so that, omitting this factor, the form of the equation is

$$((a, b, c, f, g, h)^2 \between x, y, z, w)^8 = 0;$$

viz., the equation is of the order 8 in the coordinates $(x, y, z, w)$, and of the degree 2 in the coordinates $(a, b, c, f, g, h)$ of each of the lines. It would not be very difficult to actually develope the equation; in fact, starting from the term $w^8[(a, b, c)^2]$ the other terms are obtained therefrom by changing $a, b, c$ into $a + \frac{1}{w}(hy - gz)$, $b + \frac{1}{w}(-hx + fz)$, $c + \frac{1}{w}(gx - fy)$ respectively; the equation may therefore be written in the symbolic form

$$w^8 . \exp. \frac{1}{w}\{(hy - gz)\delta_a + (-hx + fz)\delta_b + (gx - fy)\delta_c\} . [(a, b, c)^2] = 0,$$

or what is the same thing

$$w^8 . \exp. \frac{1}{w}\{x(g\delta_c - h\delta_b) + y(h\delta_a - f\delta_c) + z(f\delta_b - g\delta_a)\} . [(a, b, c)^2] = 0,$$

where exp. $\theta$ (read exponential) denotes $e^\theta$, and $[(a, b, c)^2]$ represents a determinant as above explained. The equation contains, it is clear, the four terms.

$$x^8[(a, -h, g)^2] + y^8[(-h, b, -f)^2] + z^8[(-g, f, c)^2] + w^8[(a, b, c)^2].$$

I am not sure whether this surface of the eighth order has been anywhere considered.

Mr. J. J. Walker next communicated a Note on his paper read at the January Meeting of the Society.

## *On Conditions for, and Equations of, Corresponding Points in certain Involutions.* By J. J. Walker.

The principal points discussed in the following short paper are:

1. The condition to be satisfied in order that a given binary quadric should determine two corresponding points in one of the three involutions determined by a quartic; from which condition is deduced the cubic giving the three points corresponding to an assigned one in each of those involutions respectively, the sextic determining their double points, and the cubic of centres:

2. The corresponding questions when the three involutions are determined by two quadrics instead of one quartic—any previously published discussion of which case, as far as I am aware, having been confined to the particular instance in which the roots of each quadric correspond, leaving unconsidered those in which a root of one corresponds to a root of the other:

3. The condition to be satisfied in order that two binary cubics should determine an involution in such a manner that each root of one corresponds to a root of the other; and, when this condition is fulfilled, the equations which give the point corresponding to any seventh assigned one in the involution, the double points or foci, and the centre. The involution condition is found under two distinct forms, in one of which it is expressed as a function of the coefficients of the quadric and quartic covariants of the two cubics; the particular case of one cubic being the cubic covariant of the other is treated of, and the application of the involution condition to some questions in connexion with the theory of cubic curves considered.

The questions, as thus enumerated, are discussed both by the application of elementary principles only, and likewise, where the results are invariant, by the principles of the Higher Algebra, which afford so much readier demonstrations.

4. It will be convenient to premise, for subsequent reference, a few elementary principles. If the distances of four points $p$, $q$, $r$, $s$, from any origin $o$ be given explicitly equal to $\alpha$, $\beta$, $\gamma$, $\delta$, and $h_1$, $h_2$, $h_3$ be the distances from origin of the centres of the three involutions determined by these four points, in which $pq$, $pr$, $ps$ are respectively corresponding points, while $h_1 \pm k_1$, $h_2 \pm k_2$, $h_3 \pm k_3$ are the distances of the pairs of double points respectively from origin; then those of any two corresponding points will be connected by the relation

$$x'x - h(x'+x) + h^2 - k^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

suffixes being written to indicate in which of the three systems the points are supposed to stand; and, in particular,

$$\alpha\beta - h_1(\alpha+\beta) + h_1^2 - k_1^2 = 0,$$
$$\gamma\delta - h_1(\gamma+\delta) + h_1^2 - k_1^2 = 0,$$

from which equations

$$h_1 = \frac{\alpha\beta - \gamma\delta}{\alpha+\beta-\gamma-\delta} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$h_1^2 - k_1^2 = \frac{\alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta)}{\alpha+\beta-\gamma-\delta} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and corresponding values for $h_2$, $h_2^2 - k_2^2$ are obtained by interchanging $\beta$ and $\gamma$; for $h_3$ and $h_3^2 - k_3^2$ by interchanging $\beta$ and $\delta$, in (2) and (3).

