The Electrical Distribution induced on a Circular Disc placed in any Field of Force. By H. M. Macdonald. Read February 14th, 1895. Received, in revised form, April 22nd, 1895.

The potential due to the inducing system can be expanded in a series of the form

$$\Sigma\Sigma\, A_{n\mu} J_n(\mu r)\cos(n\phi+a_\mu)$$

for points on the disc, and the component of the force due to it perpendicular to the plane of the disc can be represented by a similar series. It will be sufficient to obtain the potential V due to the induced charge for the term of the series $J_n(\mu r)\cos(n\phi+a_\mu)$, where n and μ are unrestricted. The solution of the problem has already been obtained in two particular cases, when $n=0$ (Gallop, *Quarterly Journal*, Vol. xxi., p. 229) and $n=1$ (Basset, *Camb. Phil. Soc. Proc.*, Vol. v., p. 425).

1. *To Determine a Function W_n such that for Points on the Disc*

$$W_n = J_n(\mu r),$$

and for Points in its Plane not on it

$$\frac{\partial W_n}{\partial z}=0.$$

Let I denote the integral

$$\int_0^\infty e^{-\kappa z} J_{n-\frac{1}{2}}(\kappa r') J_n(\kappa r)\,\kappa^{\frac{1}{2}}\,d\kappa\,;$$

then, if $r > r'$,

$$I=\frac{J_{n-\frac{1}{2}}\left(r'\dfrac{\partial}{\partial z}\right)}{\left(\dfrac{\partial}{\partial z}\right)^{n-\frac{1}{2}}}\int_0^\infty e^{-\kappa z}\kappa^n J_n(\kappa r)\,d\kappa,$$

that is,
$$I=\frac{2^n r^n\,\Pi\left(n-\frac{1}{2}\right)}{\sqrt{\pi}}\,\frac{J_{n-\frac{1}{2}}\left(r'\dfrac{\partial}{\partial z}\right)}{\left(\dfrac{\partial}{\partial z}\right)^{n-\frac{1}{2}}}\,\frac{1}{(r^2+z^2)^{n+\frac{1}{2}}},$$

since $$\int_0^\infty e^{-\kappa z}\kappa^n J_n(\kappa r)\,d\kappa = \frac{\Pi(n-\frac{1}{2})}{\sqrt{\pi}}\,\frac{2^n r^n}{(r^2+z^2)^{n+\frac{1}{2}}}$$

(Sonnine, *Math. Ann.*, Bd. XVI.).

Hence, when $z=0$,

$$I_0 = \frac{\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,r^n r'^{n-\frac{1}{2}}\int_0^\pi \frac{\sin^{2n-1}\theta\,d\theta}{(r^2-r'^2\cos^2\theta)^{n+\frac{1}{2}}},\quad r>r'$$

that is, $$I_0 = \frac{2\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,\frac{r^n}{r'^{n-\frac{1}{2}}}\int_0^{r'}\frac{(r'^2-x^2)^{n-1}}{(r^2-x^2)^{n+\frac{1}{2}}}\,dx,$$

or $$I_0 = \frac{2\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,\frac{1}{r^n r'^{n-\frac{1}{2}}\sqrt{r^2-r'^2}}\int_0^{r'}\frac{y^{2n-1}\,dy}{\sqrt{r'^2-y^2}},$$

whence $$I_0 = \sqrt{\frac{2}{\pi}}\,\frac{r'^{n-\frac{1}{2}}}{r^n\sqrt{r^2-r'^2}},$$

when $r>r'$;

also $$\left(\frac{\partial I}{\partial z}\right)_0 = 0,$$

when $r>r'$.

Again, when $r<r'$,

$$I = \frac{J_n\left(r\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^n}\int_0^\infty e^{-\kappa z}\kappa^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa r')\,d\kappa;$$

that is, $$I = \frac{2^{n+\frac{1}{2}}\,\Pi(n)}{\sqrt{\pi}}\,r'^{n-\frac{1}{2}}\,\frac{J_n\left(r\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^n}\,\frac{z}{(r'^2+z^2)^{n+1}};$$

therefore $$I_0 = 0,$$

and $$\left(\frac{\partial I}{\partial z}\right)_0 = \sqrt{\frac{2}{\pi}}\,\frac{r^n}{r'^{n-\frac{1}{2}}(r'^2-r^2)^{\frac{3}{2}}},\quad r<r'.$$

