{matrix} \frac{dz_1}{dx_1} & \cdots\cdots & \frac{dz_1}{dx_n} \\ \cdots & \cdots & \cdots \\ \frac{dz_m}{dx_1} & \cdots\cdots & \frac{dz_m}{dx_n} \end{matrix} \right\| \qquad \text{..................................(1).}$$

Each equation is algebraically homogeneous with respect to the Jacobians, and besides them involves only the independent variables $x_1 \ldots x_n$. Should the given system not be homogeneous, it can be reduced to that form by transformation, the new system having m dependent, and, generally, $m+n$ independent variables. It will be convenient to keep m, n as the numbers of z's and x's in the transformed equations, and it appears that n must be greater than m.

* "Proceedings of London Mathematical Society," Vol. ix., p. 41, *et sqq.*

The equations being in the homogeneous form, we may, instead of seeking the values of $$\frac{dz_i}{dx_j},$$ investigate the values of the Jacobians (1). To do this, it is only necessary to write down the relations which are identically satisfied by the Jacobians, and then, supposing these to be given by μ equations,

$$F_1 = 0,\ F_2 = 0 \ldots F_\mu = 0,$$

to transform these identities into their expressions in terms of F.

Since all the determinants of (1) involve $z_1 \ldots z_m$, any one of them will be distinguished by the subscripts of the x's it involves; we shall so indicate the Jacobians in the sequel, unless otherwise stated.

4. The first set of relations are purely algebraical, and arise from the fact that the Jacobians are determinants of a matrix. They may be thus expressed. Take any set of m suffixes, say, $1, 2 \ldots m$; and let $1', 2' \ldots m'$ represent groups, each of $m-1$ suffixes, and so arranged that each of $$11',\ 22' \ldots mm'$$ is composed of $1, 2 \ldots m$, and is reducible to this order by an even number of transpositions.* Let p be any subscript, and p' any group of $m-1$ subscripts, selected from $1, 2 \ldots m, m+1 \ldots n$. Then we have the identity $$11'.pp' = 1p'.p1' + 2p'.p2' + \ldots + mp'.pm' \quad \ldots\ldots\ldots\ldots (2),$$ where pp' represents a Jacobian involving x's with the m subscripts pp', and similarly for the other symbols.

The number of independent equations (2) may be thus determined:—

The Jacobians of the system (1), are in number $\dfrac{n\,!}{m\,!\ n-m\,!}$, $= \mu$, say. We shall show that these μ Jacobians are, in virtue of (2), equivalent to a system of $m(n-m)+1$ only; the reduced system consisting of an arbitrarily selected Jacobian, say $(12 \ldots m)$, and the $m(n-m)$ formed therefrom by replacing one of the m symbols $1, 2 \ldots m$ by one of the $n-m$ symbols $m+1 \ldots n$. Hence the equations (2) are equivalent to

$$\mu - m(n-m) - 1 \quad \ldots\ldots\ldots\ldots (3)$$

independent relations.

It remains to show that the reduced system above described, is really equivalent to the complete system (1); that, the reduced system being known, all the determinants of the complete system are also

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* When $m = 2$, this arrangement is not possible, since $1'$, &c., are single subscripts. In this case the equation (2) is replaced by

$$12.pp' = 1p'.p2 - 2p'.p1$$

(or an equivalent), p, p' being any two subscripts.

The equation (2), and the special case just noted, are included under one form in a paper by Prof. Cayley, *Quart. Journ.*, xv., pp. 55—57.

known in virtue of (2). Suppose, then, we know the Jacobian (12...m), and those derived from it by a single replacement (say, its *primary derivatives*). In (2), let p be any one of the $n-m$ symbols $m+1 \ldots n$, and suppose p' to be made up of one of these $n-m$ symbols, together with $m-2$ of the range $1, 2 \ldots m$. Then pp' is a *secondary derivative* of (12 ... m), and, by properly choosing p, p', it may be made any one we please of the secondary derivatives. All the remaining symbols in (2) are primary derivatives (since each involves only one subscript outside the range $1, 2 \ldots m$), and are therefore known. Hence we can determine all the secondaries, when all the primary derivatives are known. Similarly, the tertiary derivatives can be found from the secondaries, and so on; so that, as stated above, all the determinants of the matrix (1) are known, if only we know the value of any one of them, and all its primary derivatives.

