and $$\wp(iu\sqrt{3}) = \tfrac{1}{3}\sqrt[3]{28} = 1{\cdot}012,$$

so that, to the nearest integral value of r,

$$iu\sqrt{3} = \tfrac{171}{180}\omega_2\,(\pm 2\omega_2), \quad u = \tfrac{57}{180}\omega_2'\,(\pm \tfrac{2}{3}\omega_2');$$

$$\wp u = -2{\cdot}226, \quad +{\cdot}996, \quad -1{\cdot}803;$$

and $$x = 1{\cdot}573, \quad {\cdot}198, \quad 3{\cdot}16.$$

On the Cremonian Congruences which are contained in a Linear Complex. By Dr. T. Archer Hirst, F.R.S.

[*Read May 13th*, 1886.]

1. In his well-known memoir,* published in the *Monats Bericht* of the Academy of Berlin (17th January, 1878), Kummer drew attention to the existence of two different, and equally general, congruences of the third order and third class. One of these is contained in a linear complex; the other, which for distinction might be termed the skew cubic congruence, is such that the three rays thereof, proceeding from an arbitrary point in space, are not, in general, coplanar. The properties of the latter congruence were fully developed by Kummer; whilst those of the former were only very briefly alluded to by him.

2. A year ago, in a paper communicated to the London Mathematical Society, I had occasion to study a special case of the above-mentioned skew cubic congruence.† It was of the Cremonian

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* *Über diejenigen Flächen, welche mit ihren reciprok polaren Flächen von derselben Ordnung sind und die gleichen Singularitäten besitzen.*

† *On Congruences of the Third Order and Class,* "Proceedings of the London Mathematical Society," Vol. xvi., pp. 232—38, 1885.

I may here mention that, in 1882, Dr. Roccella published, at Piazza Armerina, in Sicily, an interesting thesis entitled, *Sugli enti geometrici dello spazio di retti generati dalle intersezione dè complessi corrispondenti in due o più fasci proiettivi di complessi lineari*, in which, amongst other things, he speaks of a congruence of the third order and class, definable as *the locus of a right line constantly incident with three corresponding rays of three given projective pencils, arbitrarily situated in space.* This congruence, as I have recently shown, in a communication to the *Circolo Matematico di Palermo* ("Rendiconti," t. i., *seduta del* 21 *febrajo* 1886), is itself a special case of the one studied by me, and referred to in the text.

I am also informed by Prof. Sturm, of Münster, that he has been led, still more recently, and quite independently, to a somewhat similar, purely descriptive method of generating the congruence described in my paper of 1885. In place of one of the three projective pencils employed by Roccella, he simply substitutes a quadric regulus, one of whose generators coincides with its corresponding ray in one of the two remaining projective pencils.

type; that is to say, a congruence whose rays determine a Cremonian, or birational, correspondence between the points of two planes. Its investigation naturally raised the question as to the existence and generation of a Cremonian cubic congruence of Kummer's first, or non-skew type; and this enquiry, just as naturally, led to the wider one which forms the subject of the present paper.

In it I propose to consider, *first* under what conditions a congruence, contained in a linear complex, will be Cremonian (Arts. 3—7); *secondly*, how such congruences may be generated, and what varieties they present (Arts. 8—23); and, *finally*, what special properties those of the third order and class possess (Arts. 24—26).

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3. The order and class of every congruence contained in a linear complex are necessarily equal to one another; and their common value, say n (>2), also indicates, if the congruence be Cremonian, the degree of the birational correspondence whence such congruence proceeds.

The truth of the first part of this statement is sufficiently obvious; that of the second follows from the first, and from the fact that the degree of the correspondence is, by definition, the order of the curve which corresponds, in it, to a right line. This order is, in fact, clearly equal to the number of congruence-rays situated in any plane passing through that line; in other words, to the class n of the congruence (*C. C.*, Art. 3).*

4. The order of a Cremonian congruence ordinarily exceeds its class by two, and order and class only become equal to one another when, in consequence of the presence of two self-respondent points C and D, on the intersection of the generating planes α and β, two pencils of congruence-rays become detached from those which connect corresponding points. (*C. C.*, Art. 20.)