Again, if $p'$, $q'$, $r'$, $s'$ be the inverse points, with respect to $o$, of $p$, $q$, $r$, $s$, so that their distances from $o$ are to be represented by $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, $\frac{1}{\delta}$ respectively; and if $h'_1$, $h'_2$, $h'_3$, $h'_1 \pm k'_1$, ... represent the distances from $o$ of the centres and double points in the involutions in which $p'q'$, $p'r'$, $p's'$ are respectively corresponding points,

$$h_1' = \frac{\alpha\beta - \gamma\delta}{\alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta)} = \frac{h_1}{h_1^2 - k_1^2} \quad \ldots\ldots\ldots\ldots\ldots(4),$$

$$h_1'^2 - k_1'^2 = \frac{\alpha+\beta-(\gamma+\delta)}{\alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta)} = \frac{1}{h_1^2 - k_1^2} \quad \ldots\ldots\ldots\ldots(5).$$

5. If the distances of the points $p, q, r, s$ from $o$ be supposed not given explicitly, but that $\alpha, \beta, \gamma, \delta$ are roots of a quartic

$$(abcde \between x1)^4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

then the three points which correspond to an assigned one, whose distance from $o$ is $x'$, in the three involutions respectively determined by the points $p, q, r, s$, will be given implicitly by a cubic, which I proceed to form.

Substituting in (1) the values of $h$ and $h^2-k^2$ from (2), (3),

$$(\alpha+\beta-\gamma-\delta)\,x'x - (\alpha\beta-\gamma\delta)(x'+x) + \alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta) = 0$$

determines the point corresponding to that whose distance from $o$ is $x'$ in the first involution. Hence, if for shortness we write

$$l_1 = \alpha+\beta-\gamma-\delta,$$
$$m_1 = \alpha\beta-\gamma\delta,$$
$$n_1 = \alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta),$$

and $l_2, m_2, n_2$ represent what $l_1, m_1, n_1$ respectively become when $\beta$ and $\gamma$ are interchanged; $l_3, m_3, n_3$ what the same expressions become when $\beta$ and $\delta$ are interchanged, the cubic sought will be

$$\{l_1x'x - m_1(x'+x) + n\}\{l_2x'x - m_2(x'+x) + n_2\}\{l_3x'x - m_3(x'+x) + n_3\} = 0,$$

or $$l_1l_2l_3\,.\,x'^3x^3 - \Sigma l_1l_2m_3\,.\,x'^2x^2(x'+x) + \Sigma l_1m_2m_3\,.\,x'x(x'+x)^2 + \Sigma l_1l_2n_3\,.\,x'^2x^2$$
$$- m_1m_2m_3(x'+x)^3 - \Sigma l_1m_2n_3\,.\,x'x(x'+x) + \Sigma m_1m_2n_3\,.\,(x'+x)^2$$
$$+ \Sigma l_1n_2n_3\,.\,x'x - \Sigma m_1n_2n_3(x'+x) + n_1n_2n_3 = 0 \quad \ldots\ldots(7).$$

6. Let

$$G = a^2d - 3abc + 2b^3,$$
$$6F = a^2e + 2abd - 9ac^2 + 6b^2c,$$
$$3E = abe - 3acd + 2b^2d,$$
$$2D = b^2e - ad^2,$$

and G′, F′, E′, D′ represent what G, F, E, D respectively become when $a$ and $e$, $b$ and $d$ are interchanged; then it will be found that the symmetric functions of the roots of the quartic, which enter into the above equation, are expressed in terms of G, F .... D′ thus:

$$l_1l_2l_3 = -32G,$$
$$\Sigma l_1l_2m_3 = 96F,$$
$$\Sigma l_1l_2n_3 = \Sigma l_1m_2m_3 = -96E,$$
$$m_1m_2m_3 = 32D,$$
$$\Sigma l_1m_2n_3 = 192D,$$
$$\Sigma l_1n_2n_3 = \Sigma m_1m_2n_3 = 96E',$$
$$\Sigma m_1n_2n_3 = -96F',$$
$$n_1n_2n_3 = 32G'.$$

Substituting in (7) and dividing by $-32$, it becomes

$$Gx'^3x^3+3Fx'^2x^2(x'+x)+3E\{(x'+x)^2+x'x\}x'x+D\{(x+x')^2+6x'x\}(x'+x)-$$
$$-3E'\{(x'+x)^2+x'x\}-3F'(x'+x)-G'=0\ldots(8),$$

which may be re-arranged, replacing $x'$ by $\frac{x'}{y'}$ and $x$ by $\frac{x}{y}$,

$$0=\big((G,F,E,D\between x'y')^3,\ (F,E,D,-E'\between x'y')^3,\ (E,D,-E',-F'\between x'y')^3,$$
$$(D,-E',-F',-G'\between x'y')\between xy)^3\ldots(9),$$

which is the cubic it was proposed to investigate, and of which a quite different demonstration will be given after noticing two special cases.