The relations (2) may be used in two ways. We may seek the value of each Jacobian of the complete set, and regard the equations (2) as so many data towards that end. Or we may use them at once to eliminate from a system of equations proposed for solution all the Jacobians, except an arbitrarily chosen one, and its primary derivatives. The latter appears to be the more convenient process, since we have not only a smaller number of Jacobians to determine, but the difficulties of notation are materially diminished. From the results of Art. 5, it will, however, be seen that the balance of convenience is in favour of retaining the complete set of Jacobians.

5. Besides the algebraical identities above discussed, there is another set, involving in their expression the differential coefficients of the Jacobians. They may be thus described:—Take any determinant of the matrix

$$\left\| \begin{matrix} \frac{d}{dx_1}, & \frac{d}{dx_2} & \cdots\cdots & \frac{d}{dx_n} \\ \frac{dz_1}{dx_1}, & \frac{dz_1}{dx_2} & \cdots\cdots & \frac{dz_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dz_m}{dx_1}, & \frac{dz_m}{dx_2} & \cdots\cdots & \frac{dz_m}{dx_n} \end{matrix} \right\| \qquad \cdots\cdots\cdots\cdots (4),$$

and develope it by the ordinary rule as a sum of products of the elements of the top line with their conjugate minors. For instance, supposing $m = 2$, we should have

$$\frac{d}{dx_i}(jk) - \frac{d}{dx_j}(ik) + \frac{d}{dx_k}(ij) = 0.$$

It is known* that these equations, combined with the algebraical equa-

* See "Proceedings of London Mathematical Society," Vol. x., pp. 55, *et sqq.*

tions of Art. 4, are necessary and sufficient to ensure that, for instance,

$$(jk),\ (ik),\ (ij) = \frac{d\,(z_1,\, z_2)}{d\,(x_i,\, x_j,\, x_k)};$$

in other words, that there are functions z_1, z_2 such that (jk) is the Jacobian of them, with respect to x_j, x_k.

In the general case, we obtain

$$\frac{n\,!}{\overline{m+1}\,!\ \overline{n-m-1}\,!}$$

equations; viz., one equation is obtained for every group of $m+1$ columns, $i, j, k \ldots l$, that can be selected from the n columns of (4); the equation being

$$\frac{d}{dx_i}(jk \ldots l) - \frac{d}{dx_j}(ik \ldots l) + \frac{d}{dx_k}(ij \ldots l) - \&c. = 0 \quad \ldots\ldots(5).$$

These equations are mutually independent.

We can now choose one of the two alternatives suggested at the end of Art. 4. Suppose $n = 4$, $m = 2$. Then we have *one* algebraical relation

$$12\,.\,34 - 13\,.\,24 + 14\,.\,23 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a);$$

and *four* differential relations

$$\left.\begin{array}{l}
\frac{d}{dx_2}\,.\,34 - \frac{d}{dx_3}\,.\,24 + \frac{d}{dx_4}\,.\,23 = 0 \\
\frac{d}{dx_1}\,.\,34 - \frac{d}{dx_3}\,.\,14 + \frac{d}{dx_4}\,.\,13 = 0 \\
\frac{d}{dx_1}\,.\,24 - \frac{d}{dx_2}\,.\,14 + \frac{d}{dx_4}\,.\,12 = 0 \\
\frac{d}{dx_1}\,.\,23 - \frac{d}{dx_2}\,.\,13 + \frac{d}{dx_3}\,.\,12 = 0
\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(b).$$

