An Essay on the Geometrical Calculus — (*continuation*). *By* EMANUEL LASKER. Received May 3rd, 1897. Communicated May 13th, 1897, by Mr. TUCKER.

In the former part of this essay* the calculus of plane spaces and plane forms has been established. It was shown that the symbols of plane spaces and plane forms may be composed with each other as if the operation of composition denoted multiplication only; that $\xi\eta$ is not $= \eta\xi$ always, but $= \pm\eta\xi$, according to the rule of signs. The conceptions of the normal form of a space, of the plane space I at infinity and the spherical manifoldness $\mathfrak{J}$ at I, have been introduced, and a multitude of metrical relations shown to take their origin therefrom. In this part, homogeneous algebraical forms of the plane space symbols are considered and shown to be algebraically equivalent to the algebraical formations of geometry. A sign $\times$ (and reciprocally $*$) is introduced as an extension of the conception of composition, and its principal laws are discussed. Some properties of the intersections of surfaces are explained, especially of the group of points common to k surfaces in the space S_k. Finally, a few applications are given to show that the symbolism used will yield good results without much effort.

Let the space in which we operate be a straight line l. Let $A, B, \ldots L$ denote any point on that line, $\mathfrak{G}$ a group of such points, for instance, A, B, and C. Then $\mathfrak{G}$ will be denoted by $A.B.C$, where the symbol . is expressive of the fact that the various points thus connected are to be considered as a group, collectively. The number of such points is called the *order* of the group. In the natural extension of the symbolism used for plane spaces, $[\mathfrak{G}D]$, or sometimes simply $\mathfrak{G}D$, where D may be any point on l, will denote the magnitude which is the product of the various magnitudes formed by the points of the group $\mathfrak{G}$ with D;

$$\mathfrak{G}D = AD.BD.CD.$$

Any equation between groups such as

$$\mathfrak{G} + \mathfrak{G}' = \mathfrak{G}'',$$

or

$$\mathfrak{G}.\mathfrak{G}' = \mathfrak{G}'',$$

* Pp. 217–260 *supra*.

signifies that both the right- and left-hand sides composed with an arbitrary point symbol on l are equal. If $\mathfrak{G}$ contains the point A,

then $$\mathfrak{G}A = 0,$$

and inversely, if $$\mathfrak{G}A = 0,$$

then $\mathfrak{G}$ must contain the point A, since a product can only vanish when any one of its factors vanishes. If P, Q are any two points on the line l, and λ, μ two parameters,

$$\mathfrak{G}(\lambda P + \mu Q) = 0$$

will be an equation for λ, μ whose roots determine the position of the points $\lambda P + \mu Q$ of the group $\mathfrak{G}$. The definitions given allow us therefore to treat such groups $\mathfrak{G}$ as algebraical forms of two homogeneous variables and to reduce any equation between groups to algebraical identities.

Let u be any group, or, as we shall sometimes say, *point-form*, of the n^{th} order. Let $A_1, \dots A_n$, be any n points on l, and $a_1, \dots a_n$, any constants. Then $u(a_1A_1 + \dots + a_nA_n)$ is a magnitude determined by u, the position of the A, and expressible as a homogeneous rational integral function of the $a_1, \dots a_n$, of the n^{th} order. It will therefore contain a term $Ca_1, \dots a_n$, where C is a magnitude determined by u and the points $A_1, \dots A_n$, alone. We denote the $n!^{\text{th}}$ part of C by $u \times A_1 . A_2 . \dots A_n$. The sign $\times$ is a symbol of operations whose properties we propose now to study. First of all, it is clear from its definition that

$$(u+u') \times A_1 . \dots A_n = u \times A_1 . \dots A_n + u' \times A_1 . \dots A_n.$$

Secondly, if we develop

$$u(a_1A_1 + \dots + a_nA_n + a_{n+1}A_{n+1})$$

as a homogeneous form of the a_i, we shall obtain altogether $n+1$ terms which linearly contain the $n+1$ parameters, the form being only of the n^{th} order; and, from the supposition

$$a_{n+1} = 0,$$

it is clear that the factor of $n!\, a_1 \dots a_n$ is again $u \times A_1 \dots A_n$; that of $n!\, a_1 \dots a_{n-1}a_{n+1}$ is therefore similarly $u \times A_1 \dots A_{n-1}A_{n+1}$, &c. If now we identify a_{n+1} with a_n, we obtain

$$u \times A_1 . A_2 . \dots A_{n-1}(A_n + A_{n+1}) = u \times A_1 . \dots A_{n-1}A_n + u \times A_1 . \dots A_{n-1} . A_{n+1}.$$

It follows then that the symbol $\times$ in regard to linear changes of the forms operated upon has all the properties of an ordinary multiplication symbol, and that therefore no error will be produced by treating the $A_1, \ldots A_n$ collectively, that is, as a group v.

The fact that $u \times v$ is a magnitude which changes linearly in a corresponding manner to the u and v will be expressed by saying that the operation $\times$ is distributive; or, in symbols, the operation $\times$ is distributive because

$$(\lambda u + \mu u') \times v = \lambda (u \times v) + \mu (u' \times v),$$

and

$$u \times (\lambda v + \mu v') = \lambda (u \times v) + \mu (u \times v'),$$

λ, μ denoting constants.

To give an instance, let

$$u = A.B, \quad v = C.D,$$

$$u(\alpha C + \beta D) = A(\alpha C + \beta D).B(\alpha C + \beta D)$$

$$= \alpha^2 AC.BC + \alpha\beta (AC.BD + AD.BC) + \beta^2 AD.BD;$$

therefore

$$u \times v = \tfrac{1}{2}(AC.BD + AD.BC).$$

Or let

$$u = A.B.C, \quad v = D.E.F;$$

then

$$u \times v = \tfrac{1}{6}(AD.BE.CF + AD.BF.CE + AE.BF.CD + AE.BD.CF + AF.BD.CE + AF.BE.CD).$$

If we transpose u and v, the magnitude $u \times v$ changes sign when n is odd, but remains unchanged when n is even. Therefore $u \times u$ is always 0 when n is odd, but may be distinct from 0 when n is even.

It is also immediately seen that uP, where P is any point on l, is the same as $u \times P^n$. Therefore $u \times P^n = 0$ only if P is one of the points of the group u.

$u \times v = 0$ is a single condition for the coefficients of u and v; if u therefore is fixed, $u = A_1 \ldots A_n$, then v restricted by the linear condition $u \times v = 0$ will only contain n independent parameters; and it therefore follows that

$$v = c_1 A_1^n + \ldots + c_n A_n^n,$$

where the c are arbitrary constants.

When v is of lower degree than u, say of the m^{th}, $n - m$ being $= \lambda$, then $v \times u$ will be represented by the form w,

$$v \times u = w,$$

so that identically $$v.P^{\lambda}\times u = wP,$$
where P remains arbitrary.

Let, for instance, $$u = A.B,$$
and $$v = C;$$
then $$v\times u = \tfrac{1}{2}(CA.B+CB.A).$$

$v\times u$ may also be defined as that form which is represented by the group of the points P for which
$$v.P^{\lambda}\times u = 0.$$
If we replace, in the equation
$$v.P^{\lambda}\times u = wP,$$
P by $a_1.P_1+\ldots+a_{\lambda}.P_{\lambda}$, and, in the development of $v.P^{\lambda}\times u$ and wP as rational integral functions of the a, equate the respective coefficients of $a_1\ldots a_{\lambda}$ to each other, then we obtain, denoting further $P_1.\ldots P_{\lambda}$ by t,
$$v.t\times u = w\times t.$$

If v and u are both of the same order, v and u will be called conjugate or harmonic to each other whenever
$$v\times u = 0.$$
If v is of inferior order to u, then $v\times u$ will be spoken of as the *polar* of v with respect to u, and, if this polar vanishes identically (equivalent to $\lambda+1$ conditions), then v will be called *apolar* with respect to u.

If u has a double point A, n being the order of u, then
$$A^{n-1}\times u = 0,$$
as is immediately clear from its manner of formation. This equation is equivalent to two conditions; therefore it is also the general condition for A to be a double point of u. In the same manner
$$A^{n+1-\lambda}\times u = 0,$$
whenever A is a λ-fold point of u.

If $$u = A^{\lambda_1}B^{\lambda_2}\ldots L^{\lambda_k},$$
$\lambda_1+\lambda_2+\ldots+\lambda_k$ being of course $=n$, and, if v be a form of the n^{th} order, such that
$$u\times v = 0,$$

then v must be of the form

$$A^{n+1-\lambda_1} v' + B^{n+1-\lambda_2} v'' + \ldots + L^{n+1-\lambda_k} v^{(k)},$$

where $v', v'', \ldots v^{(k)}$ are arbitrary forms of orders $\lambda_1 - 1, \lambda_2 - 1, \ldots \lambda_k - 1$, respectively. For this form satisfies the equation

$$u \times v = 0,$$

and contains

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = n;$$

that is the requisite number of arbitrary constants.

If

$$v \times u = 0,$$

where the order of v (m) is smaller than that of u (n), and

$$v = A^{\lambda_1} \ldots L^{\lambda_k},$$

then

$$v \times u = 0$$

restricts u $(n-m+1)$-fold; u contains therefore only m arbitrary constants. Hence

$$u = A^{n+1-\lambda_1} u' + B^{n+1-\lambda_2} u'' + \ldots,$$

where again $u', u'', \ldots$ are arbitrary forms of orders $\lambda_1 - 1, \lambda_2 - 1, \ldots$, respectively.

If u is given of the n^{th} order, v any form of the m^{th} order, then

$$v \times u = 0,$$

being equivalent to $n-m+1$ conditions, restricts the $m+1$ constants of v $(n-m+1)$-fold, so that $2m-n$ arbitrary constants remain. One factor being necessarily arbitrary, $2m-n$ must be at least $= 1$; so that a general form u of the n^{th} order where n is odd $= 2m+1$ determines exactly one form v of order $\frac{n-1}{2}$, apolar to u; v can be found by the solution of $\frac{n-1}{2}$ linear equations. v being found, we may express u in a specially simple manner; for instance, if v is the product of $\frac{n-1}{2}$ distinct points, it may be expressed as the sum of $\frac{n-1}{2}$ n^{th} powers of multiples of these points. This expression for u is sometimes called its *canonical* form.