Writing

$$W_n = \int_0^\infty e^{-\kappa z}d\kappa\int_0^a J_{n-\frac{1}{2}}(\mu r')\,J_{n-\frac{1}{2}}(\kappa r')\,J_n(\kappa r)\,r'\kappa^{\frac{1}{2}}\mu^{\frac{1}{2}}dr',$$

when $z=0$, and $r<a$,

$$W_n = \int_0^r J_{n-\frac{1}{2}}(\mu r')\, r' \mu^{\frac{1}{2}} \sqrt{\frac{2}{\pi}}\, \frac{r'^{n-\frac{1}{2}}}{r^n \sqrt{r^2-r'^2}}\, dr',$$

from the foregoing; that is,

$$W_n = \sqrt{\frac{2\mu r}{\pi}} \int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(\mu r \sin\theta) \sin^{n+\frac{1}{2}}\theta\, d\theta\,;$$

therefore $$W_n = J_n(\mu r),$$

when $z=0$ and $r<a$.

Again, when $z=0$ and $r<a$,

$$\frac{\partial W_n}{\partial z} = -\int_0^\infty d\kappa \int_0^a J_{n-\frac{1}{2}}(\mu r')\, J_{n-\frac{1}{2}}(\kappa r')\, J_n(\kappa r)\, \kappa^{\frac{3}{2}} \mu^{\frac{1}{2}} r'\, dr'\,;$$

that is,

$$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) + \int_0^\infty d\kappa \int_a^\infty J_{n-\frac{1}{2}}(\mu r')\, J_{n-\frac{1}{2}}(\kappa r')\, J_n(\kappa r)\, \kappa^{\frac{3}{2}} \mu^{\frac{1}{2}} r'\, dr',$$

or $$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) - \int_a^\infty \sqrt{\frac{2}{\pi}}\, \frac{r^n}{r'^{n-\frac{1}{2}}(r'^2-r^2)^{\frac{3}{2}}}\, J_{n-\frac{1}{2}}(\mu r')\, \mu^{\frac{1}{2}} r'\, dr',$$

from the above. Hence

$$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) - \sqrt{\frac{2\mu}{\pi}}\, r^n \int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')\, dr'}{r'^{n-\frac{3}{2}}(r'^2-r^2)^{\frac{3}{2}}},$$

when $z=0$ and $r<a$.

Similarly, when $z=0$ and $r>a$,

$$W_n = \sqrt{\frac{2\mu r}{\pi}} \int_0^{\sin^{-1}(a/r)} J_{n-\frac{1}{2}}(\mu r \sin\theta) \sin^{n+\frac{1}{2}}\theta\, d\theta,$$

$$\frac{\partial W_n}{\partial z} = 0.$$

Hence, if the potential at the point r, ϕ of the disc (radius a) is

$$A_{n\mu} J_n(\mu r) \cos(n\phi + a_\mu),$$

the potential at any point on the positive side of the plane of the disc is given by

$$A_{n\mu} W_n \cos(n\phi + a_\mu),$$

where W_n is the integral defined above, and the potential on the negative side is obtained from this by changing the sign of z.

The surface density of the distribution on either side of the disc necessary to produce this potential is given by

$$\sigma = \frac{A_{n\mu}\cos(n\phi+a_\mu)}{4\pi}\left\{\mu J_n(\mu r)+\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')\,dr'}{r'^{\,n-\frac{1}{2}}(r'^2-r^2)^{\frac{1}{2}}}\right\};$$

for the case $n=0$, this agrees with the result found by Gallop, *Quarterly Journal*, Vol. XXI., p. 234.

2. *To find the Potential at any Point due to any Inducing System.*

It will be sufficient to consider the case where the inducing system lies wholly on the positive side of the disc; in this case, for points in the immediate neighbourhood of the plane of the disc, the potential of the inducing system can be represented by a series of the form

$$\Sigma\Sigma\, A_{n\mu}\,e^{\mu z}\,J_n(\mu r)\cos(n\phi+a_\mu) \equiv V_1.$$

Let V be the potential due to the induced charge on the disc; then over the disc

$$V+V_1=0,$$

and it is easy to verify that

$$V=-\Sigma\Sigma\, A_{n\mu}\,W_n\cos(n\phi+a_\mu),$$

for this satisfies the above condition, and makes $\frac{\partial V}{\partial z}$ continuous for all points in the plane of the disc not on it. The density of the distribution induced on the positive side of the disc is given by