Say, we select 13 as principal Jacobian; then all the Jacobians except 24 are included in the *reduced system*. To eliminate this, we must replace it, in (b1), (b3), by its value deduced from (a). This being done, (b1), after simplification, becomes an equation linear with respect to the differential coefficients of 34, 13, &c.; but the co-factors of these differential coefficients are binary products of 34, 13, &c. Also the four equations cannot be mutually dependent, for each contains a differential coefficient not contained in any of the others; viz., $\frac{d}{dx_3}\,13$, $\frac{d}{dx_4}\,13$, $\frac{d}{dx_1}\,13$, and $\frac{d}{dx_2}\,13$ occur, each in only one equation. Hence the effect of removing 24, is to increase the complexity of the differential equations, without diminishing their number. Besides this, it is not obvious what is the general form of the differential equations when only the Jacobians of a reduced system are explicitly involved; and the advantages of working with a smaller set of Jacobians seem to be more than counterbalanced by the disadvantages of losing such a simple general form as (5). Hence we elect to retain the full set of Jacobians, and to look upon (2) as given relations between them. The question is, it may be remarked, purely one of convenience, as the process of solution is virtually the same whichever course we adopt.

6. If the μ Jacobians $(12 \dots m)$, $(ij \dots l)$, &c., instead of being given explicitly in terms of $x_1 \dots x_n$, be given by means of μ equations

$$F_1 = 0, \ F_2 = 0 \ \dots \ F_\mu = 0,$$

the functions F must satisfy certain conditions in order that the resulting values of $(12 \dots m)$, &c., may satisfy the relations (5). It is easy to write these conditions down by making use of the following notation. The Jacobian of $F_1 \dots F_\mu$, with respect to the μ quantities $(12 \dots m)$, $(ij \dots l)$, &c., will be denoted by $\left| \ \right|$; the Jacobian formed from this by writing x_k instead of $(ij \dots l)$, will be written $\left| \begin{matrix} ij \dots l \\ k \end{matrix} \right|$; and similarly for others. Making use of the theorem

$$\left| \ \right| \times \frac{d}{dx_k}(ij \dots l) = \left| \begin{matrix} ij \dots l \\ k \end{matrix} \right|,$$

we obtain at once the following equivalent to (5),

$$\left| \begin{matrix} jk \dots l \\ i \end{matrix} \right| - \left| \begin{matrix} ik \dots l \\ j \end{matrix} \right| + \left| \begin{matrix} ij \dots l \\ k \end{matrix} \right| - \&c. = 0 \quad \dots\dots\dots\dots\dots (6),$$

each term being, as explained above, a Jacobian of $F_1 \dots F_\mu$. The two systems (5), (6) are exactly equivalent; either is a consequence of the other.

7. The application of these results to the solution of a system of equations in the homogeneous form presents no difficulty. Suppose the system to consist of r equations

$$F_1 = 0, \ F_2 = 0, \ \dots \ F_r = 0.$$

We must supplement these by others

$$F_{r+1} = 0, \ \dots \ F_\mu = 0,$$

of which some are already known, being the equations (2). There are, therefore, only $m(n-m)+1-r$ functions, F, to be determined. To obtain these, we have only the datum that they must be such as to give values of $(12 \dots m)$, $(ij \dots l)$, &c., satisfying (5); in other words, we are only given that these unknown F's must satisfy the equations (6). These equations, as they stand, involve Jacobians of the μ^{th} order; but it is easy to make each of them lineal in Jacobians of the μ'^{th} order, μ' being the number of unknown F's; this is done by developing each Jacobian in (6), as a sum of products of conjugate minors, one of the minors in each term involving the μ' unknown F's, while the other contains all the known F's, and is therefore itself known.