If $u_1, \ldots u_k$ are forms of the n^{th} order, then any form expressible by $c_1 u_1 + \ldots + c_k u_k$ is said to belong to the *involution* $u_1 \ldots u_k$; and

$k-1$ is called its degree of manifoldness. We suppose, of course, that no such identity as

$$c_1 u_1 + \ldots + c_k u_k = 0$$

exists (that the u_i are *linearly independent*). The k equations

$$u_1 \times v = 0, \quad u_2 \times v = 0, \quad \ldots \quad u_k \times v = 0$$

restrict the form v, supposed to be of the n^{th} order, k-fold; v will therefore be a member of an involution whose degree of manifoldness will be $n-k$, and which may well be called the *reciprocal* of the involution $u_1 \ldots u_k$, or the involution *conjugate* to the involution of the u. By means of this conception many truths concerning involutions can immediately be derived from truths known to hold for reciprocal involutions. For example, let

$$k = n;$$

the involution of the v will be of manifoldness 0, and v will be uniquely determined. It is therefore immediately seen that, in general, an involution of manifoldness $n-1$ will contain n n^{th} powers (of points belonging to v); and that the group of these will be harmonic to all members of the involution $u_1 \ldots u_n$, which may also be defined by this property. Or let the degree of manifoldness of the involution $u_1 \ldots u_k$ be $n-2$, and let v_1, v_2 form the conjugate involution. Then the condition that the involution $u_1 \ldots u_k$ should contain an n^{th} power is equivalent to the condition that v_1 and v_2 should have a point in common.

The equation

$$v.t \times u = (v \times u) \times t$$

may be geometrically illustrated as follows. Let v be a point-group of order m, u of order n. Then all groups of order n conjugate to u and comprising the m points v form an involution; and so do the system of the $n-m$ points which combined with v are conjugate to u. This latter involution is reciprocal to a certain group of order $n-m$, which is exactly $v \times u$, the polar of v to u.

To obtain these results it is not in the least essential that the symbols used should be points situated on a certain line. They may be any point or plane space symbols situated anywhere. For the definitions given for the operation $\times$ will again apply. If u is a form of order n of S_a symbols, ξ any plane space, $u\xi$ will denote the result of the composition of all terms contained in u with ξ. Let, for instance, u be a point-form in a plane S_2, and let l be any line

$$l = a_1 l_1 + \ldots + a_n l_n;$$

then $u\,(a_1 l_1 + \ldots + a_n l_n)$ will be a magnitude developable as a rational integral function of the a, and will contain a term $a_1 \ldots a_n$, whose coefficient will depend only upon the mutual situation of u and the l_i, and whose $n!^{\text{th}}$ part may be denoted by $u \times l_1 l_2 \ldots l_n$. It follows in the same manner as before that any linear change of the l may be treated as if the sign $\times$ denoted multiplication, and that the l may be treated as if they were multiplied together. It is true that the line-forms v in a plane cannot generally be represented as the product of lines, but nothing prevents us from extending our definitions also to general forms v, since v can be represented as the sum of such products. Hence $u \times v$ is a magnitude uniquely determined by the point-form u and the line-form v; and the operation $\times$ is distributive. To give an instance, let

$$u = A.B + C.D,$$

and

$$v = a.b + c.d,$$

the A, B, C, D denoting points; the a, b, c, d denoting lines. Then $u \times v$ will be

$$u \times v = \tfrac{1}{2}\,(Aa.Bb + Ab.Ba + Ca.Db + Cb.Da + Ac.Bd + Ad.Bc + Cc.Dd + Cd.Dc).$$

Or let u be a point-form of the order n, and P any point in the plane of operation. Putting

$$P = a_1 P_1 + \ldots + a_n P_n,$$

uP will be a line-form developable as a power-series of the a; whose coefficients are again line-forms. The coefficients of $a_1 \ldots a_n$ will be a line-form which depends solely upon the situation of u and $P_1 \ldots P_n$. Since the calculus is just the same whatever the symbols may signify, whether lines or points, the results obtained will also be the same. $u \times v$, where u and v are any two point-forms of order n, is therefore a line-form uniquely determined by u and v, and the operation $\times$ is also in this instance distributive.

After these explanations, we may announce the general result. The operation $\times$ is applicable to forms of symbols of any manifoldness. If u is of order n, $u\xi$ is the same as $u \times \xi^n$. The operation $\times$, which in the reciprocal geometry will be written $*$, is an extension of that of composition, and is always distributive; $u \times v$ differs from $v \times u$ either not at all or only in sign.

The result of $u \times v$ polarized with any new form w, written $(u \times v) \times w$, can only differ in sign, if at all, from $u \times (v \times w)$ or $(u \times w) \times v$, &c. It suffices to show this when u, v, w are products of plane forms.

Let, for instance,

$$u = \xi_1 . \xi_2,$$
$$v = \eta_1 . \eta_2,$$
$$w = \zeta_1 . \zeta_2,$$

where the ξ, η, ζ are any plane space symbols. Then

$$u \times v = \tfrac{1}{2} (\xi_1\eta_1 . \xi_2\eta_2 + \xi_1\eta_2 . \xi_2\eta_1)$$

and

$$(u \times v) \times w = \tfrac{1}{4} (\xi_1\eta_1\zeta_1 . \xi_2\eta_2\zeta_2 + \xi_1\eta_1\zeta_2 . \xi_2\eta_2\zeta_1 + \xi_1\eta_2\zeta_1 . \xi_2\eta_1\zeta_2 + \xi_1\eta_2\zeta_2 . \xi_2\eta_1\zeta_1).$$

The law of formation makes the statement immediately obvious, so that the wording of the proof seems unnecessary.

We come now to the problem of finding geometrical equivalents to the algebraical forms introduced and identities established. A certain liberty of choice will always exist, but the conception of the old geometers cannot in any way be improved upon. Accordingly, if $h+1$ is the manifoldness of the space containing all symbols of a form u, whose symbols may be S_h forms, then u will be represented by the manifoldness of points P for which uP vanishes, and u will be called a surface. And reciprocally, if u is a point-form, it will find its geometrical equivalent in the manifoldness of all spaces Σ of h manifoldness for which $u\Sigma$ will vanish. But, if u can be represented as the product of points, this group of points—each point counted in its proper multiplicity—is a more direct representation of u. In any case, a S_h form or point-form u and its geometrical equivalent determine each other uniquely, if an arbitrary factor is left out of consideration.

If u is a form of S_a symbols, where a differs from 0 or h, then we might in the same manner represent it by the manifoldness of plane *forms* Σ of manifoldness $h-a$, for which

$$u\Sigma = 0,$$

and to obtain a visible representation we should have to define Σ again by a number of plane *spaces* Σ' of manifoldness a belonging to the involution reciprocal to Σ. So then u might ultimately find its geometrical equivalence in the manifoldness of a group of plane spaces Σ', which stand in a certain relation to each other, in virtue of which the reciprocal Σ to the involution determined by them makes $u\Sigma$ vanish. This representation would certainly have the advantage of defining u uniquely, a factor being left out of consideration. But it is wholly unsuited to aid the geometrical imagination, especially when u is itself defined as the intersection of surfaces or as a curve or geometrical formation of some kind in general. For this reason we must represent a form u by the manifoldness of plane *spaces* Σ, for which

$$u\Sigma = 0.$$

Remembering that certain relations are identically satisfied by the coordinates of a plane space other than a point or a S_h—in space, for instance, if A, B, C, D represent a pyramid,

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC$$

applied to *any* line would vanish—we must bear in mind, that certain forms $\mathfrak{S}_1 \dots \mathfrak{S}_k$ will in the light of the above definition vanish identically. The definition given above will therefore create a correspondence between u and its geometrical equivalent only modulo $\mathfrak{S}_1 \dots \mathfrak{S}_k$. This fact, however, does not touch the validity of our equations, if it is understood that they are always to be read modulo, the fixed system of moduli $\mathfrak{S}_1 \dots \mathfrak{S}_k$.

If, then, u is a geometrical formation, the manifoldness of Σ belonging to it will be that of those spaces Σ which have with u a point in common; for this is true if u reduces to a product of plane spaces, and, since u can always be represented as the sum of such products, in many ways, the general truth of the proposition is easily made evident. For that reason a geometrical formation $\mathfrak{C}$ will define the form u of which it is the representative (modulo $\mathfrak{S} \dots \mathfrak{S}_k$), and we might consequently apply our equations directly to the formations whose symbols we use.*

The geometrical representation which we now have agreed to use will immediately lead to some notable consequences. Let, for instance, the space in which we operate be the plane. A point-group of the second order, such as

$$u = aA^2 + bB^2 + cC^2,$$

will, in general, not be represented by two points, unless its discriminant abc vanishes.

If we compose u with any point P,

$$uP = a \,.\, AP^2 + b \,.\, BP^2 + c \,.\, CP^2,$$

* [A geometrical "formation," as the word is used in this essay, denotes always a manifoldness of points, forming a curve, a surface, &c., in space of any degree of manifoldness. According to the principle established above, an algebraical form (of the symbols used in the geometrical calculus) will correspond to it. It is, however, easily seen that the converse is not true. A form like, for instance, $aAB^2 + bCD^2 + cEF \,.\, GH$, where A, B, C, D, E, F, G, H may be situated in space S_3, will not generally define a conic in space, but a line-manifoldness of the second order. Hence it will be understood that we may speak of algebraical forms which exist as geometrical formations and of such as do not.]

this will represent a line-group which will obviously contain P. If this line-group contains also some other point Q, then

$$uPQ = 0,$$

from which it follows that PQ belongs to the manifoldness of straight lines represented by u. In other words, uP represents the two lines through P which belong to the manifoldness u. Let in a similar manner u be a line-form, a curve, in the plane, and l any line; then in the reciprocal geometry u/l will denote the point-group which u and l have in common.

If $\mathfrak{C}$ is a curve in space S_3, $\mathfrak{C}P$ must denote, according to the definitions given, the manifoldness of points Q for which

$$\mathfrak{C}PQ = 0,$$

which is evidently the cone standing on $\mathfrak{C}$ whose vertex is P. If S is any plane in space, $\mathfrak{C}/S$ will similarly denote the point-group common to $\mathfrak{C}$ and S. Generally, if $\mathfrak{C}$ is any geometrical formation, ξ any space, P any point on $\mathfrak{C}$, then $\mathfrak{C}\xi$ will contain the space $P\xi$; for, if Σ has the point Q in common with $P\xi$, then

$$\xi\Sigma = 0;$$

therefore

$$\mathfrak{C}\xi\Sigma = 0.$$

In the same manner $\mathfrak{C}/\xi$ denotes the intersection of $\mathfrak{C}$ by ξ.