5. Only one of the n congruence-rays proceeding from an arbitrary point A of α, say, passes through the corresponding point B of β; the remaining ones connect points of β, situated on $\overline{\alpha\beta}$, with the several points in α which correspond to them. Ordinarily, the latter envelope a congruence-curve in α of the class $n-1$.

But, if the congruence under consideration form part of a linear complex, these $n-1$ rays must coincide with one another, otherwise

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* As in my last paper, I shall refer thus to my memoir *On Cremonian Congruences*, published in Vol. XIV. of the "Proceedings of the London Mathematical Society," pp. 259—301, 1883.

they could not always lie, with the n^{th} ray $\overline{AB}$, in one and the same plane. In other words, the above-mentioned congruence-curve becomes a congruence-pencil, each of whose rays is to be counted $n-1$ times; viz., once for every point of such ray, which is thereby connected with its corresponding point on $\overline{\alpha\beta}$.

Since the last-mentioned point has more than one, it must have an infinite number of corresponding points in α, all which points must be situated on a curve of the order $n-1$. It is, in fact, a principal point, B_{n-1}, of β of the same order, $n-1$, of multiplicity, and being necessarily the only point of this kind in β, it must likewise be the centre of the congruence-pencil in α above alluded to.

6. From the above analysis we readily conclude that the birational correspondence between α and β whence a Cremonian congruence contained in a linear complex proceeds must be of the isographic (*de Jonquières*) type, with its principal multiple points A_{n-1} and B_{n-1} situated on the intersection $\overline{\alpha\beta}$; to which multiple points correspond, in β and α respectively, principal curves b^{n-1} and a^{n-1} of the order $n-1$, and having B_{n-1} and A_{n-1} for $(n-2)$-ple points. Exclusive of the above principal curves, therefore, to every right line a, in α, passing through A_{n-1}, corresponds a right line b, in β, which passes through B_{n-1}. Not merely do the corresponding points of these lines a and b form two projective rows, but the lines themselves are corresponding rays of two projective pencils A_{n-1} (α) and B_{n-1} (β). Of these pencils, moreover, $\overline{\alpha\beta}$ is a self-respondent ray, since it contains the two self-respondent points C and D (Art. 4).

7. Conversely, every isographic correspondence between α and β whose principal multiple points are situated on $\overline{\alpha\beta}$, and in which the latter line is self-respondent, generates a congruence which is contained in a linear complex.*

* Although not essential to the present enquiry, a brief consideration of the interesting problem, "To determine a Cremonian Congruence which shall be contained in a *given* linear complex," here merits a place.

A plane α, and in it a point A_{n-1}, being arbitrarily assumed, the centre B_{n-1} of the complex-pencil situated in α will necessarily lie in the plane β of the complex-pencil of which A_{n-1} is the centre; and every ray a of the pencil A_{n-1} (α) will have for "conjugate polar," relative to the given complex, a perfectly determinate ray b of the pencil B_{n-1} (β) [see Plücker's *Neue Geometrie des Raumes*, p. 28]. Not only are a and b, however, corresponding rays of the two projective pencils A_{n-1} (α) and B_{n-1} (β), having a common self-respondent complex-ray $\overline{\alpha\beta}$; they are likewise, as is well known, the directrices of a linear congruence contained in the given complex. The latter, indeed, is simply the aggregate of all such congruences.

From this it follows at once that, if an isographic correspondence of the degree n could be established between α and β, of which A_{n-1} and B_{n-1} were the multiple principal points, and every pair of rays a, b corresponding lines, the Cremonian Congruence thereby generated would satisfy the conditions of the problem.

The establishment of such an isographic correspondence presents no difficulty.

For from an arbitrary point P, in space, the corresponding rays a and b are projected by the planes of two projective pencils which have in $(P, \overline{\alpha\beta})$ a self-respondent plane. The intersections of all other pairs of corresponding planes, therefore, are coplanar; they form, in fact, a plane pencil of rays $P(\pi)$, and belong to a linear complex. Amongst them, of course, are the several rays, passing through P, of the Cremonian congruence which the assumed isographic correspondence between α and β generates.