7. To obtain the equation which determines the three centres of the involutions, it is only necessary to divide (8) by $x'^3$, and then suppose it to become infinitely great: thus there results, replacing $x$ by $\frac{x}{y}$,

$$(G,F,E,D\between xy)^3=0\ \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

the roots of which, expressed in terms of those of the given quartic, are

$$h_1=\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta},\quad h_2=\frac{\alpha\gamma-\beta\delta}{\alpha+\gamma-\beta-\delta},\quad h_3=\frac{\alpha\delta-\beta\gamma}{\alpha+\delta-\beta-\gamma}.$$

The above is a new "reducing" cubic, by the aid of which the quartic might be solved. Its connection with Lagrange's reducing cubic may be exhibited by transforming the roots. For, writing

$$\tfrac{1}{4}\{(\alpha+\beta)^2-(\alpha-\beta)^2\}\ \text{for}\ \alpha\beta,\quad \tfrac{1}{4}\{(\gamma+\delta)^2-(\gamma-\delta)^2\}\ \text{for}\ \gamma\delta,$$

$$4h_1=\frac{(\alpha+\beta)^2-(\gamma+\delta)^2+(\gamma-\delta)^2-(\alpha-\beta)^2}{\alpha+\beta-\gamma-\delta}$$
$$=\frac{(\alpha+\beta+\gamma+\delta)(\alpha+\beta-\gamma-\delta)+(\alpha+\gamma-\beta-\delta)(\beta+\gamma-\alpha-\delta)}{\alpha+\beta-\gamma-\delta}$$
$$=-\frac{4b}{a}-\frac{l_1l_2l_3}{l_1^2}=-\frac{4b}{a}+\frac{32G}{l_1^2},$$

or
$$8G\left(h_1+\frac{b}{a}\right)^{-1}=l_1^2;$$

but the roots of Lagrange's cubic are $l_1^2,\ l_2^2,\ l_3^2$.

8. Again, the double points, or foci, of the three involutions will be determined by making $x'=x$ in (8), which gives the sextic, replacing $x$ by $\frac{x}{y}$,

$$(G,F,E,D,-E',-F',-G'\between xy)^6=0\ \ldots\ldots\ldots (11).$$

This is the sextic covariant of the quartic, the geometrical significance of which has already been pointed out by Dr. Salmon (Higher Algebra, § 210, 2nd ed.)

9. Multiplying equation (8) by $a'^3$, then substituting $c'$ for $a'x'x$, $-2b'$

for $a'(x'+x)$, the condition that $(a'b'c' \between xy)^2$ should determine two points, conjugate one to the other in one of the three involutions determined by the quartic $(abcde \between xy^4)$, is at once obtained; viz.,

$$Gc'^3 - 6Fb'c'^2 + 3E(4b'^2 + a'c')c' - 4D(2b'^2 + 3a'c')b'$$
$$- 3E'a'(4b'^2 + a'c') + 6F'a'^2b - G'a'^3 = 0 \ldots (12).$$

This must be an invariant of the system of the quartic ($u$) and the quadratic ($v$), and the coefficients of each quantic enter in the third degree. I find that it is formed thus:—

The quartic covariant $\frac{du}{dx}\frac{dv}{dy} - \frac{dv}{dx}\frac{du}{dy}$ is

$$(ab' - a'b,\ ac' + 2bb' - 3a'c,\ 3(bc' - a'd),\ 3cc' - 2b'd - a'e,\ c'd - b'e \between xy)^4;$$

and the cubic invariant of this covariant is (to a numerical factor) precisely the condition (12).

10. An *à posteriori* proof of the condition may now be readily obtained from the following considerations:—

It is well known that the Anharmonic Ratio of a pencil of lines is unaffected by any linear transformation of its equation; consequently, if the equation of six lines can be so transformed that the terms containing odd powers of the variables disappear, the pencil must be one of involution, since the rays of the transformed are equally inclined in pairs to the axis if these are rectangular, or in general intercept on any transversal parallel to one axis equal segments in pairs on either side of the point where the other axis meets the transversal.