To give an instance of the working of this, let u be a conic, and a, b be any two lines in its plane intersecting in P. Through P draw any third line $\lambda a+\mu b$. Its two points of intersection with u are $u/\lambda a+\mu b$, or, what is equivalent to this, $u * (\lambda a+\mu b)^2$. If L and M are these two points of intersection, then it follows that

$$L \,.\, M = \lambda^2 \,.\, \mathfrak{A}+\lambda\mu \,.\, \mathfrak{B}+\mu^2 \,.\, \mathfrak{C},$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are fixed point-forms of the second order. So, then, if $L_1 \,.\, M_1$, $L_2 \,.\, M_2$, $L_3 \,.\, M_3$, $L_4 \,.\, M_4$ are any four such point-groups, a linear relation will connect them; and, if we compose this with L_4, it appears that

$$L_1 L_4 \,.\, M_1 L_4, \quad L_2 L_4 \,.\, M_2 L_4, \quad L_3 L_4 \,.\, M_3 L_4$$

are linearly dependent. Or, in other words, the three point-pairs which are common to a conic u and three concurrent lines projected from any point of u are in involution.

Or, let u be a surface of the second order in space, and l be any line upon it. Any plane through l will be of the form $\lambda a+\mu b$, a and b

representing any two distinct planes through l. This plane will cut u in another line l'. Now

$$u/\lambda \dot{a} + \mu b \equiv l \, . \, l',$$

or, what is immediately seen to be equivalent to this, four such lines l' are linearly dependent. The lines of the quadric, as easily follows, form two groups, belonging to two conjugate involutions of plane line-forms of manifoldness 2.

This proceeding, which is also applicable to the intersection of curved spaces, gives many interesting results, and can be generalized without difficulty.

If u and v are given forms, what does $u \times v$ signify? To answer this question for the plane will be sufficient to indicate the general idea. Let u and v both be point-forms of the n^{th} order in the plane. $u \times v = w$ is then a line-form of the n^{th} order. Let P be any point upon the curve w; therefore

$$w \times P^n = 0;$$

then we have, since

$$w = u \times v,$$

$$u \times v \times P^n = 0.$$

In the geometry of lines through P, AP composed with BP is, according to our previous definitions, identical with ABP in point geometry. It therefore follows that $u \times v \times P^n$ in point geometry is the same as $uP \times vP$ in the geometry of lines through P. Hence $u \times v$ is the locus of points P which have the property that the two line-groups uP and vP are conjugate to each other.

As a corollary, if u is a conic, $u \times u$ is its reciprocal. If u is a point-group of the third order on a line, $u^2 \times u^2$ is its discriminant. Hence, u representing a plane curve of the third order, $u^2 * u^2$ is the point-form for the tangent lines of the cubic u. Similar laws exist for surfaces of any order.

We shall, of course, speak again of involutions of forms, and, in connexion with the equation $u \times v = 0$, of conjugate forms and conjugate and reciprocal involutions respectively.

Let u now be any *surface*, in the space S_k, and the order of u be n. Let A, B be any two points in S. The line AB will cut u in n points, to be found by evaluating u/AB, or else by the following method. Any point of the line AB may be represented by $\alpha A + \beta B$, α, β being parameters. For the coordinates α, β of the points $\alpha A + \beta B$ situated upon u, we obtain an equation of the n^{th} degree,

$$u \times (\alpha A + \beta B)^n = \alpha^n \, . \, u \times A^n + n \, . \, \alpha^{n-1} \, . \, \beta \, . \, u \times A^{n-1} \, . \, B + \ldots = 0.$$

If A is a point upon u, $\beta = 0$ will be one root of the equation, the equation of the other $n-1$ roots reducing to

$$n \,.\, a^{n-1} \,.\, u \times A^{n-1} \,.\, B + (n)_2\, a^{n-2} \beta \,.\, u \times A^{n-2} \,.\, B^2 + \ldots = 0;$$

$$u \times A^{n-1} \,.\, B = 0$$

is therefore the necessary condition for the line AB to have two consecutive points in A in common with u, to touch u. This equation is a single condition restricting the position of the point B. All the points B of this kind are therefore situated in a certain plane space of $k-1$ manifoldness, which is said to touch u at the point A. We may evolve its form by calculating $A^{n-1} \,.\, P \times u$, P being left undetermined, *i.e.*, by calculating the polar of A^{n-1} to u.

If, however, any line through A intersects u in two coincident points, then the equation for B must be an identity; *i.e.*, $A^{n-1} \times u$ must vanish identically, and, *vice versâ*, $A^{n-1} \times u = 0$ is the necessary and sufficient condition for A to be a double point upon u.

In the space of $k-1$ manifoldness $A^{n-1} \times u$, we may subject B to the further condition that $A^{n-2} \,.\, B^2 \times u$ should also vanish. In that case the line AB will have three consecutive points in common with u. The equation for B being then of the second order, the points B will form in the space $A^{n-1} \times u$ a surface of the second order. If C is any point upon AB, C^2 will be linearly dependent upon A^2, $A \,.\, B$, B^2. Hence, if

$$u \times A^n = 0, \quad u \times A^{n-1} \,.\, B = 0, \quad u \times A^{n-2} \,.\, B^2 = 0,$$

then generally $$u \times A^{n-2} \,.\, C^2 = 0.$$

The surface of the second order in question is therefore a cone whose vertex is A. If, further, we select upon that cone only such points B for which also

$$A^{n-3} \times B^3 \times u = 0,$$

then AB will have four consecutive points in common with u. These lines AB are therefore found by the intersection of the cone with a certain surface of the third order; or else by their property that they are wholly contained by the surface $A^{n-3} \times u$. Generally AB will have h consecutive points in common with u whenever it is wholly contained by the polar of A^{n+1-h} to u.

If A is a double point upon u, $A^{n-1} \times u$ will vanish identically, as we have seen. The points B for which $A^{n-2} \,.\, B^2 \times u = 0$ will then be such that AB has three coincident points in common with u. The equation for B being of the second order, such points form a surface of the second order, which is, as immediately follows from the co-existence of

$$A^n \times u = 0, \quad A^{n-1} . B \times u = 0, \quad A^{n-2} . B^2 \times u = 0,$$

a cone whose vertex is A—the cone of contact at A. If also $A^{n-2} \times u$ identically vanishes, then A will be a triple point upon u, and generally the condition for A to be a λ-fold point upon u is that $A^{n+1-\lambda} \times u$ should vanish identically, and the cone of contact is evidently $A^{n-\lambda} \times u$.

This is indeed only a restatement of Joachimsthal's method.

We have made use of the conception of the polar of a point-form v of order m to a surface u of order n. It may be defined as that surface w of order $n-m$ whose points P satisfy the condition

$$v . P^{n-m} \times u = 0.$$

As before, it follows, if $\quad v \times u = w,$

that $\quad v . t \times u = w \times t.$

If $v \times u$ identically $= 0$, v will be called apolar to u. This equation expressing the vanishing of a surface of the $(n-m)^{\text{th}}$ order is equivalent to a $(n+k-m)_k$-fold condition.

If $\mathfrak{C}$ is a curve in space, and l any line intersecting it, we shall have

$$\mathfrak{C} \times l^n = 0.$$

If, therefore, P is any point on l, Q any point in space,

$$\mathfrak{C} \times P^n \times Q^n = 0.$$

From this it follows that $\mathfrak{C} \times P^n$ (or $\mathfrak{C}P$) must identically vanish.

If l is any line intersecting $\mathfrak{C}$ twice, then $l^{n-1} . \mathfrak{C} = 0$ identically, and, if l has a points in common with $\mathfrak{C}$, then

$$l^{n+1-a} \times \mathfrak{C} = 0.$$

For let P be any point on l not on $\mathfrak{C}$, S any plane through l, and a, b any two lines through P lying upon S. Obviously there are n (different or coincident) lines through P in S intersecting the curve $\mathfrak{C}$. These lines are $\lambda a + \mu b$, where λ and μ are to be determined by the equation of the n^{th} order

$$\mathfrak{C} \times (\lambda a + \mu b)^n = 0.$$

Everything else follows as before.

If P is a double point upon $\mathfrak{C}$, then P^{n-1} will vanish identically, since $(PQ)^{n-1} \times \mathfrak{C}$ vanishes identically wherever Q may be situated. For a λ-fold point upon $\mathfrak{C}$, we have, in a similar manner,

$$P^{n+1-\lambda} \times \mathfrak{C} = 0.$$

Whenever $P^{n-1}.\,Q \times \mathfrak{C}$ vanishes identically, the line PQ will touch $\mathfrak{C}$ in P, and *vice versâ*. The curve $\mathfrak{C}$ is a singly infinite series of points. It is therefore possible to represent its points Π by a power-series in a variable parameter λ,

$$\Pi \equiv P + \lambda P' + \lambda^2 P'' + \ldots,$$

where P is one of the curve-points corresponding to the value $\lambda = 0$. Since

$$\Pi^n \times \mathfrak{C} = 0,$$

we may develop $\Pi^n \times \mathfrak{C}$ according to powers of λ, and equate each coefficient of the various powers of λ to zero. It therefore follows that

$$P^{n-1}.\,P' \times \mathfrak{C} = 0.$$

But for infinitesimal values of λ the point Π consecutive on $\mathfrak{C}$ to P may be considered as being situated upon the line joining P and P'. So then the statement is verified.