The same result will be arrived at if the projective pencils $A_{n-1}(\alpha)$, $B_{n-1}(\beta)$ be cut by an arbitrary plane π. The corresponding rays a, b then determine on $\overline{\pi\alpha}$, $\overline{\pi\beta}$, two projective rows which have in $(\pi, \overline{\alpha\beta})$ a self-respondent point. The connectors of all other pairs of corresponding points, therefore, are concurrent, say in P, and in the pencil $P(\pi)$ will necessarily be found the several rays of the Cremonian congruence under consideration.

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8. If the isographic correspondence by which this congruence is generated be of the degree n, and C and D be the only two self-respondent points thereof, the congruence itself will be of the order as well as of the class n (Art. 4).

9. If, however, three, and consequently all, points of $\overline{\alpha\beta}$ be self-respondent, a special linear complex may be detached from the aggregate of rays joining corresponding points (*C. C.*, Art. 22), and there will remain a congruence $(n-1, n-1)$, which, as in the above more general case, is itself contained in a linear complex.*

10. Of Cremonian congruences contained in a linear complex we have, consequently, two distinct types. Those of the *first* type, described in Art. 8, have four singular points on $\overline{\alpha\beta}$; those of the *second*, only two, as explained in Art. 9.

11. From the mode in which the congruence (n, n) of the *first type* has been generated in Art. 6, it is obvious that it possesses two $(n-1)$-fold congruence-pencils, in the generating planes α and β, having their respective vertices B_{n-1} and A_{n-1} situated on $\overline{\alpha\beta}$. It has, moreover, $2(n-1)$ pairs of congruence-pencils whose centres are the associated principal single points of the isographic correspondence

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* This congruence may also be generated, after the manner of Roccella and Sturm (see Note to Art. 2), as the locus of a right line which is constantly incident with three corresponding generators of three projective forms; viz., two plane pencils having a self-respondent ray, and a unicursal scroll.

whence the congruence proceeds, and whose planes pass through the principal lines corresponding to those centres. An additional pair of congruence-pencils has its centres at the self-respondent points C and D, situated on $\overline{\alpha\beta}$; its planes, γ and δ, intersect each other in that same line $\overline{\alpha\beta}$. (*C. C.*, Art. 21.) The congruence itself, moreover, may be regarded as the aggregate of all the quadric reguli having, for directrices, the several pairs of corresponding rays a, b of the projective pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$, and whose generators join corresponding points of these rays (Art. 6). The reguli respectively conjugate to these form, in the aggregate, another congruence (n, n) (*C. C.*, Art. 36a). This associated congruence, however, is not Cremonian.

12. The common focal surface, however, of the latter, and of the congruence (n, n) in which we are more immediately interested, is the envelope of the system of quadric surfaces upon which the several systems of conjugate reguli are situated. It is of the order as well as of the class $4(n-1)$ (*C. C.*, Arts. 10 and 20), touches each of the planes α and β along the principal curve of the order $n-1$ which that plane contains, and likewise *cuts* it along the $2(n-2)$ tangents which can be drawn to that curve from the principal multiple point to which it corresponds. The line $\overline{\alpha\beta}$, moreover, is a double one on the focal surface under consideration.

13. In support of these statements, I observe that the quadric $(a, b)^2$, and therefore the focal surface it envelopes, touches the planes α and β respectively, at the points A_0 and B_0, where its directrices a and b cut, ulteriorly, the principal curves a^{n-1} and b^{n-1}. For to these points correspond, respectively, the principal points B_{n-1} and A_{n-1}, so that $a_0 \equiv \overline{A_0 B_{n-1}}$ and $b_0 \equiv \overline{B_0 A_{n-1}}$ are generators of the quadric $(a, b)^2$.

The latter, of course, likewise touches the focal surface along the quartic curve (characteristic), in which it is intersected by the next succeeding quadric of the system. Now this quartic curve clearly breaks up into a cubic and the generator a_0 or b_0 whenever the latter happens to touch the principal curve a^{n-1} or b^{n-1} at A_0 or B_0. Hence it follows that all such generators a_0 and b_0 lie wholly on the focal surface, and in the composite section of that surface, made by either of the planes α or β, they count as an element of the order $2(n-2)$; this being, in general, the class of a^{n-1} or b^{n-1}.