8. Supposing the system (6) to be solved, then we can, from the μ equations
$$F = 0,$$
find $(12 \ldots m)$, $(ij \ldots l)$, &c., by the ordinary processes of algebra. It is, however, unnecessary to determine the whole system; for, as will appear immediately, the completion of the solution only requires a knowledge of what we have called a reduced system, consisting of one Jacobian and its primary derivatives. Say, we select $(12 \ldots m)$ as the principal Jacobian, and represent it by $\left|\,\right|$; any prime derivative formed by replacing herein x_k by x_j, may be indicated by $\left|\begin{smallmatrix} k \\ j \end{smallmatrix}\right|$. Then, I say that $z_1, z_2 \ldots z_m$ are the m solutions of the system

$$\left|\,\right| \frac{dz}{dx_j} - \left|\begin{matrix} 1 \\ j \end{matrix}\right| \frac{dz}{dx_1} - \left|\begin{matrix} 2 \\ j \end{matrix}\right| \frac{dz}{dx_2} - \ldots - \left|\begin{matrix} m \\ j \end{matrix}\right| \frac{dz}{dx_m} = 0 \quad \ldots\ldots\ldots(7),$$
$$j = m+1,\ m+2, \ldots n.$$

The same thing is stated in other words, when we assert that $z_1, z_2 \ldots z_m$ are m integrals of the system

$$\left|\,\right| dx_i + \left|\begin{matrix} i \\ m+1 \end{matrix}\right| dx_{m+1} + \left|\begin{matrix} i \\ m+2 \end{matrix}\right| dx_{m+2} + \ldots + \left|\begin{matrix} i \\ n \end{matrix}\right| dx_n = 0 \ldots(8),$$
$$i = 1, 2 \ldots m.$$

The truth of (7) is immediately seen by observing that it is equivalent to the system

$$\left\| \begin{matrix} \frac{dz}{dx_1}, & \frac{dz}{dx_2} & \ldots\ldots & \frac{dz}{dx_n} \\ \frac{dz_1}{dx_1}, & \frac{dz_1}{dx_2} & \ldots\ldots & \frac{dz_1}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dz_m}{dx_1}, & \frac{dz_m}{dx_2} & \ldots\ldots & \frac{dz_m}{dx_n} \end{matrix} \right\| = 0,$$

which is evidently satisfied when z is any one of the functions $z_1, z_2 \ldots z_m$, or any function of them; and is not satisfied for any other form of z.

In order that (7) or (8) may give m values for z, certain conditions must be satisfied by the coefficients $\left|\,\right|$, $\left|\begin{smallmatrix} i \\ j \end{smallmatrix}\right|$; and, supposing our reasoning to have been correct, these conditions must be satisfied identically in virtue of (2) and (5). It is easy to see that this is so:

in fact, the working has been given in a former paper of mine ("Proceedings of London Mathematical Society," Vol. x., pp. 66, 67): and thus we have an *a posteriori* verification of our results.

9. In the process of Art. 2, and in that developed in the succeeding articles of this paper, the solution of any system of equations involving

$$\frac{dz_i}{dx_j}$$

(say, for distinction, the *original* system) has been reduced to dependence upon the solution of a system of linear equations (say, the *auxiliary* system), concerning which some remarks will be made. The auxiliary system, in its most general form, may be described as consisting of ρ equations, each of which is linear in the Jacobians of

the matrix $$\frac{d(F_1, F_2 \dots F_\mu)}{d(\xi_1, \xi_2 \dots \xi_\nu)}, \ (\nu > \mu).$$

Such equations are met with in the following cases:—

(A.) When the original system involves the first partial differential coefficients of one dependent variable with respect to n independent variables. In this case,

$$\mu = n, \ \nu = 2n+1, \ \rho = \tfrac{1}{2}n(n-1).$$

(B.) When the original system is of the kind discussed in the present paper. Adopting the method of Art. 2, we have

$$\mu = mn, \ \nu = mn+m+n, \ \rho = \tfrac{1}{2}mn(n-1).$$

(C.) When the second mode of solution is adopted for the systems of this paper, we are led to an auxiliary system, in which

$$\mu = \frac{n!}{m!\,n-m!}, \text{ or } m(n-m)+1, \ \nu = \frac{n!}{m!\,n-m!}+n, \ \rho = \frac{n!}{m+1!\,n-m-1!}.$$