If Q is not a point on that tangent-line, then $P^{n-1}.\,Q \times \mathfrak{C}$ will represent a plane form of the order n, that is (in space S_3), a *surface*. This surface is represented by the cone of order $n-1$ whose vertex is P, and which contains the curve $\mathfrak{C}$, in conjunction with the plane composed of the tangent-line PP' and Q. For let R be any other curve point, and $aP + \beta R$ any point collinear with P and R; then

$$P^{n-1}.\,Q \times (aP + \beta R)^n$$

being identically $= n\,.\,a\beta^{n-1}\,(PR)^{n-1}.\,QP + \beta^n.\,P^{n-1}.Q \times R^n$,

and both $\mathfrak{C} \times R^n$ and $\mathfrak{C} \times (PR)^{n-1}$ vanishing identically,

$$\mathfrak{C} \times P^{n-1}.\,Q \times (aP + \beta R)^n = 0.$$

In other words, $\mathfrak{C} \times P^{n-1}.\,Q$ contains all the points $aP + \beta R$, *i.e.*, the cone in question. $\mathfrak{C} \times P^{n-1}.\,Q$ contains also any point

$$S \equiv aP + \beta P' + \gamma Q,$$

since $\mathfrak{C} \times P^{n-1}.\,Q$ identically $= \mathfrak{C} \times P^{n-1}.\,(\lambda P + \mu P' + Q)$,

and $P^{n-1}.\,S \times S^n$ vanishes identically.

The more complex formations of geometry may be treated in the same manner. We shall, however, for the present, abstain from rigidly formulating the general laws whose existence is indicated above.

II. A geometrical formation $\mathfrak{C}$ is either irreducible or it is complex. $\mathfrak{C}$ will be called irreducible if algebraically the form representing it is irreducible. If $\mathfrak{C}$ is algebraically reducible, it is geometrically complex. If $\mathfrak{C}$ is the product of several forms

$$\mathfrak{C} \equiv \mathfrak{C}' . \mathfrak{C}'' . \dots \mathfrak{C}^{(k)},$$

then, since $$\mathfrak{C}P = 0,$$

either $$\mathfrak{C}'P = 0 \quad \text{or} \quad \mathfrak{C}''P = 0, \ \&c.$$

$\mathfrak{C}' . \mathfrak{C}'' . \dots \mathfrak{C}^{(k)}$ containing an infinity of points, at least one of them, say $\mathfrak{C}'$, will contain an infinity of points. $\mathfrak{C}'$ will therefore geometrically exist. This line of thought being followed further makes it evident that, if a form $\mathfrak{C}$ which has existence as a formation is reducible, its various factors $\mathfrak{C}' . \mathfrak{C}'' . \dots \mathfrak{C}^{(k)}$ will also represent forms having geometrical existence; or that, in other words, a reducible formation $\mathfrak{C}$ is always a complex of several irreducible formations (some of which may be identical).

If $\mathfrak{C}$ is irreducible, a surface U contains $\mathfrak{C}$ when containing all of its points. U contains $\mathfrak{C}' . \mathfrak{C}''$ when it contains $\mathfrak{C}'$ as well as $\mathfrak{C}''$. U contains $\mathfrak{C}^{\lambda}$ when it contains $\mathfrak{C}$, and besides $\lambda - 1$ formations *consecutive* to $\mathfrak{C}$ (*not* necessarily coincident with $\mathfrak{C}$). But, if U contains every point of $\mathfrak{C}$ as a λ-fold point, then it may be said to contain $\mathfrak{C}$ λ-fold, *i.e.*, to contain λ formations *coincident* with $\mathfrak{C}$. This definition of "containing" will only be of importance when the work of Brill and Nöther is consulted.

Let now $\mathfrak{C}$ be an irreducible curve in any space S of order n. A surface U in S of order N will have $n . N$ points in common with it, since this is the number of points U would have in common with $\mathfrak{C}$ if U were the product of planes, and since this number must be independent of the exact values of the coefficients of U. But, if N is chosen large enough, $n . N$ will fall short of the degree of manifoldness of surfaces U of order N, and other surfaces U', U'', ... of order N will therefore exist which also contain these $n . N$ points of intersection.

If U and U' are any two of these surfaces, a surface $aU + bU'$ may be constructed where the constants a, b are so adjusted that

$$aU + bU' = V$$

will contain, besides the $n . N$ points, yet another of the points of $\mathfrak{C}$. $\mathfrak{C}$ will then have $n . N+1$ points in common with V, and, $\mathfrak{C}$ being irreducible, it is in keeping with one of the fundamental principles of Algebra to conclude that 𝔈 must be wholly contained by V.

It may in a similar manner be shown that generally an involution of surfaces of order N exists of which each member contains any given geometrical formation 𝔈, and which is defined by this property; on the supposition only that N is chosen large enough.

Referring to Dr. Salmon's classical treatises upon the theory of curves in space, and on the order of restricted systems of equations, we shall make use of the following fundamental proposition: That, if 𝔈 be the *complete* intersection of surfaces $u, v, \ldots w$, any surface S containing 𝔈 must be of the form

$$S = a\,.\,u+b\,.\,v+\ldots+c\,.\,w,$$

the $a, b, \ldots c$ denoting forms.

A proof of this proposition may be given as follows:—The proposition is true if the complete intersection of $u, v, \ldots w$ is a group of points, as is implicitly verified by the discussion which follows, upon the supposition of the truth of Bézout's theorem only. It is algebraically evident that in this instance the $a, b, \ldots c$ contain the coefficients of $u, v, \ldots w$ and S rationally. If, then, some of the variables in $u, v, \ldots w$ and S are treated as parameters, the truth of the proposition follows quite generally.

We may express the substance of this proposition by the statement that the geometrical substrate of the system of moduli $u, v, \ldots w$ whose resultant does not identically vanish is their complete intersection.*

The form of the complete intersection 𝔈 of the surfaces $u, v, w, \ldots$ can be found as follows:—Any $k+1$ surfaces in space S_k will have a point in common if a magnitude, the resultant of the system, vanishes. If, then, $u, v, \ldots w$ are h surfaces, join any $l = k+1-h$ arbitrary plane spaces $S_1, S_2, \ldots S_l$ to them. The resultant R of this system does not contain the coefficients of $S_1, S_2, \ldots S_l$ *per se*, but only in such combinations as are determinants of the matrix $S_1, S_2, \ldots S_l$. In other words, R depends only on the coefficients of $u, v, \ldots w$, and the coordinates of the space $S_1 \mid S_2 \mid \ldots \mid S_l$ in regard to some pyramid of reference $a, b, \ldots c$. Let these coordinates be called

* It might be seen and verified in a similar manner, that generally to any geometrical formation 𝔈 belongs a system of moduli $u_1 \ldots u_k$, and *vice versâ*. The work of Brill and Nöther has to a certain extent modified the fundamental proposition, but not so as to embarrass us in its use.

$p_1, p_2, \dots p_N$, and let

$$R = F(p_1, p_2, \dots p_N).$$

Let the border-spaces of h manifoldness of the pyramid $a, b, \dots c$ be $\xi_1, \xi_2, \dots \xi_N$. Let

$$X \equiv S_1 \mid S_2 \mid \dots \mid S_l = p_1\xi_1 + \dots + p_N\xi_N.$$

Let $\bar{\xi}_i$ be the space residual to ξ_i; then

$$p_1 = X\bar{\xi}_1 \cdot [a, b, \dots c],$$

$$p_2 = X\bar{\xi}_2 \cdot [a, b, \dots c],$$

&c.,

and $\mathfrak{C}$ is therefore in our notation $F(\bar{\xi}_1, \bar{\xi}_2, \dots \bar{\xi}_N)$.

This proceeding is not of much practical value, since it involves the necessity of the introduction of a pyramid of reference. Theoretically it is sufficient to show some of the most important properties of the form belonging to $\mathfrak{C}$, which again are sufficient to make the direct evaluation unnecessary. These are: If

$u_1 \dots u_k$ are the k surfaces,

$\lambda_1 \dots \lambda_k$ their orders,

$\mathfrak{C}$ will be a form whose coefficients are rational integral functions of the coefficients of the u_i, containing those of u_1 in the order $\lambda_2\lambda_3 \dots \lambda_k$, &c., and, considered as functions of the coefficients of all the u_i, are of the order $\lambda_1 \lambda_2 \dots \lambda_k$.

If $\mathfrak{C}$ vanishes identically, $u_1 \dots u_k$ must have in common a formation of higher manifoldness than in general, and *vice versâ*.

Another method for the formation of $\mathfrak{C}$ is this. According to Clebsch's work, the invariants and covariants of a system of surfaces may be symbolically expressed—in the notation used here by means of the symbol $*$ and of the symbols of Algebra, applied in some specified manner to the set of surfaces under consideration. If u, for instance, is a quadric in space S_h, $u * u * u \dots * u$ ($h+1$ times) is its discriminant, $u * u$ therefore the manifoldness of its tangent-lines, $u * u * u$ the manifoldness of its tangent-planes, &c. If u and v are two point-pairs upon a straight line, $4(u * v)^2 - 3 \cdot u^2 * v^2$ is their resultant; and this form is therefore also the product of the four points common to u and v, if u, v denote conics; the curve of intersection if they denote quadrics in space, &c. If, then, the resultant of k forms of orders $\lambda_1, \dots \lambda_k$ in space of manifoldness $k-1$, in its symbolical expression is known, this same expression gives also the intersection L of surfaces $u_1, \dots u_k$ of orders $\lambda_1, \dots \lambda_k$ in any space.

If U is any surface of order a in the space S_k, the manifoldness of surfaces of order n containing it is $p \cdot u$, where p is an arbitrary surface of order $n-a$. The number of (homogeneous) constants contained by p is $(n-a+k)_k$; or, if we denote $(n+k)_k$ by $\varphi(n)$, it is $\phi(n-a)$. The order of the condition that a surface of order n should contain all points of u is therefore $\phi(n)-\phi(n-a)$; or $\Delta_a\phi(n)$, by the introduction of a symbol of operation Δ_a, whose definition is obvious.

The reciprocal of the involution of surfaces v containing u is formed by the n^{th} powers of the points upon u, which follows from the definition of the involution v. Hence the n^{th} powers of any $\Delta_a\phi(n)+1$ points upon u will be linearly dependent.

If u and v are two surfaces of orders a, β, any surface of order n containing their intersection is of the form

$$p \cdot u + q \cdot v.$$

This form would contain $\phi(n-a)+\phi(n-\beta)$ constants, were it not that this number is diminished by the existence of identical relations, such as

$$pu+qv=0.$$

In fact this relation will be satisfied whenever

$$p=r \cdot v, \quad q=-r \cdot u,$$

where r denotes any surface of order $n-a-\beta$. So then the number of independent constants in the identical relations reduces to

$$\phi(n-a)+\phi(n-\beta)-\phi(n-a-\beta),$$

and it follows that the n^{th} powers of any $\Delta_a\Delta_\beta\phi(n)+1$ points upon the intersection of u and v must be linearly connected.