Now, if the quartic and quadratic can be transformed simultaneously into $Ax^4 + 4Cx^2y^2 + Ey^4$ and $A'x^2 + C'y^2$, the cubic invariant of their quartic covariant above must vanish, since every term in (12) vanishes separately* on the suppositions $b=0$, $d=0$, $b'=0$. The vanishing of this invariant is therefore the condition of such a transformation being possible, *i.e.*, of the two forms determining six points, or lines, in involution.

The condition (12) having been established by this reasoning, by reversing the substitutions, *i.e.*, putting $a'x'x$ for $c'$, and $-a'(x'+x)$ for $2b'$, the general equation (8) is arrived at.

11. The quadratic covariant $\overline{1.2}^2$ of $uv$ is

$$(ac' + a'c - 2bb')x^2 + 2(bc' + a'd - 2b'c)xy + (cc' + a'e - 2b'd)y^2 \ldots (13).$$

Now, when $u$, $v$ are transformed so that the terms containing odd powers of $x$ and $y$ disappear, it is plain that the term $xy$ in the above covariant also vanishes. Hence it follows that if $v$ determine two corresponding points in one of the three involutions determined by $u$, then $\overline{1.2}^2$ will determine another pair of corresponding points in the

* For every term in D, E, G contains either $b$ or $d$ as a factor.

same involution. This consideration makes it possible to discriminate, as it were, that particular one of the three involutions to which the points determined by $v$ belong. It is known that when two pairs of corresponding points are determined in an involution by two quadratics $(a'b'c' \between xy)^2$, $(a''b''c'' \between xy)^2$, then any other pair $(x'x)$ of corresponding points will satisfy

$$2(a'b''-a''b')\,x'x+(a'c''-a''c')\,(x'+x)+2\,(b'c''-b''c')=0 \quad\ldots(14);$$

so that, if the covariant $\overline{1.2}^2$ (13) be written $(a''b''c'' \between xy)$, the foci of the involution particularized will be determined by

$$(a'b''-a''b')\,x^2+(a'c''-a''c')\,xy+(b'c''-b''c')\,y^2=0 \quad\ldots\ldots (15),$$

and the centre of the same involution by

$$2\,(a'b''-a''b')\,x+(a'c''-a''c')\,y=0 \quad\ldots\ldots\ldots\ldots\ldots (16).$$

12. In the next place, the four points $p, q, r, s$ may be supposed to be determined by two quadrics $u \equiv (abc \between xy)^2$ and $v \equiv (a'b'c' \between xy)^2$. But here one of the three involutions can be distinguished at once from the other two; viz., that in which one pair of corresponding points is determined by one quadric, the other pair by the other. This case has naturally attracted most attention, and has been fully discussed; but for completeness the results arrived at are here noticed and demonstrated, as they readily may be, in accordance with the method of proof above employed.

If $w \equiv (a''b''c'' \between xy)^2$ is to determine a third pair of points, a certain condition must be satisfied identical with that for it being possible to transform $u$, $v$, $w$ by the same substitutions into

$$Ax^2+Cy^2,\quad A'x^2+C'y^2,\quad A''x^2+C''y^2.$$

But for these forms the invariant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

evidently vanishes; this, then, or

$$a''(b'c-bc')+b''(ac'-a'c)+c''(a'b-ab')=0 \quad\ldots\ldots\ldots (17),$$

is the required condition.

If, in (17), $b''$ be replaced by $-\frac{a''}{2}(x'+x)$, $c''$ by $a''x'x$, the equation connecting two corresponding points is obtained in the form used above (14); and the foci are determined by making $x'=x$, the centre by making $x'$ infinite.

13. In the other two cases a root of one quadric corresponds to one of the other; consequently, the forms into which the quadrics should be changed to determine corresponding points equidistant from any origin, on either side of it, are

$$Ax^2+2Bxy+Cy^2 \text{ and } kAx^2-2kBxy+kCy^2.$$

If a third quadratic $w \equiv (a''b''c'' \between xy)^2$ is to determine a pair of corresponding points in one of these two involutions, it must be possible, by the same substitutions as transform $u$, $v$ into the forms above, to transform $w$ into $\quad A''x^2+C''y^2$.