When the larger value of μ is adopted, all the Jacobians in the auxiliary system belong to a reduced system; this is not the case when the smaller number is selected. The case (A) is the particular case of (B), (C) corresponding to $m=1$; and the values of μ, &c., found by putting $m=1$, agree with those given under (A), with one exception. In (C) it is necessary that the variables $z_1 \dots z_m$ should not appear in the auxiliary system. If, then, we reduce to the case of $m=1$, we find $\nu = 2n$, the variable z being excluded.

(D.) When the original system consists of partial differential equations with one dependent variable and n independent variables, and the order of the highest differential coefficient is r, we get a similar auxiliary system.* The number of F's is equal to the number of

* "Proceedings of London Mathematical Society," Vol. ix., p. 89.

differential coefficients of order r; viz.,

$$\mu = \frac{\overline{n+r-1}!}{\overline{n-1}!\, r!}.$$

The variables ξ are the differential coefficients of the r^{th} and all lower orders, together with z and the n independent variables. Hence

$$\nu = \frac{\overline{n+r}!}{n!\, r!} + n.$$

The number of equations in the auxiliary system is most easily determined thus: If we differentiate each differential coefficient of the r^{th} order with respect to each of the n independent variables, we get

$$n \cdot \frac{\overline{n+r-1}!}{\overline{n-1}!\, r!}$$

different expressions. But the number of differential coefficients of of order $r+1$ is only

$$\frac{\overline{n+r}!}{\overline{n-1}!\, \overline{r+1}!}.$$

Hence there must be $n \cdot \dfrac{\overline{n+r-1}!}{\overline{n-1}!\, r!} - \dfrac{\overline{n+r}!}{\overline{n-1}!\, \overline{r+1}!}$

(differential) relations between the differential coefficients of order r. Each of these gives rise to one equation in the auxiliary system. Hence,

for this case, $$\rho = \frac{\overline{n+r-1}!\, r}{\overline{n-2}!\, \overline{r+1}!}.$$

(E.) It is clear that the most general system of differential equations, viz., a system involving differential coefficients (of order r) of m dependent variables with respect to n independent variables, cannot be solved without an auxiliary system of type now considered. Indeed, the process of Art. 2 may be applied at once to such systems. Probably also a series of processes analogous to the second method of this paper would be applicable. It is not to be expected that the solution of the auxiliary system will suffice for the complete solution of the original system; indeed, in the case of $m=1$, we know that other equations must be solved as well as the auxiliary system now under consideration; but it is at least certain that the solution of this auxiliary system is an indispensable preliminary.

It is to be noticed that in every case mentioned the auxiliary system involves Jacobians of a reduced system only: an exception occurring in case (C) only, if we take the smaller value of μ.

10. The only case in which the auxiliary system has been solved is that marked (A) above. But the solution depends upon a certain transformation which is not possible unless certain relations obtain be-

tween μ, ρ. The auxiliary system is transformed* into another system, in which each equation connects two only of F's. Such an equation may be written $$(F_i, F_j) = 0 \quad\text{.............................}(9).$$
On account of the symmetry of the auxiliary system with respect to F it is evident that the transformed system must involve all the equations that can be formed by giving i, j in (9) all values from λ to μ. There is nothing to prevent $$(F_i, F_i) = 0$$
being *identically* satisfied; but if any equation (9), in which i, j are different, should be satisfied independently of the form of F, then the same will be true of the whole system (9), and the transformation is illusory. Thus the transformed system consists of $\frac{1}{2}\mu(\mu-1)$ different equations; and accordingly ρ must be at least equal to $\frac{1}{2}\mu(\mu-1)$. It appears that another necessary condition for the transformation is that all the Jacobians involved must be the primary derivatives of a Jacobian which does not vanish. A reference to Art. 9 will show that these conditions are fulfilled only in the case (A).