If u, v, w are any three surfaces of orders a, β, γ, the form

$$pu+qv+rw$$

would contain $\phi(n-a)+\phi(n-\beta)+\phi(n-\gamma)$ constants,

but for the existence of identities

$$pu+qv+rw=0.$$

Supposing the intersection of u, v, and w not to vanish identically, the intersection of u and v will not be contained by w. If, then, the above identity holds good, the intersection of u and v must be wholly situated upon r; therefore

$$r = au+bv,$$

and, similarly,

$$q = cu-bw,$$

$$p = -cv-aw;$$

a, b, c are perfectly arbitrary forms. The diminution to be effected would therefore appear to be $\phi(n-\alpha-\beta)+\phi(n-\alpha-\gamma)+\phi(n-\beta-\gamma)$ were it not that some of these identities are counted several times, since

a might be changed into $a+A.v$,

simultaneously b „ „ „ $b-A.u$,

c „ „ „ $c-A.w$,

without adding to the number of identities, A denoting any form of order $n-\alpha-\beta-\gamma$. So then the diminution is only

$$\phi(n-\alpha-\beta)+\phi(n-\beta-\gamma)+\phi(n-\gamma-\alpha)-\phi(n-\alpha-\beta-\gamma);$$

and therefore the order of the condition that a surface of order n should contain the intersection of u, v, w is $\Delta_\alpha\Delta_\beta\Delta_\gamma\phi(n)$. Repeating this process, we obtain the general theorem: If $u_1 \ldots u_p$ are any h forms of orders $\lambda_1, \lambda_2, \ldots \lambda_h$ whose resultant does not identically vanish, then the order of the condition that a surface of order n should contain their intersection is

$$N = \Delta_{\lambda_1}, \Delta_{\lambda_2}, \ldots \Delta_{\lambda_h}\phi(n),$$

or, what is the same thing, the n^{th} powers of any $N+1$ points upon the intersection of the U are linearly dependent.

Now, it will be noticed, k being the manifoldness of the space in which these surfaces are situated, that $\phi(n)$ is a function of n of order k; and that N is some integer function of n of order $k-h$. It appears, therefore, at least when $\mathfrak{C}$ is a geometrical formation generated by the intersection of surfaces, that the order of the condition for a surface of order n to contain $\mathfrak{C}$ is an integer function N of n, whose degree is equal to the degree of manifoldness of $\mathfrak{C}$; a linear function, for instance, for curves. We shall discuss this result later, and show that it is valid without any restriction on the nature of the generation of $\mathfrak{C}$; and, that, moreover the coefficients of the function N of n are numbers in intimate relation to $\mathfrak{C}$.

If $h = k$, N is a constant whose value is found, according to elementary theorems of the calculus of differences,

$$N = \lambda_1 \ldots \lambda_k.$$

Hence it follows that this is the number of points of intersection common to k surfaces of orders $\lambda_1 \dots \lambda_k$ in space S_k—another demonstration of this famous proposition. But the order of the condition that a surface of order n should contain these points will not be equal to their number whenever n is so small that some of the identities counted above do not exist; for instance, when n is smaller than

$$n' = \lambda_1 + \lambda_2 + \dots + \lambda_k.$$

When $n = n'-1$, $\phi(n-n')$ will be 0; also, when

$$n = n'-2 \;\dots\; n = n'-k.$$

But, when $$n = n'-k-1,$$

then $$\phi(n-n') \text{ will be } \mp 1;$$

and therefore the order of the condition that a surface of order

$$\lambda_1 + \lambda_2 + \dots + \lambda_k - k - 1$$

should contain the $\lambda_1 . \lambda_2 . \dots \lambda_k$ points of intersection is not $\lambda_1 . \lambda_2 . \dots \lambda_k$, but, since one of the identities which were counted above will cease to exist, only $\lambda_1 \dots \lambda_k - 1$. So, then, any surface ν of order

$$\lambda_1 + \dots + \lambda_k - k - 1,$$

containing $\lambda_1 \dots \lambda_k - 1$ of the points of intersection of k surfaces of orders $\lambda_1 + \dots + \lambda_k$ in space S_k, will also contain the last one. In other words, the $(\lambda_1 + \dots + \lambda_k - k - 1)^{\text{th}}$ powers of these points are linearly dependent.

If, finally, $h = k+1$, then $N = 0$. Hence the *Theorem:* If the resultant of any $k+1$ surfaces $u_1 \dots u_{k+1}$ in space S_k does not vanish, any surface V of order n, n being assumed large enough, is expressible in the form

$$V = p_1 u_1 + \dots + p_{k+1} u_{k+1}.$$

The orders of $u_2 \dots u_{k+1}$ being denoted by $\lambda_1 \dots \lambda_{k+1}$, this theorem will hold good if n is at least $= \Sigma\lambda_i - k$. If, however, $n = \Sigma\lambda_i - k - 1$, it is seen, as above, that one condition has to be satisfied for the above identity to exist. For instance, if u_1, u_2, u_3 are three conics in a plane, V any cubic in the same plane, V will not be expressible in the form

$$V = p_1 u_1 + p_2 u_2 + p_3 u_3$$

(the p denoting lines), unless the six points of intersection of V and u_1 and the four points common to u_2 and u_3 are situated upon one cubic. If $V = S^3$, the cube of a line, A, B, C, D the four points common to u_2 and u_3, and

$$aA + bB + cC + dD = 0,$$

the linear equation connecting them, then $S^3 - p_1 . u_1$ will contain A, B, C, D; hence

$$\frac{(SA)^3}{u_1 A} = p_1 A,$$

$$\frac{(SB)^3}{u_1 B} = p_1 B,$$

$$\dots \quad \dots \quad \dots,$$

and
$$a\frac{(SA)^3}{u_1 A} + b\frac{(SB)^3}{u_1 B} + c\frac{(SC)^3}{u_1 C} + d\frac{(SD)^3}{u_1 D} = 0;$$

S must therefore belong to the manifoldness

$$\theta = \frac{a}{u_1 A}.A^3 + \frac{b}{u_1 B}.B^3 + \frac{c}{u_1 C}.C^3 + \frac{d}{u_1 D}.D^3.$$

Hence generally the condition for V to be expressible modulo u_1, u_2, u_3 is to be conjugate to θ.

Generally to any set of $k+1$ surfaces of orders $\lambda_1 \dots \lambda_{k+1}$ in the space S_k must belong a point-form θ of order $\lambda_1 + \dots + \lambda_{k+1} - k - 1$ in intimate relation to it (which is, in fact, that $u_1 \dots u_{k+1}$ are apolar to θ) to be found in the manner given above. If the resultant of the forms vanish, θ will reduce to the $(\lambda_1 + \dots + \lambda_{k+1} - k - 1)^{\text{th}}$ power of the point common to the surfaces. θ will vanish identically when $u_1 \dots u_{k+1}$ have more than one point in common.

Any form V of order $\lambda_1 + \dots + \lambda_{k+1} - k$ will, as we have seen, be expressible by $p_1 u_1 + \dots + p_{k+1} u_{k+1}$. If the coefficients of the u_i undergo continuous changes until the resultant R of the u_i vanishes, this will cease to be true, for V must then obey the one condition to pass through the point common to the u to be thus expressible. This can only be explained by the supposition that, in consequence of the vanishing of R, a *new* relation such as

$$p_1 u_1 + \dots + p_{k+1} u_{k+1} = 0$$

is created. Hence, if the form of order $\lambda_1 + \dots + \lambda_{k+1} - k$,

$$p_1 u_1 + \dots + p_{k+1} u_{k+1},$$

vanishes, unless p_i belongs to the system of moduli $u_1 \dots u_{k+1}$ (u_i excepted), one condition must be fulfilled, viz., that R should vanish, and *vice versâ*.

III. If we have any group of points $A_1, \dots A_h$ in the space S_k, any $k+2$ of them will be connected by a linear equation, and the h points therefore by $h-k-1$ such equations. This system of equations will have the following form :—

$$a_1A_1+a_2A_2+\ldots+a_hA_h=0,$$
$$b_1A_1+b_2A_2+\ldots+b_hA_h=0,$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$l_1A_1+l_2A_2+\ldots+l_hA_h=0.$$

Multiplying the first by α_1, the second by α_2, ..., the last by α_l, where $l=h-k-1$, we can write the whole system of equations in one line

$$\mathfrak{A}_1 . A_1+\mathfrak{A}_2 . A_2+\ldots+\mathfrak{A}_h . A_h=0,$$

where
$$\mathfrak{A}_1=a_1 . \alpha_1+b_1 . \alpha_2+\ldots+l_1 . \alpha_l,$$
$$\mathfrak{A}_2=a_2 . \alpha_1+b_2 . \alpha_2+\ldots+l_2 . \alpha_l,$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

The α_1, ... α_l will be perfectly arbitrary and independent of each other. We may therefore use the geometrical calculus, interpreting the α_i as corners of a pyramid in an auxiliary space. The $\mathfrak{A}_i$ will consequently be $\equiv$ points in some space $\mathfrak{S}_{l-1}$ of perfectly arbitrary situation. It will also be noticed that, on account of the perfect freedom in the choice of the α, the group $\mathfrak{A}$ may be subjected to any linear transformation without ceasing to make the equation connecting the A_i and $\mathfrak{A}_i$ true.

From (θ) $$A_1 . \mathfrak{A}_1+A_2 . \mathfrak{A}_2+\ldots+A_h . \mathfrak{A}_h=0,$$

we may deduce any relation connecting the A_i, for instance, the one connecting A_1, A_2, ... A_{k+2} by composing the $\mathfrak{A}$ in (θ) with $\mathfrak{A}_{k+3}\mathfrak{A}_{k+4}\ldots\mathfrak{A}_h$. The A and $\mathfrak{A}$ being situated in totally different spaces, any operation may be performed on the one group, while the symbols of the other group are treated as constants.