Each of the latter curves, moreover, being the curve of contact between its plane and the focal surface, counts as another element of the section, made with the latter by the former, of the order $2(n-1)$;

so that, the order of the total section being $4(n-1)$, the residual element thereof can only be of the order

$$4(n-1)-2(n-1)-2(n-2)=2.$$

This proceeds from $\overline{\alpha\beta}$, which is a double line on the focal surface.

14. The self-respondent points C and D of Arts. 4 and 11 are nodes of the focal surface, at each of which the quadric cone of contact breaks up into a pair of right lines. One of these, at both points, is $\overline{\alpha\beta}$; the other, we will denote by c at C, and by d at D.

In fact, confining our attention for the present to the point C, if c' and c'' be two right lines, in α and β respectively, each of which touches, at C, the curve corresponding to the other, two of the n congruence-rays in the plane (c', c'') will coincide with the intersection of the latter and γ. (*C. C.*, Art. 19.) But c' and c'' are obviously corresponding rays of two pencils which have in $\overline{\alpha\beta}$ a self-respondent element, so that the quadric cone enveloped by the plane (c', c'') breaks up into two pencils, one of which has $\overline{\alpha\beta}$, and the other c for its axis.

15. The focal surface, like the congruence (n, n) itself, is self-reciprocal. Hence we may infer from the above that γ and δ are double planes of that surface, and that the conic of contact, in each of them, breaks up into a pair of right lines; viz., $\overline{\alpha\beta}$ and $\bar{c}$ in γ, $\overline{\alpha\beta}$ and $\bar{d}$ in δ.

16. It is worthy of note, also, that of any congruence-ray of the pencil $C(\gamma)$ or $D(\delta)$, one focus is fixed at C or D, and the other, variable with the ray, moves on the line $\bar{c}$ or $\bar{d}$; whilst one focal plane is fixed at γ or δ, and the other, variable with the ray, turns around c or d.

17. The points A_{n-1} and B_{n-1} are multiple ones on the focal surface. At each of them the cone of contact breaks up into the right line $\overline{\alpha\beta}$, axis of a pencil of planes, and a unicursal cone of the class $n-1$ which touches α $[\beta]$ along the $n-2$ tangents to the principal curve a^{n-1} $[b^{n-1}]$ at its multiple point A_{n-1} $[B_{n-1}]$.

In fact, confining our attention for a moment to the point A_{n-1}, the cone of contact thereat is the envelope of the plane (a, b_0) (Art. 13) which touches the quadric $(a, b)^2$ at A_{n-1}. Now, to each ray a, in α, corresponds one, and only one, ray b_0, in β; whilst to each ray b_0 —since it cuts b^{n-1} in $n-1$ points B_0, to each of which proceeds a ray b —correspond $n-1$ rays a. Of this $(1, n-1)$ correspondence between the rays a and b_0, however, $\overline{\alpha\beta}$ is a self-respondent element, so that, by a well-known theorem, the plane (a, b_0) of two corresponding rays envelopes a cone, of the class n, which breaks up into the pencil of planes whose axis is $\overline{\alpha\beta}$, and a cone of the class $n-1$ having α for a $(n-2)$-ple tangent plane. The generators of contact with the latter plane are the rays a which touch a^{n-1} at its $(n-2)$-ple point A_{n-1} (Art. 12); since these correspond, as may be easily verified, to the $(n-2)$ rays b touching b^{n-1} at its multiple point B_{n-1}, with which latter $n-2$ of the points B_0 coincide when b_0 falls on $\overline{\alpha\beta}$.

18. From Plücker's formulæ we conclude, further, that the cones of contact at A_{n-1} and B_{n-1}, which, as we have just seen, are of the class $n-1$, and of the order $2(n-2)$, have each $3(n-3)$ cuspidal edges, and $2(n-3)(n-4)$ double ones.

These cuspidal edges, it may be observed, are tangents at A_{n-1} and B_{n-1} to a curve of regression,* and in like manner we may infer that the double edges, above alluded to, give the directions, at the last named points, of a double curve on the focal surface.