Possibly the following process, which is at any rate general, may serve to give a solution in particular cases. It is of little use in the general case, because our process is stopped almost immediately by difficulties; but it may be that these disappear in certain particular instances; moreover the mere statement of the problem in a different form may not be without value.

Consider the equations as a system for the determination of one of the F's, say F_μ. Then each equation is a linear homogeneous equation of the first order, the coefficients of $\frac{dF_\mu}{dx_i}$ being a function (linear) of the Jacobians of $F_1 \dots F_{\mu-1}$. In order that this system may have a common solution, certain solutions must be satisfied, the form of these being well known. These conditions serve as a system of equations for the determination of $F_1 \dots F_{\mu-1}$, which, however, involve differential coefficients of the second order. The conditions that such a system, regarded as equations for $F_{\mu-1}$ say, should have a common solution, are not known; but they evidently will form a system connecting $F_1 \dots F_{\mu-2}$. Proceeding in this way, we arrive ultimately at a system of equations involving only one F; or, it may be, at a system of identities each involving μ' F's, viz. $F_1 \dots F_{\mu'}$. (The former case corresponds to $\mu' = 0$.) $F_1 \dots F_{\mu'}$ are then perfectly arbitrary; and, leaving them thus arbitrary, we pass back through the successive systems above indicated, and from their solution determine $F_{\mu'+1} \dots F_\mu$ in turn. There is the double

* This transformation and its converse are actually completed in Arts. 3, 4, 5 of a paper "On a General Method of solving Partial Differential Equations," *Proc. L. M. S.*, Vol. ix., pp. 76 *et sqq.*

difficulty of forming these systems and then solving them; but this does affect one result: viz., that the auxiliary system may be reduced to systems involving $\mu-1, \mu-2, \ldots \mu'$ of the F's. These systems are not generally of the first order; though, as in case (A) of Art. 9, one of them may be so; and it seems not unlikely, bearing this case in mind, that a considerable simplification in the form of the subsidiary equations may be found when they can be written down.

The lack of symmetry in the above process is another objection to it; but it is not unprecedented.

Note on the Integral Solution of $x^2-2Py^2=-z^2$ *or* $\pm 2z^2$ *in certain Cases.* By SAMUEL ROBERTS, F.R.S.

[*Read March* 11*th*, 1880.]

1. I assume that P is a non-square number, the prime factors of which are of the form $8m+1$. The number is therefore uneven, and susceptible of the three forms $\alpha^2+16\beta^2$, $\alpha^2+8\beta^2$, and $\alpha^2-8\beta^2$.

The number of such representations is finite in the first two cases, and infinite in the last case.

The forms $\alpha^2+16\beta^2$ are comprised in

$$(8k\pm s)^2+16l^2,$$

where k, l remain unchanged in species throughout all the system for a given value of P, and s remains constant, having one of the values 1, 3.

In fact, if a^2+16b^2, $c^2+16d^2 \ldots m^2+16n^2$ are representations of the prime factors of P (and these may be identical in groups, so that one or more prime factors enter in powers higher than the first), then, giving positive values to the letters, we may write

$$(a\pm i4b)(c\pm i4d)\ldots(m\pm i4n) = \pm A \pm i4B \ldots\ldots\ldots\ldots (a)$$

with $P=A^2+16B^2$. If, therefore, $ac\ldots m \equiv s$ (mod. 8), we shall have, independently of the ambiguity of the signs,

$$A = 8k\pm s, \; B = l,$$

where k, l remain unchanged in species because the species of $\pm\mu\pm\nu\pm\pi\pm$ &c. is the same as that of $\mu+\nu+\pi+$&c. Moreover, all the representations of P as the sum of two squares can be obtained from (a). The sign affecting s in two representations of the same number may be different.