$$\sigma=-\frac{1}{4\pi}\frac{\partial}{\partial z}(V+V_1);$$

that is,

$$\sigma=-\frac{1}{4\pi}\Sigma\Sigma\, A_{n\mu}\left\{2\mu J_n(\mu r)+\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')}{r'^{\,n-\frac{1}{2}}(r'^2-r^2)}\,dr'\right\}$$
$$\times\cos(n\phi+a_\mu);$$

the density on the negative side is given by

$$\sigma=-\frac{1}{4\pi}\Sigma\Sigma A_{n\mu}\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')}{r'^{\,n-\frac{1}{2}}(r'^2-r^2)^{\frac{1}{2}}}\,dr'\cos(n\phi+a_\mu).$$

Thursday, March 14th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Franklin Pierce Matz, M.A., M.Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, U.S.A., was elected a member.

The President announced that he had written letters of condolence to Lady Cockle and Mrs. Cayley, and had received their acknowledgements of receipt of the same, which he communicated to the meeting.

Professor Hill read a paper by Mr. F. H. Jackson, entitled "Certain II Functions," and the President (Mr. Kempe, Vice-President, in the Chair) communicated a paper on "The Perpetuant Invariants of Binary Quantics." Lt.-Col. Cunningham gave a proof that $2^{107}-1$ is divisible by 7487.

The President read a letter from Rev. T. C. Simmons announcing what the writer believed to be a "New Theorem in Probability."

The following presents were received:—

"Proceedings of the Royal Society," No. 342.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 2; Leipzig, 1895.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. IX., No. 2; 1894-95.

"Proceedings of the Physical Society of London," Vol. XIII., Pt. 4; March, 1895.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 2; Coimbra, 1895.

"Bulletin des Sciences Mathématiques," Tome XIX., Mar., 1895; Paris.

"Bulletin de la Société Mathématique de France," Tome XXII., No. 10; Paris, 1895.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 5; New York, 1895.

"Rendiconti del Circolo Matematico di Palermo," Tomo IX., Fasc. 1 and 2; 1895.

Brioschi, F.—"Notizie sulla Vita e sulle Opere di A. Cayley," 4to pamph.; Roma, 1895.

Donisthorpe, W.—"A System of Measures," 4to; London, 1895. From the Author.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 39-53; Oct., 1894 to Dec., 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 2, 3, 4; Roma, 1895.

Lie, Sophus.—"Untersuchungen über Unendliche Continuirliche Gruppen," Roy. 8vo; Leipzig, 1895.

"Educational Times," March, 1895.

"Annales de la Faculté des Sciences de Toulouse," Tome IX., Fasc. 1; Paris, 1895.

"Philosophical Transactions of the Royal Society," Vol. CLXXXV., Pt. 1.

"Indian Engineering," Vol. XVII., Nos. 4-7.

The Perpetuant Invariants of Binary Quantics. *By* Major P. A. MacMahon, R.A., F.R.S. Read March 14th, 1895. Received 16th May, 1895.

It was in Vol. v. of the *American Journal of Mathematics* that Sylvester first proposed the problem of the enumeration of the perpetuants of given degree and weight.* Of a given degree Cayley's rule gives a generating function which enumerates the asyzygetic seminvariants. A knowledge of the perpetuants of lower degrees leads to the generating function for the compound seminvariants of the given degree. Since these forms are not linearly independent, it is necessary to find the generating function of the syzygies which connect them. We have, then, the means for arriving at the generating function of the perpetuants. It is merely necessary to subtract the generating function of the syzygies from that of the compound forms, and then subtract the difference from that of the asyzygetic forms. This procedure was adopted by Sylvester. For the first four degrees no syzygies arise, and the perpetuant generating functions were found to be

$$x^0,\quad \frac{x^2}{1-x^2},\quad \frac{x^3}{(1-x^2)(1-x^3)},\quad \frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},$$

respectively; the enumeration of the perpetuants being given, for a weight w, by the coefficient of x^w in the developments.

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* "On Sub-Invariants, *i.e.*, Semi-Invariants to Binary Quantics of Unlimited Order," *Amer. Math. Jour.*, Vol. v., p. 79.