19. It is scarcely necessary to add that the $2(n-1)$ pairs of associated principal single points A_1, B_1 of the isographic correspondence between α and β are also nodes of the focal surface; at these points the quadric cones of contact are both touched by the planes $(A_1B_1B_{n-1})$ and $(B_1A_1A_{n-1})$; which planes, moreover, are singular tangent planes of the focal surface; that is to say, each touches the latter along a conic which passes through A_1 as well as B_1.†

20. Proceeding, now, to the Cremonian congruences of the *second type* (Art. 10), which are contained in a linear complex, and whose order and class are $n-1$, when the generating isographic correspondence is of the degree n (Art. 9), I observe that α and β contain congruence-pencils whose centres are B_{n-1} and A_{n-1} respectively, and with every ray of which $n-2$, in place of $n-1$, congruence-rays coincide. This difference, it need scarcely be said, arises from the fact that the principal curves a^{n-1} and b^{n-1} now pass, respectively, through B_{n-1} and A_{n-1}, so that, exclusive of the latter, they are only cut in $n-2$ other points by every ray of the pencils $B_{n-1}(\alpha)$ and $A_{n-1}(\beta)$.

In addition to these $(n-2)$-fold pencils, the congruence $(n-1, n-1)$ now under consideration possesses $2(n-1)$ pairs of ordinary ones. The centres of each pair are associated principal single points, and their planes pass respectively through the principal lines corresponding to those centres.

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* See Arts. 12 and 13 of my paper, *On Congruences of the Third Order and Class*, "Proceedings of the London Mathematical Society," Vol. xvi., p. 235, 1885.

† *Ibid.*, Art. 10.

The congruence itself is again the aggregate of all quadric reguli whose directrices are corresponding rays a and b of the pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$. When these directrices are coincident in $\overline{\alpha\beta}$, however, their corresponding points likewise coincide, so that the regulus no longer degenerates, as in Art. 11, to a pair of pencils $C(\gamma)$, $D(\delta)$. It is easy to see, for instance, that the generators a_0 and b_0 of this regulus, which lie in the planes α and β respectively, are the tangents at B_{n-1} and A_{n-1} of the principal curves a^{n-1} and b^{n-1}.

21. The focal surface of our congruence $(n-1, n-1)$ is again the envelope of the several quadrics $(a, b)^2$ on which the above reguli are situated. Its order and class, however, are now $4(n-2)$.* It touches α and β, as before, along the principal curves a^{n-1} and b^{n-1}, which latter, it must be remembered, have not only $(n-2)$-ple points at A_{n-1} and B_{n-1}, but, as stated in Art. 20, also pass, respectively, through B_{n-1} and A_{n-1}. This focal surface likewise *cuts* the planes α and β along the $2(n-3)$ tangents which can be drawn from B_{n-1} and A_{n-1} respectively, to touch *elsewhere* the principal curves a^{n-1} and b^{n-1}. The section of the focal surface with each of the planes α and β is thus seen to be of the already-stated order, viz.:

$$2(n-1)+2(n-3) = 4(n-2).$$

22. The singularities of the points A_{n-1} and B_{n-1} on the focal surface are again precisely correlative to those of the planes α and β. At each of these multiple points the cone of contact is of the class $n-1$. That with vertex at $A_{n-1}[B_{n-1}]$ touches the plane $\alpha[\beta]$ $n-2$ times, and $\beta[\alpha]$ once; both, in fact, along the tangents to the principal curves which pass through $A_{n-1}[B_{n-1}]$. This cone, moreover, besides *touching* the plane $\beta[\alpha]$ once, as above stated, *cuts* it along the $2(n-3)$ tangents that can be drawn from $A_{n-1}[B_{n-1}]$ to touch the principal curve $b^{n-1}[a^{n-1}]$ elsewhere. In fact,

$$2+2(n-3) = 2(n-2)$$

is, in general, the order of the cone in question.

The singularities described in Art. 18 reappear, without change, in the cones now under consideration.

23. The singularities presented by the focal surface at the $2(n-1)$ pairs of associated principal single points A_1 and B_1 are precisely the same as those described in Art. 19. They may, however, be elucidated somewhat differently, thus:—

The system of quadrics $(a, b)^2$ includes $2(n-1)$ point-and-plane

* *C. C.*, Art. 23, in which m is to be replaced by unity.

pairs. The points of each, A_1, B_1, are associated principal single points; the planes of each $(A_1B_1B_{n-1})$, $(B_1A_1A_{n-1})$ connect these points with the principal lines which respectively correspond to them. These planes are cut by the next succeeding quadric $(a, b)^2$ of the system in a pair of conics which intersect in A_1 and B_1; they are, in fact, singular tangent planes of the focal surface, and these are their conics of contact therewith. Correlatively A_1 and B_1 are nodes of the focal surface, the quadric cones of contact at which touch both the singular planes just referred to.