Consider now the invariant of the system $uvw$

$$(ac''+a''c-2bb'')^2(a'c'-b'^2)-(a'c''+a''c'-2b'b'')^2(ac-b^2) \;...(18).$$

This vanishes identically for the special forms

$$(A, B, C \between xy)^2, \;(kA, -kB, kC \between xy)^2, \;(A'', 0, C'' \between xy)^2;$$

and it is therefore the condition that $(a''b''c'' \between xy)^2$ should determine two corresponding points in one of those two involutions determined by $(abc \between xy)^2$ and $(a'b'c' \between xy)^2$, in which either root of the former corresponds to a root of the latter.*

Replacing $-\frac{2b''}{a''}$ by $x'+x$, and $\frac{c''}{a''}$ by $x'x$, in (18),

$$\{ax'x+b(x'+x)+c\}^2(a'c'-b'^2)-\{a'x'x+b'(x'+x)+c'\}^2(ac-b^2)=0 ...(19)$$

gives the two points corresponding to an assigned one, at a distance $x'$ from origin, in the two involutions respectively.

Again, making $x'=x$, in (19),

$$(ax^2+2bx+c)^2(a'c'-b'^2)-(a'x^2+2b'x+c')^2(ac-b^2)=0 \;......(20),$$

or the two quadrics

$$(a'c'-b'^2)^{\frac{1}{2}}(ax^2+2bxy+cy^2) \pm (ac-b^2)^{\frac{1}{2}}(a'x^2+2b'xy+c'y^2)=0,$$

determine the foci of the involutions; and, making $x'$ infinite in (19), after division by $x'^2$,

$$(ax+by)^2(a'c'-b'^2)-(a'x+b'y)^2(ac-b^2) \;............ (21),$$

or $$(a'c'-b'^2)^{\frac{1}{2}}(ax+by) \pm (ac-b^2)^{\frac{1}{2}}(a'x+b'y),$$

determines the two centres.

14. The equation (19) might have been obtained directly by forming the product of

$$(\alpha+\alpha'-\beta-\beta')x'x-(\alpha\alpha'-\beta\beta')(x'+x)+\alpha\alpha'(\beta+\beta')-\beta\beta'(\alpha+\alpha'),$$

and $$(\alpha+\beta'-\alpha'-\beta)x'x-(\alpha\beta'-\alpha'\beta)(x'+x)+\alpha\beta'(\alpha'+\beta)-\alpha'\beta(\alpha+\beta'),$$

* It is easily verified that this condition is satisfied when $(a''b''c'' \between xy)^2=0$ is the equation (15) determining the foci of the involution in which the roots of $u$ correspond, also those of $v$; thus affording a proof of M. Chasles' theorem, that these foci are corresponding points in each of the other two involutions. Otherwise, this appears from the fact that (15)—which is a covariant of $u$, $v$—loses the term $xy$ when $u$ and $v$ are transformed as above.

($\alpha, \beta$ being supposed to be the roots of $u$; $\alpha', \beta'$ those of $v$,) and expressing the symmetric functions of the roots in terms of the coefficients; since this product must evidently vanish if $x'$, $x$ are to determine corresponding points in one of the two involutions. The result would be readily indentified with (19).

15. The case of an involution determined by two cubics, $u = (abcd \between xy)^3$, $v = (a'b'c'd' \between xy)^3$, in such a manner that the roots of one correspond each to one of those of the other, has never heretofore been considered, as far as I am aware; yet it is a case which frequently presents itself in the theory of cubic curves, particularly those of the third class. A certain condition must be fulfilled by the coefficients of $u\ v$, which I shall refer to as the "involution condition of a pair of cubics." It may be shown *à priori* that it must be of the sixth degree in the coefficients of each cubic, from the following considerations:—Let the roots of $u$ be $\alpha$, $\beta$, $\gamma$, those of $v$, $\alpha'$, $\beta'$, $\gamma'$; and suppose $\alpha$, $\beta$, $\alpha'$, $\beta'$ to be given. These determine two involutions, in which $\alpha$ corresponds to $\alpha'$ or $\beta'$; consequently, if further $\gamma$ be supposed to be given, and $\gamma'$ is to be determined so as to correspond to it, the condition must be of the second degree in $\gamma'$, *i. e.*, of the second degree in the coefficients of $v$, since these contain the root $\gamma'$ implicitly in the first degree. But as there are three combinations of $\alpha$, $\beta$, $\gamma$ taken two and two, the general condition must be made up of the product of three such partial conditions, *i. e.*, be of the sixth degree in the coefficients of $u$; and since the same reasoning will apply where $\alpha'$, $\beta'$, $\gamma'$ are given, and $\alpha, \beta, \gamma$ successively to be determined from the other two of the three being supposed given, the coefficients of $u$ must also enter in the sixth degree.