If A, B, C, D are four points on a line, they are connected by two relations, and we shall have to introduce four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ on some other line, so that

$$a . A . \mathfrak{A}+b . B . \mathfrak{B}+c . C . \mathfrak{C}+d . D . \mathfrak{D}=0,$$

the a, b, c, d denoting constants. Composing with C and $\mathfrak{D}$, we obtain

$$a . AC . \mathfrak{AD}+b . BC . \mathfrak{BD}=0;$$

similarly, composing with $\mathfrak{C}$ and D,

$$a . AD . \mathfrak{AC}+b . BD . \mathfrak{BC}=0;$$

hence
$$\frac{AC}{AD} : \frac{BC}{BD} = \frac{\mathfrak{AC}}{\mathfrak{AD}} : \frac{\mathfrak{BC}}{\mathfrak{BD}};$$

or the necessary and sufficient condition for the two groups to be corresponding in the above-mentioned manner is that their cross-ratios should be equal.

(If the group A is given, the group $\mathfrak{A}$ defined by the relation
$$\Sigma a_i \,.\, A_i \,.\, \mathfrak{A}_i = 0$$
may be linearly constructed, $l+1$ of them being arbitrary. To k points A_i on a straight line correspond k points $\mathfrak{A}_i$ of a S_{k-3}. If, of the k points A, $k-1$ remain fixed, the last one describing its straight line, while also, of the k points $\mathfrak{A}$, the $k-1$ arbitrarily to be assumed remain fixed, the last point $\mathfrak{A}$ will describe a certain rational curve passing through the fixed points $\mathfrak{A}$. *Vide Nature*, 17th October, 1895.)

There is no reason why no more than two different space-symbols should be used. Our definitions will apply also in this case, the general rule being that any operation may be performed upon the symbols of any one space, while the symbols of the other spaces are treated as constants.

A form like the following,
$$a_1 \,.\, A_1 \,.\, A_1' \,.\, A_1'' \dots A_1^{(k)} + \dots + a_h \,.\, A_h \,.\, A_h' \,.\, A_h'' \dots A_h^{(k)},$$
where the A_i belong to one space, the A_i' to another, &c., is called a $(k+1)$-linear form. The point symbols $A_i^{(s)}$ and their compositions are called "different space-symbols."

An equation between point-forms, such as, for instance,
$$a_1 u_1 + a_2 u_2 + \dots + a_h u_h = 0,$$
imposes a condition upon the coefficients of $u_1, u_2, \dots u_h$. In fact, if the order of the $u_1 \dots u_h$ is n, the a denoting unknown constants, the order of the condition expressed by this equation is $\phi(n)+1-h$ (since a form of order n in S_k vanishes, the form containing h homogeneous parameters). If, for instance, it is known that three point-pairs $A \,.\, A'$, $B \,.\, B'$, $C \,.\, C'$ in a plane are linearly dependent, the six points must satisfy a four-fold condition.

If, in an equation $\quad a_1 u_1 + \dots + a_h u_h = 0,$

the point-symbols implicitly contained in the $u_1 \dots u_h$ are subjected to one and the same linear transformation, and they are projected into some other space, only the constants $a_1 \dots a_h$ will be affected, but the projections $u_1' \dots u_h'$ of $u_1 \dots u_h$ will again be linearly dependent. This follows immediately if we compose the equation with some point P

outside the space S of the u, and then cut it by a space Σ contained in SP, and of the same manifoldness as S.

If we polarize the equation

$$a_1 u_1 + \ldots + a_h u_h = 0$$

with $P_1 . P_2 . \ldots P_n$, where $P_1 \ldots P_n$ are *not* contained in S, and $SP_1 \ldots P_n$ is different from 0, the point-symbols $A_1 \ldots A_m$ implicitly contained in the $u_1 \ldots u_h$ will combine with the $P_1 \ldots P_n$, and we shall obtain a relation between

$$\begin{array}{llll} A_1 P_1, & A_2 P_1, & \ldots & A_m P_1, \\ A_1 P_2, & A_2 P_2, & \ldots & A_m P_2, \\ \ldots & \ldots & \ldots & \ldots \\ A_1 P_n, & A_2 P_n, & \ldots & A_m P_n, \end{array}$$

which evidently represent in the geometry of lines through $P_1, P_2, \ldots P_n$ *different space-symbols*. Cutting these by spaces $\Sigma_1, \Sigma_2, \ldots \Sigma_n$ of the same manifoldness as S in $SP_1, SP_2, \ldots SP_n$, respectively, we obtain a relation between

$$\begin{array}{llll} A'_1, & A'_2, & \ldots & A'_m, \\ A''_1, & A''_2, & \ldots & A''_m, \\ \ldots & \ldots & \ldots & \ldots, \\ A^{(n)}_1, & A^{(n)}_2, & \ldots & A^{(n)}_m, \end{array}$$

where the $A', A'', \ldots$ are projections of the $A_1, A_2, \ldots A_m$, but situated in different spaces.

Thus from the original equation we obtain another which is evidently an n linear form of different space-symbols denoting point-groups in each separate space which are projectively identical with the original point-group. Therefore it follows that we may perform any operation with any one of the point-groups $A', A'', \ldots$, treating the others as constants, and afterwards identify again the symbols of that point-group with the original point-group A (which is but a specialization).

The polarized form of the power of a point P is especially simple. That of P^n is evidently $P' . P'' \ldots P^{(n)}$.

To give a few instances: an equation

$$(\theta) \quad aA^2 + bB^2 + cC^2 + dD^2 + eE^2 + fF^2 = 0,$$

where the $a, \ldots f$ are unknown constants, expresses one condition, which is evidently that the six points $A, \ldots F$ are situated upon one conic. Writing (θ),

$$aA^2 + bB^2 + cC^2 = -dD^2 - eE^2 - fF^2,$$

it appears that A, B, C and D, E, F form self-conjugate triangles to some conic. Polarizing each side of the last equation with itself, we obtain

$$abAB^2+bcBC^2+caCA^2 = deDE^2+\dots;$$

hence AB, BC, CA, DE, EF, FD touch one conic. Polarizing (θ) in regard to the cubic u formed by $DE.EF.FD$, which obviously contains D, E, F as double points, so that

$$D^2\times u = 0, \quad E^2\times u = 0, \quad F^2\times u = 0,$$

we have

$$aA^2\times u+bB^2\times u+cC^2\times u = 0,$$

or the three polar lines of A, B, C to the triangle D, E, F (which can be linearly constructed) are concurrent. Polarizing (θ),

$$aA.\mathfrak{A}+bB.\mathfrak{B}+\dots+fF.\mathfrak{F} = 0,$$

further composing with E and $\mathfrak{F}$,

$$aAE.\mathfrak{A}\mathfrak{F}+bBE.\mathfrak{B}\mathfrak{F}+c.CE.\mathfrak{C}\mathfrak{F}+d.DE.\mathfrak{D}\mathfrak{F} = 0,$$

or the cross-ratio of the lines AE, BE, CE, DE is equal to that of AF, BF, CF, DF.

If we put

$$aA^2+bB^2 = h.A'.B',$$
$$cC^2+dD^2 = k.C'.D',$$
$$eE^2+fF^2 = l.E'.F',$$

then

$$h.A'.B'+k.C'.D'+l.E'.F' = 0.$$

It is obvious that the line joining C' and E', for instance, must contain either A' or B'. The figure must therefore be that of a triangle $A'C'\mathfrak{F}'$, cut by a straight line $B'D'F'$, B' to be on $C'E'$, &c. Its reciprocal may be derived from the identity which connects four points upon a straight line

$$AD.BC+AB.CD+AC.DB = 0,$$

which must also exist for four points in a plane, since this form would contain all points of the plane, and must therefore identically vanish. The equation connecting $A'.B'$, &c., may be interpreted as meaning that if, $A'.B'$ and $C'.D'$ are conjugate to any conic, $E'.F'$ will be so also. If L, M, N are the three corners of the triangle formed by AB, CD, EF, it is at once seen that

$$A'.B' \equiv aL^2-bM^2,$$
$$C'.D' \equiv bM^2-cN^2,$$
$$E'.F' \equiv cN^2-aL^2,$$

a, b, c denoting parameters. $A'B'$ are therefore the double points of the involution formed by AB and LM, &c.

Let P, Q, R be any three points, and p, q, r their polars in respect to some conic u. If, then, $p'p''$, $q'q''$, $r'r''$ are points upon p, q, r, respectively, $P.p'$, $P.p''$, $Q.q'$, $Q.q''$, $R.r'$, $R.r''$ will be conjugate to the same conic; therefore linearly dependent. It follows that points P', Q', R' must exist upon p, q, r, so that $P.P'$, $Q.Q'$, $R.R'$ are linearly dependent. Now P' is collinear with QR; hence it is the cut of QR and p. The relation is then expressed thus, QR/p, RP/q, PQ/r are collinear, and, the configuration being its own reciprocal, $P\,q/r$, $Q\,r/p$, $R\,p/q$ are concurrent. This may also be proved in a different manner thus:—Let u be any conic, A, B, C the corners of a triangle. Then $(AB*u)\times C+(BC*u)\times A+(CA*u)\times B$ is a line-form $\times$. Composed with C it is

$$(BC*u)*AC+(CA*u)*BC = BC.AC*u+CA.BC*u = 0.$$

Hence $\times A = \times B = \times C = 0$. $\times$ will therefore vanish identically. This may also be generalized. If $u \equiv \mathfrak{J}$, this shows that the four heights of a tetrahedron belong to one quadric.

The curve whose points have the property that the cross-ratios of the twice four points

$$A,\ B,\ C,\ D,$$
$$A',\ B',\ C',\ D',$$

stand in a certain proportion $\beta : \alpha$ is the numerator of

$$\alpha\left(\frac{AC}{BC}:\frac{AD}{BD}\right)-\beta\left(\frac{A'C'}{B'C'}:\frac{A'D'}{B'D'}\right)$$

(the line-symbols written in their normal form), a curve of the fourth order. If $A\ldots D$, $A'\ldots D'$ are situated upon a conic u, $\beta : \alpha$ being the proportion of their cross-ratios with respect to the points of u, then the curve above defined must degenerate into two conics u, u'; u' contains the eight points $AC/A'C'$, $AC/B'D'$, $BD/A'C'$, ..., and is thus defined.

If u, v, w are three conics having two points in common, the form $a.u+b.v+c.w$, the a, b, c denoting lines, can only denote an involution of manifoldness 7; hence some identity such as

$$a.u+b.v+c.w = 0$$

must exist, and consequently the lines a, b, c joining the two variable points of intersection of v, w; w, u; u, v respectively are concurrent. If now u is a conic, A, B, C, D, E, F six points upon it, then u, $AC.BD$ and $AE.BF$ have the points A and B in common; therefore the cut of CD and EF must be situated on the line joining the cuts of AC/BF and BD/AE, which is Pascal's theorem.