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24. In accordance with the scheme proposed in Art. 2, I now pass to a brief consideration of the special properties of the two different Cremonian congruences of the third order and class which are contained in a linear complex. The first is obtained by putting $n = 3$ in Arts. 11—19; the second, by making $n = 4$ in Arts. 20—23. For both these congruences, it will be observed (Arts. 12 and 21), the focal surface is of the eighth order and class; in other respects, however, the congruences in question differ from each other materially.

25. That of the *first type* has doubled congruence-pencils in the planes α and β, with centres at B_2 and A_2, respectively, on $\overline{\alpha\beta}$. It has also four pairs of congruence-pencils whose centres A_1 and B_1 are associated principal single points of the cubic correspondence between α and β, and whose planes $(A_1B_1B_2)$ and $(B_1A_1A_2)$ pass through the principal lines corresponding to those centres. It has, moreover, a fifth pair of congruence-pencils whose centres are the self-respondent points C and D, and whose respective planes γ and δ pass through the intersection $\overline{\alpha\beta}$.

The focal surface has $\overline{\alpha\beta}$ for a double line, and the five pairs of singular points and planes above enumerated have, for it, precisely the properties described in Arts. 14, 15, 16, and 19.

This focal surface touches α and β along the principal *conics* passing respectively through A_2 and B_2, and it cuts these planes, moreover, along the tangents to these conics which proceed from B_2 and A_2 respectively.

Correlatively, the points A_2 and B_2 are nodes on the focal surface; at each of which the cone of contact breaks up into $\overline{\alpha\beta}$, regarded as the axis of a pencil of planes, and a cone of the second class. The cone whose vertex is at A_2 [B_2], for instance, not only touches α [β] along the tangent at A_2 [B_2] to the principal conic, but it likewise cuts β [α] along the two tangents from A_2 [B_2] to the principal conic in the latter plane. These tangents, as we have just seen, lie wholly on the focal surface.

26. The congruence (3, 3) of the *second type* has, like that of the first type, doubled congruence-pencils in the planes α and β, the centres of which are at B_3 and A_3 respectively. But, instead of having five, it has six pairs of congruence-pencils, the vertices of which are all associated principal single points of the quartic, isographic correspondence whence the congruence proceeds, and the planes of which pass, as usual, through the principal lines corresponding to those points.

On the focal surface, these points and planes are singular ones of the kind already described in Art. 23 and elsewhere, and the surface in question touches the planes α and β along the principal cubics which these planes contain. Besides touching α [β], however, along this cubic, which has a double point at A_3 [B_3] and passes through B_3 [A_3], it cuts it along the two tangents to this cubic which can be drawn from B_3 [A_3] to touch the curve elsewhere.

Correlatively, A_3 and B_3 are nodes on the focal surface, at which the cones of contact are of the third class and fourth order. Of each cone, one of the planes α and β is a double, and the other an ordinary tangent plane. The cone whose vertex is A_3, for instance, touches α along the tangents at the double point A_3 of the principal cubic in α, and it also touches β along the tangent at A_3 to the principal cubic in β. At the same time it cuts the latter plane along the remaining two tangents that can be drawn from A_3 to the cubic just referred to. The last-mentioned generators of the cone of contact at its singular point A_3 lie, indeed, wholly on the focal surface.

On the Airy-Maxwell Solution of the Equations of Equilibrium of an Isotropic Elastic Solid, under Conservative Forces. By W. J. Ibbetson, M.A., F.R.A.S.

[*Read May 13th,* 1886.]

Sir G. B. Airy was the first to propose* a very elegant method of solving the equations of stress in two dimensions, the very obvious extension of which to three dimensions is due to Clerk Maxwell.†

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* *British Association Report*, Cambridge, 1862, p. 82; and *Phil. Trans.* for 1863, p. 49.

† *Edinburgh Trans.*, Vol. xxvi., p. 31.