In the next place, it may be observed, that if two roots of $u$ become equal, the condition should imply that two of $v$ necessarily become so too. Hence, calling the discriminants of $u$ and $v$, $\Delta$, $\Delta'$ respectively, it might be inferred that the condition would be of the form

$$\Delta\Theta_2 + \Delta\Delta'\Phi_0 + \Delta'\Theta_1 = 0,$$

where $\Theta_2$ must be of the sixth degree in the coefficients of $v$ and of the second in those of $u$; $\Phi_0$ of the second degree in those of both, and $\Theta_1$ of the sixth and second degree in those of $u$ and $v$ respectively. The analogy of (18) would suggest the probability that the middle term would be wanting, and that $\Theta_1$, $\Theta_2$ would be the squares of invariants of the third degree in coefficients of one cubic and of the first in those of the other. Consider now the invariants which are the coefficients of the successive powers of $\lambda$ in the discriminant of $u + \lambda v$,* which may be written

$$(\Delta, 2\Theta, \Phi, 2\Theta', \Delta' \between 1, \lambda)^4,$$

* It is evident that we should not look for those invariants of the system of two cubics which are combinants, as likely to enter into $\Theta_1$, $\Phi_0$, $\Theta_2$.

$$\Theta = ad'^2a' + a^2dd' + 6(ac'^2c' + b^2db') + 4(c^3a' + b^3d') - 3bc(bc' + cb')$$
$$- 3(abcd' + abdc' + acdb' + bcda'),$$
$$\Theta' = a'd'^2a + a'^2d'd + 6(a'c'^2c + b'^2d'b) + \ldots\ldots - 3(\ldots + b'c'd'a),$$
$$\Phi = a^2d'^2 + d^2a'^2 + \&c.$$

The most probable form of condition that suggests itself for examination is, then, $\Delta\Theta'^2 - \Delta'\Theta^2$.

Now it is evident that if $u, v$ determine an involution, they must be transformable into

$$(A, B, C, D \between xy)^3 \text{ and } (kA, -kB, kC, -kD \between xy)^3$$

by the same linear substitutions. But for these forms it easily appears that $\Delta' = k^4\Delta, \; \Theta' = 8k^3(AC^3 - B^3D) = k^2\Theta$;

so that, eliminating $k$ between these two equations,

$$\Delta\Theta'^2 - \Delta'\Theta^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

which is therefore the involution condition of the cubics $u, v$.

16. This condition may be obtained under a different form by considering the covariants $\overline{1.2}$ and $\overline{1.2}^2$, which are respectively

$$(ab' - a'b, \; 2(ac' - a'c), \; ad' - a'd + 3(bc' - b'c), \; 2(bd' - b'd), \; cd' - c'd \between xy)^4$$
and $$(ac' + a'c - 2bb', \; ad' + a'd - bc' - b'c, \; bd' + b'd - 2cc' \between xy)^2.$$

For the forms of $u, v$, which express that the roots of one are equal, but of opposite sign, to those of the other respectively, the second and fourth terms of $\overline{1.2}$ vanish, as does the middle term of $\overline{1.2}^2$; so that when $u, v$ determine an involution in which the roots of one correspond to those of the other, each to each, both $\overline{1.2}$ will determine two pairs of corresponding points in the same involution, and $\overline{1.2}^2$ a third pair. Hence the coefficients of $\overline{1.2}$ and $\overline{1.2}^2$ must satisfy the condition (12), *i.e.*, the cubic invariant of $\overline{1.2}$ on these covariants must vanish. Writing, then,

$$l = ab' - a'b, \quad 2m = ac' - a'c, \quad 6n = ad' - a'd + 3(bc' - b'c),$$
$$l' = cd' - c'd, \quad 2m' = bd' - b'd,$$
$$p = ac' + a'c - 2bb',$$
$$2q = ad' + a'd - bc' - b'c,$$
$$p' = bd' + b'd - 2cc',$$

the condition for $\overline{1.2}$ and $\overline{1.2}^2$ on $uv$ determining such an involution as that specified as above—which must plainly be identical with (22)—is obtained by writing $l$ for $a$, $m$ for $b$, $n$ for $c$, $m'$ for $d$, $l'$ for $e$, $p$ for $a'$, $q$ for $b'$, and $p'$ for $c'$, in (12); *i.e.*,

$$
\begin{aligned}
&(l^2m'-3lmn+2m^3)\,p'^3-(l^2l'+2lmm'-9ln^2+6m^2n)\,qp'^2\\
&+(lml'-3lm'n+2m^2m')\,(4q^2+pp')\,p'-2\,(l'm^2-lm'^2)\,(2q^2+3pp')\,q\\
&-(lm'l'-3l'mn'+2m'^2m)\,(4q^2+pp')\,p+(ll'^2+2l'mm'-9l'n^2+6m'^2n)\,p^2q\\
&-(l'^2m-3l'm'n+2m'^3)\,p^3=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).
\end{aligned}
$$