Let $A_1 \dots A_9$ be nine points common to two cubics. Their configuration is expressed by the fact that their cubes are linearly dependent. Eight points being given, the last one is obviously determined. A construction may, for instance, be arrived at thus: We may have

$$a_1 A_1^3 + a_2 A_2^3 + \dots + a_8 A_8^3 + a_9 A_9^3 = 0.$$

Polarizing, we obtain as a consequence

$$a_1 A_1^2 \,.\, \mathfrak{A}_1 + a_2 A_2^2 \,.\, \mathfrak{A}_2 + \dots + a_8 A_8^2 \,.\, \mathfrak{A}_8 + a_9 A_9^2 \,.\, \mathfrak{A}_9 = 0.$$

Polarizing with the A_i^2 in regard to the cubic u represented by the sides of the triangle $A_9 A_7 A_8$, and denoting the lines $A_1^2 \times u$ by α_1, $A_2^2 \times u$ by $\alpha_2, \dots,$

$$a_1 \,.\, \alpha_1 \mathfrak{A}_1 + \dots + a_5 \,.\, \alpha_5 \,.\, \mathfrak{A}_5 + a_9 \,.\, \alpha_9 \,.\, \mathfrak{A}_9 = 0.$$

Composing with $\mathfrak{A}_9$,

$$a_1 \alpha_1 \,.\, \mathfrak{A}_1 \mathfrak{A}_9 + a_2 \alpha_2 \,.\, \mathfrak{A}_2 \mathfrak{A}_9 + \dots + a_5 \alpha_5 \,.\, \mathfrak{A}_5 \mathfrak{A}_9 = 0.$$

Now $\alpha_1, \dots \alpha_5$ are known. The cross-ratio in which $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is cut by α_5 is equal to that of $\mathfrak{A}_1\mathfrak{A}_9$, $\mathfrak{A}_2\mathfrak{A}_9$, $\mathfrak{A}_3\mathfrak{A}_9$, $\mathfrak{A}_4\mathfrak{A}_9$, and that of a_1/a_4, a_2/a_4, a_3/a_4, a_5/a_4, equal to that of $\mathfrak{A}_1\mathfrak{A}_9$, $\mathfrak{A}_2\mathfrak{A}_9$, $\mathfrak{A}_3\mathfrak{A}_9$, $\mathfrak{A}_5\mathfrak{A}_9$. Hence $\mathfrak{A}_9$ can be constructed as the fourth point common to two conics through $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{A}_3\mathfrak{A}_4$ and $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{A}_3\mathfrak{A}_5$.

If seven points are given, $A_1, \dots A_7$, and the other two points A_8, A_9 are restricted to a given line l, only one solution is possible. Through the seven points an involution of three cubics is possible. They cut l in an involution whose reciprocal will be a certain point-group $P \,.\, Q \,.\, R$ conjugate to any cubic containing the seven points. Therefore $P \,.\, Q \,.\, R$ is linearly dependent upon the cubes of the seven points. Representing $P \,.\, Q \,.\, R$ in its canonical form as the sum of two cubes, we find A_8 and A_9.

If two of the points P, Q, R coincide, there is no solution, but, if $Q \equiv P$, then P^2R will be linearly dependent on the cubes of the seven points, and this will express that any cubic passing through $A_1 \dots A_7$, and P will touch l; that, in other words, A_8 and A_9 are consecutive in P upon l. It may also express that a cubic is possible through $A_1 \dots A_7$ having P as a double point, the linear dependence between $A_1 \dots A_7$ and P^2R reducing the number of conditions this implies to nine.

We may add, though without demonstration, that, if $A_1 \dots A_8$ are fixed and A_8 moves upon any curve of order λ, A_9 will generally move upon a curve of order 8λ. If, however, the curve A_8 contains the fixed points $A_1 \dots A_7$ as $(a_1 \dots a_7)$-fold points, denoting $a_1 + \dots + a_7$ by β, the order of the curve A_9 will be $8\lambda - 3\beta$, and it will contain A_1 as $3\lambda - \beta - a_1$, A_2 as $(3\lambda - \beta - a_2)$-fold points, &c.

Eight points common to three quadrics satisfy a relation

$$a_1 A_1^2 + \dots + a_8 A_8^2 = 0.$$

By reading it $\quad a_1 A_1^2 + \dots + a_4 A_4^2 = -a_5 A_5 \dots,$

it is evident that $A_1 A_2 A_3 A_4$ and $A_5 A_6 A_7 A_8$ are two self-conjugate pyramids to one and the same quadric. Polarizing

$$a_1 A_1 . \mathfrak{A}_1 + \dots + a_8 A_8 \mathfrak{A}_8 = 0,$$

and composing with $A_8 A_7$ and $\mathfrak{A}_6 \mathfrak{A}_5$ we see that $A_1 A_2 A_3 A_4$ have the same cross-ratio, whether projected from $A_5 A_6$ or from $A_7 A_8$. We also see that any quadric through six of the points $A_1 \dots A_6$ will cut $l \equiv A_7 A_8$ in an involution whose double points are B and C, so that

$$B . C = a_7 A_7^2 + a_8 A_8^2.$$

Any point-pair on l harmonic with $B . C$ will therefore, if joined to $A_1 \dots A_6$, complete the configuration. The two points A_7, A_8, though moving upon l, will coincide in B and C, and l will be a tangent to any quadric containing $A_1 \dots A_6$, B or $A_1 \dots A_6$, C.

Now a twisted cubic having seven points in common with a quadric must be wholly contained in it. So any eight points upon a twisted cubic form the configuration. Seven points $A_1, \dots A_7$ being given, $A_7 A_8$ is the straight line through A_7 cutting the twisted cubic through $A_1 \dots A_6$ twice. So, then, l can always be determined, except in the following cases:—

(1) That A_7 is collinear with any two of the fixed points $A_1 \dots A_6$. Then A_8 is any point on that line, the squares of four points on a line being always linearly dependent.

(2) That any five of the fixed points are coplanar. Then A_8 is any point upon their conic.

(3) That any six of the fixed points are coplanar. Then A_8 is any point of their plane; or, if the six given points are also upon one conic, A_8 is any point whatever.

(4) That the seven given points are upon the same twisted cubic. Then A_8 is any point of that twisted cubic.

In no other case can the configuration degenerate. If, for instance,

$A_1 \ldots A_4$ are coplanar in S, the line common to S and $A_5A_6A_7$ will be cut in involution by the conics through $A_1 \ldots A_4$; hence a point-pair $B.C$ will exist upon it which is linearly dependent upon $A_1^2 \ldots A_4^2$. But the conics passing through A_5, A_6, A_7 and containing $B.C$ as conjugate point-pair have a fourth point A_8 in common, so that $B.C$ is also linearly dependent upon the A_5^2, A_6^2, A_7^2, A_8^2.

If six of the points $A_1 \ldots A_6$ are given, while A_7 is restricted to a plane S, and A_8 to a line l, l to be within S, we shall find three point-pairs A_7, A_8 to complete the configuration. For $A_1, \ldots A_6$ determine an involution of four quadrics which cut S in an involution of conics, the reciprocal of which is formed by the two point-forms u and v. Their involution contains three point-pairs $L.L'$, $M.M'$, $N.N'$. LL' cuts l in A_8, while A_7 is $\equiv A_8 \times L.L'$. Thus the three point-pairs $A_7.A_8$ are constructed. The lines LL', MM', NN' are obviously the sides of the triangle, in which the twisted cubic through $A_1 \ldots A_6$ cuts S.

Proceeding similarly when five points $A_1, \ldots A_5$ are given, and a plane S as locus for the other points, we obtain in S one point-form u of the second order linearly dependent upon the squares $A_1, \ldots A_5$; A_6, A_7, A_8, will therefore be the corners of any triangle self-conjugate to u_7. And we shall have two solutions, if A_6, A_7, A_8 are restricted to lie on given lines in S.

A configuration of some importance, at least in the theory of surfaces of the second order in any space, is that of $2n$ points characterized by (θ),

$$a_1A_1.B_1+a_2A_2.B_2+\ldots a_nA_n.B_n=0,$$

where it is understood that $A_1, \ldots A_n$ are the corner-points of a (non-vanishing) pyramid in space S_{n-1}. The order of the condition imposed by (θ) is $\frac{n.n+1}{2}-n+1$. Since $[A_1 \ldots A_n]$ is different from 0, it follows that

$$[A_1 \ldots A_{n-1}B_n]=0.$$

Hence $[A_1 \ldots A_{n-2}B_{n-1}B_n]$ is generally different from 0, and

$$[A_1 \ldots A_{n-3}B_{n-2}B_{n-1}B_n] \text{ again} = 0.$$

Any space composed of an odd number of the B has a point in common with the space composed by the residual A. If, then, B_n and B_{n-1} are assumed anywhere in $A_1 \ldots A_{n-1}$, and $A_1 \ldots A_{n-2}A_n$, respectively, B_{n-2} will be in the cut of

$$A_1, \ldots A_{n-3}A_{n-1}A_n \text{ and } A_1, \ldots A_{n-3}B_{n-1}B_n;$$

B_{n-3} will be subject to three conditions to lie in the cut of

$$A_1 \dots A_{n-4}A_{n-2}A_{n-1}A_n\,; \quad A_1 \dots A_{n-4}B_{n-2}B_{n-1}A_n\,;$$

$$A_1 \dots A_{n-4}A_{n-2}B_{n-1}B_n\,;$$

B_{n-4} will similarly be subject to 4 conditions;

... ...

B_1 will similarly be subject to $n-1$ conditions;

raising the number of conditions to $\frac{n.\overline{n-1}}{2}+1$.

In space the two pyramids $A_1, \dots A_4$ and $B_1, \dots B_4$ are so related that each has its corner-points upon the faces of the other.

Whenever n is an even number, the relation of the pyramids is reciprocal. When n is odd, the pyramid of the B points must vanish.