This invariant may actually be identified with $\Delta\Theta'^2-\Delta'\Theta^2$ (to a numerical factor) without much labour by taking for one of the cubics ($u$, suppose) its canonical form $ax^3+by^3$.

17. The fact of the involution determined by $\overline{1.2}$, $\overline{1.2}^2$ agreeing with that determined by $u$, $v$, when the condition (22) or (23) is fulfilled, gives at once the relation between any two corresponding points, whose distances from origin are $x'$, $x$, and in particular the double points and centre of the involution. For (see § 11) if we form the quadratic covariant $\overline{1.2}^2$ of the covariants $\overline{1.2}$ and $\overline{1.2}^2$ of $u, v$, it will determine two new corresponding points in the involution. Hence, writing this new covariant, which in full is [see (13)]

$$(lp'+pn-2mq)\,x^2+2\,(mp'+m'p-2nq)\,xy+(np'+l'p-2m'q)\,y^2 \ldots (24),$$

$$(P, Q, P'\between xy)^2,$$

the relation between $x'$ and $x$ is (14)

$$2\,(pQ-Pq)\,x'x+(pP'-Pp')\,(x'+x)+(P'q-p'Q)=0\,;$$

or, substituting from (24) their values for P, Q, P′,

$$
\begin{aligned}
2(m'p^2+mpp'-3npq-lp'q+2mq^2)x'x+\{l'p^2+2(mp'-m'p)q-lp'^2\}(x'+x)&\\
-2\,(mp'^2+m'pp'-3np'q-l'pq+2m'q^2)=0 \;\ldots\ldots\; (25).&
\end{aligned}
$$

Making $x'=x$ in (25), and replacing $x$ by $\dfrac{x}{y}$,

$$
\begin{aligned}
(m'p^2+mpp'-3npq-lp'q+2mq^2)\,x^2+\{l'p^2+2(mp'-m'p)-lp'^2\}\,xy&\\
-(mp'^2+m'pp'-3np'q-l'pq+2m'q^2)\,y^2=0 \;\ldots\ldots\; (26)&
\end{aligned}
$$

determines the foci or double points of the involution; and finally, making $x'$ infinite in (25),

$$2\,(m'p^2+mpp'-3npq-lp'q+2mq^2)\,x+\{l'p^2+2\,(mp'-m'p)-lp'^2\}\,y=0 \;\ldots\ldots\ldots (27)$$

determines the centre.

18. There is one pair of cubics for which the relation (22) or (23) is satisfied, but for which the equations (25), (26), (27) become nugatory. If simultaneously

$$ac'+a'c-2bb'=0,\;\; ad'+a'd-bc'-b'c=0,\;\; bd'+b'd-2cc'=0 \ldots\ldots (28),$$

the condition (23) is plainly satisfied, since every term contains a vanishing factor. The two cubics therefore form an involution in which the roots of one correspond to those of the other, but the coefficients in (25), (26), (27) separately vanish. This case must therefore be dealt with by special considerations.

If the three equations (28) be solved linearly for $\frac{b'}{a'}$, $\frac{c'}{a'}$, $\frac{d'}{a'}$,

$$\frac{b'}{a'}=\frac{abd+b^2c-2ac^2}{a^2d-3abc+2b^3},\quad \frac{c'}{a'}=\frac{-acd-bc^2+2b^2d}{a^2d-3abc+2b^3},\quad \frac{d'}{a'}=\frac{-ad^2+3bcd-2c^3}{a^2d-3abc+2b^3}.$$

Hence $v$ is (to a factor) the cubic covariant of $u$, and each of the points determined by $v$ is harmonic conjugate to one of the three points determined by $u$ with respect to the other two. The Hessian of $u$ is

$$(ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2,$$

and the invariant of this and any quadric $(a_1,\ b_1,\ c_1 \between xy)^2$ is

$$(ac-b^2)\,c_1-(ad-bc)\,b_1+(bd-c^2)\,a_1.$$

Again, if $u$ be transformed so that it becomes

$$\mathrm{A}\,x^3+\mathrm{D}\,y^3,$$

its cubic covariant becomes $\mathrm{AD}\,(\mathrm{A}\,x^3-\mathrm{D}\,y^3)$,

[which fact verifies that the two cubics determine an involution,] and the invariant above becomes $\mathrm{AD}b_1$;

so that, if $(a_1,\ b_1,\ c_1 \between xy)$ is to determine a pair of corresponding points in the involution determined by $u$ and its cubic covariant, *i. e.*, if the same substitution transforms the quadratic to

$$\mathrm{A}_1x^2+\mathrm{C}_1y^2,$$

the invariant vanishes. Two corresponding points are therefore connected by the relation (between their distances $x'$, $x$ from origin)

$$2\,(ac-b^2)\,x'x+(ad-bc)\,(x'+x)+2\,(bd-c^2)=0,$$

and the foci of the involution are determined by

$$(ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2=0,$$

*i. e.*, by the Hessian itself of $u$;* while the centre is given by

$$2\,(ac-b^2)\,x+(ad-bc)\,y=0.$$

The above formulæ are to be understood as applying to the involution in which the corresponding points are also harmonic conjugates. There are three other involutions determined by the six points; but as these are not analytically distinguishable from one another, the equations giving their foci and centres would be an irresolvable sextic and cubic respectively, into the investigation of which it would occupy too much space here to enter.

* I am not aware that this interpretation of the Hessian of a binary cubic has been previously noticed. Other interpretations have been given by Professors J. H. S. Smith (orally to the London Mathematical Society) and Cayley (Quarterly Journal, Vol. X., p. 148).

19. As an example of a question in cubic curves to which the condition (22) may be applied, suppose it be required

"To find the locus of a point (P) such that, when tangents are drawn from it touching a cissoid in $T_1, T_2, T_3$, and meeting the same curve again in $t_1, t_2, t_3$, the pencil joining the cusp (O) with these six points may be in involution, each of the three lines $O(T_1 T_2 T_3)$ corresponding to one of the three $O(t_1 t_2 t_3)$."

The Cissoid being $x^3 - y^2 z = 0$, the equation to $O(T_1, T_2, T_3)$ is [Proceedings, Vol. II., p. 162 (8)], $x'y'z'$ being the point P,

$$z'y^3 - 3x'x^2y + 2y'x^3 = 0;$$

that to the three lines $O(t_1 t_2 t_3)$ [*ib.* p. 164 (17)]

$$4z'y^3 - 3x'x^2y - y'x^3 = 0.$$

Here $\Delta = 4z'(y'^2z' - x'^3), \quad \Delta' = 16z'(y'^2z' - x'^3);$

$\Theta$ reduces to $ad\,(ad' + a'd) + 2c^2\,(3ac' + a'c)$, when $b = 0$, $b' = 0$;

and $\Theta'$ „ $a'd'(ad' + a'd) + 2c'^2(ac' + 3a'c)$, „ „ „

so that here $\Theta = 14z'(x'^3 - y'^2z'), \quad \Theta' = 2z'(13x'^3 + 14y'^2z');$

and $\Delta\Theta'^2 - \Delta'\Theta^2 = 432x'^3z'^3\,(x'^3 - y'^2z')\,(x'^3 - 28y'^2z').$

Rejecting the irrelevant factors, the required locus is therefore

$$x^3 - 28y^2z = 0,$$

a curve generated from the cissoid by dividing its ordinates in a constant ratio.

The problem of finding the envelope of a transversal cutting two cubic curves in six points in involution, again, would be solved by substituting in them for $z$ from the equation to the transversal. Applying the condition (22) to the two resulting cubics in $x$ and $y$, the result would be the tangential equation to the required envelope, which would probably be of the 18th class.

Mr. W. K. Clifford read the following:—

## *On a Case of Evaporation in the Order of a Resultant.*

A particular case of the following theorem was required in the course of my proof that every rational equation has a root; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

*Theorem.* Let it be required to eliminate $x$ between two equations homogeneous in $x$ and certain other variables $y, z, \ldots$, in which equations, however, $x$ only occurs in virtue of the occurrence of a quantity $w = x^\alpha y^\beta z^\gamma \ldots$, where $\alpha + \beta + \gamma + \ldots = \mu$; let also $m, n$ be the orders of