Let $A_1, \dots A_8$ be eight points common to three quadrics. Join A_1A_2, A_3A_4, A_5A_6, A_7A_8. There are two lines cutting these four (the two lines of the involution reciprocal to the one formed by the four) say a and b, cutting A_1A_2 in L_1, L_2, A_3A_4 in L_3, L_4 .... Now, L_1, L_3, L_5, L_7 being collinear, their squares are linearly dependent; also those of L_2, L_4, L_6, L_8. Consequently, for any values of λ, μ, a relation will exist, viz.,

$$c_1\lambda L_1^2+c_2\mu L_2^2+c_3\lambda L_3^2+c_4\mu L_4^2+\dots=0.$$

Putting

$$c_1\lambda L_1^2+c_2\mu L_2^2=M_1.M_2,$$

$$c_3\lambda L_3^2+c_4\mu L_4^2=M_3.M_4,$$

$$\dots \quad \dots \quad \dots \quad \dots\,.$$

$M_1.M_2$, $M_3.M_4$, $M_5.M_6$, and $M_7.M_8$ will be linearly dependent. The order of the condition implied by this statement is 7. Consequently, only a singly infinite series of such quadruples of point-pairs *can* exist on the four given lines. Hence that series is exactly represented by

$$(c_1\lambda L_1^2+c_2\mu L_2^2), \quad (c_3\lambda L_3^2+c_4\mu L_4^2)\ \dots\,.$$

This gives rise to the theorem: The double-points of the involutions $(L_1.L_2,\ A_1.A_2)$, $(L_3.L_4,\ A_3.A_4)$, &c., form the configuration of two pyramids, of which each has its corner-points upon the faces of the other.

To multiply these theorems to any extent would only require some imagination. To write what may be considered a complete theory of such configurations requires however more resources than are developed in this essay. We shall leave, therefore, the further discussion of configurations for a future occasion.

Metrical relations can also be derived with ease. If u is a surface, A and B any two points, the line AB cutting the surface in the n points $w_1, w_2, \ldots w_n$, then the points w are found by the solution of

$$u \times (\alpha A + \beta B)^n = 0,$$

the n roots $\alpha : \beta$ corresponding to the n points w,

$$\alpha : \beta = -w_i B : w_i A.$$

Since

$$u \times (\alpha A + \beta B)^n = \alpha^n . uA + n . \alpha^{n-1} \beta . u \times A^{n-1} . B + \ldots + \beta^n . uB,$$

it follows that the product of the roots

$$w_1 B . w_2 B \ldots w_n B : w_1 A . w_2 A \ldots w_n A = uB : uA.$$

If A, B are on u—say, and λ points w coincide in A, μ points in B—only $n - \lambda - \mu$ points w will remain corresponding to the equation

$$(n)_\lambda \alpha^{n-\lambda-\mu} . u \times A^{n-\lambda} . B^\lambda + \ldots + (n)_\mu \beta^{n-\lambda-\mu} . u \times A^\mu B^{n-\mu} = 0;$$

hence

$$\Pi w_i B : \Pi w_i A = (n)_\mu . u \times A^\mu . B^{n-\mu} : (n)_\lambda . u \times A^{n-\lambda} . B^\lambda.$$

Similar theorems exist for any geometrical formations.

If a surface of order $2n$ contains $\mathfrak{I}^n$, points of its space will have a "power" in regard to it. If u is such a surface, P any point, l any line through it, $w_1, \ldots w_{2n}$, its cuts with u, then $\Pi w_i P$ is constant, and independent of the situation of l. We have, denoting l/I by D,

$$\Pi P w_i : \Pi D w_i = uP : uD.$$

But, D being the I of l, all Dw are 1; and, $\mathfrak{I}^n$ being the cut of I and u, uD is a constant multiple of $(ID)^n$, which is 1. Putting

$$uD = k,$$

we have

$$\Pi P w_i = k . uP.$$

To give a few examples: If any surface u cuts the sect AB in n points $w_1, \ldots w_n$, then the ratio $\Pi A_w : \Pi B_w$ may be called the ratio in which u cuts AB. This ratio, which is $= uA : uB$, being given, $= c$, it follows that $A^n - c . B^n$ is conjugate to u; it is, therefore, equivalent to a linear condition for u.

A conic which cuts five given sects in given proportions is therefore generally uniquely determined. A conic cuts any six given sects in six ratios $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ between which a six-linear relation exists; for, $\alpha, \beta, \gamma, \delta, \epsilon$ being given, ζ is uniquely determined. This may evidently be generalized. The equation illustrating the situation of six points on a conic

$$aA^2+bB^2+cC^2+dD^2+eE^2+fF^2=0$$

may be interpreted as meaning that any conic cutting the sect AB in a given ratio $\left(-\frac{b}{a}\right)$ and CD in another $\left(-\frac{d}{c}\right)$ will cut EF in a certain ratio $\left(-\frac{f}{e}\right)$. Similarly, any curve of the n^{th} order cutting n of the $n+1$ given sects AA', ... LL' in certain ratios will also cut the last one in a known ratio if A, A', ... L, L' are upon one conic. The identity $(aA^n-bB^n)+(bB^n-cC^n)+(cC^n-aA^n)=0$ gives Carnot's theorem. All this might be much generalized and varied.

Finally, as regards the calculus with $\mathfrak{J}$, it follows, from our previous results, that all plane spaces ξ in contact with $\mathfrak{J}$ are to be regarded as isotropic spaces. If the $\mathfrak{J}$ of a S_k is defined as a point-form, a space S_a is isotropic, when

$$S_a^2\times\mathfrak{J}\times\mathfrak{J}\ldots(k-a\text{ times})=0,$$

and in its normal form, when the magnitude on the left-hand side is $=1$. $S_a\times\mathfrak{J}\times\mathfrak{J}\ldots$ is any S_{k-a} perpendicular to S_a, since any two points at I perpendicular to each other are conjugate to $\mathfrak{J}$. If S_a is in its normal form, so also

$$S_{k-a}=S_a\times\mathfrak{J}\times\mathfrak{J}\ldots.$$

For $[S_{k-a}\,S_a]$ is the same as

$$S_{k-a}\times S_a=S_a.S_a\times\mathfrak{J}\times\mathfrak{J}\ldots=1.$$

But the magnitude formed by the two spaces S_{k-a} and S_a perpendicular to each other is 1, it being a product of the sines of angles, all of which are right angles. S_a is in its normal form. Hence $[S_{k-a}\,S_a]$ cannot be equal to 1 unless S_{k-a} is also in its normal form. These statements may be regarded to express all cosine theorems, &c., in fact, all metrical relations based upon the measurement of angles in their simplest form. To give only one instance, D_1, D_2, D_3, D_4, denoting points at the I of our space in their normal form (points of a sphere), from the identity

$$D_1D_2.D_3D_4+D_1D_3.D_4D_2+D_1D_4.D_2D_3=0,$$

we conclude [considering that D_1D_2 is not in its normal form, but multiplied by $\sin(D_1, D_2)$, and similarly for the other line-symbols], by polarizing with $\mathfrak{J}$,

$$\begin{aligned}&\sin(D_1, D_2).\sin(D_3, D_4).\cos(D_1D_2, D_3D_4)\\+&\sin(D_1, D_3).\sin(D_4, D_2).\cos(D_1D_3, D_4D_2)\\+&\sin(D_1, D_4).\sin(D_2, D_3).\cos(D_1D_4, D_2D_3)=0.\end{aligned}$$

Thursday, June 10th, 1897.

Prof. E. B. ELLIOTT, F.R.S., President, in the Chair.

Seven members present.

Mr. W. W. Taylor exhibited numerous models of the regular convex and star solids.

Major MacMahon, Vice-President, having taken the Chair, communicated papers by Mr. H. MacColl, "The Calculus of Equivalent Statements" (Sixth Paper); by Dr. G. A. Miller, "On the Primitive Substitution Groups of Degree Fifteen."

Mr. Love, Hon. Sec., read "A Generalized Form of the Binomial Theorem," which had been sent by the Rev. F. H. Jackson. The Chairman (Major MacMahon) stated that the form was a known one.

The following present was made to the Society's Album:—

Cabinet likeness of Mr. W. Esson, F.R.S. (now Savilian Professor of Geometry, Oxford).

The following presents were made to the Library:—

Carruthers, G. T.—"The Origin of the Celestial Laws and Motions," 8vo; London, 1897.

Parasada Ganesh.—"On the Potential of a Solid Ellipsoid of Revolution at an External Point" (4 copies), 8vo; Allahabad, 1897.

"Proceedings of the Royal Society," Vol. LXI., Nos. 371-373.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 4, 5; Leipzig, 1897.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLI., Pt. 3, 1896-97.

"Jahresbericht der Deutschen Mathematiker Vereinigung," Bd. V., Heft 1, 1896; Leipzig, 1897.

"Proceedings of the Physical Society," Vol. XV., Pt. 5, No. 80; May, 1897.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," 1897, Heft 1.

"Wiskundige Opgaven," Amsterdam, Deel VII., St. 3; 1897.

"Nieuw Archiev voor Wiskunde," Deel III., St. 2; Amsterdam, 1897.

"Bulletin of the American Mathematical Society," Vol. III., No. 8; May, 1897.

"Adresse présentée à M. Mittag-Leffler le 16 Mars 1896, 1846-1896," 8vo; Paris, 1896.

"Proceedings of the Canadian Institute," Vol. I., No. 1, Pt. 1, February, 1897; Toronto.

"Bulletin des Sciences Mathématiques," 1897, Mai, Tome XXI.

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On the Primitive Substitution Groups of Degree Fifteen. By G. A. MILLER, Ph.D. Received June 2nd, 1897. Read June 10th, 1897.

If any group (G) of order g contains a non-self-conjugate sub-group (G_1) of order g_1 that does not include any self-conjugate sub-group of G, with the exception of identity, then is G simply isomorphic to a transitive substitution group (G') of degree $g \div g_1$. When G_1 is a maximal sub-group of G, *i.e.*, when it is not contained in a larger sub-group of G, G' is a primitive group. When this condition is not satisfied, G' is non-primitive.*

It is a singular fact that we can find all the primitive groups of degree 15 which do not contain the alternating group of this degree by means of these well-known principles. The four groups

$$(+abcdef)_{24}, \quad (abcdef)_{48}, \quad (abcdefg)_{168}, \quad (abcdefgh)_{1344}\dagger$$

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* Dyck, *Mathematische Annalen*, Vol. XXII., p. 94.

† Noether, *Mathematische Annalen*, Vol. XV., p. 90. We follow the notation employed by Professor Cayley in his lists of substitution groups published in the *Quarterly Journal of Mathematics*, Vol. XXV.