Power Sum Residues Modulo 144:

A Four-Value Partition Theorem for Prime Power Sums

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Abstract

Let p be an odd prime and define the power sum $S(p,k) = 1^k + 2^k + \cdots + (p-1)^k$. We prove that $S(p,k) \bmod 144$ is independent of k for all odd $k \geq 3$ if and only if p mod 144 belongs to one of exactly 32 residue classes, and in that case the value is one of only six possibilities: {0, 1, 9, 64, 73, 81}.

The stronger form of the theorem — constant for all odd $k \geq 1$ — holds for exactly 8 residue classes with lock values $\{0, 1, 64, 81\} = \{0, 1, 2^6, 3^4\}$.

The proof uses three ingredients: (1) modular periodicity to reduce to m = p mod 144; (2) the Chinese Remainder Theorem decomposition $144 = 16 \times 9$ to handle the two components independently; (3) a Type A/B classification of elements mod 16 based on whether $i^2 \equiv 1$ or $i^2 \equiv 9 \bmod 16$, together with a complete residue system argument mod 9.

The four lock values for all odd k are exactly the pure prime powers of the prime factorization $144 = 2^4 \times 3^2$: the values $\{0, 1, 2^6, 3^4\}$ arise as the unique CRT combinations of {0,1} mod 9 and {0,1} mod 16. All results are computationally verified for k up to 199 and all residue classes mod 144.

> **Discovery note —** This theorem was discovered through investigation of harmonic structure in the 144-Hz frequency identity. The result is independent of that framework — it is a statement about power sums and modular arithmetic that stands on its own.

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1. Introduction and Main Theorem

The power sum $S(m,k) = \Sigma_{i=1}^{m-1} i^k$ has been studied extensively in connection with the Bernoulli numbers, the Erdős–Moser conjecture, and Wieferich primes. The standard result (Von Staudt–Clausen and related work) characterizes S(p,k) mod p for prime p. The behavior mod fixed composite moduli such as 144 has received less systematic attention.

We investigate a specific question: for which primes p is S(p,k) mod 144 independent of the exponent k? The answer has an unexpectedly clean structure.

Main Theorem — Let p be an odd prime. Then S(p,k) mod 144 is the same for all odd $k \geq 3$ if and only if p mod 144 belongs to one of exactly 32 residue classes. The value in that case lies in {0, 1, 9, 64, 73, 81}.

Theorem A (Stronger Form) — S(p,k) mod 144 is the same for all odd $k \geq 1$ if and only if p mod 144 belongs to one of exactly 8 residue classes. The value in that case lies in {0, 1, 64, 81} = $\{0, 1, 2^6, 3^4\}$.

The distinction between $k \geq 1$ and $k \geq 3$ arises because $S(p,1) = p(p-1)/2$ carries arithmetic information beyond what higher odd powers reveal. From k = 3 onward the cancellation mechanism described in Section 3 operates fully.

Condition on k	Universal classes	Lock values	Notes
All odd $k \geq 1$	8 classes	{0, 1, 64, 81} = $\{0,1,2^6,3^4\}$	Theorem A — strongest form
All odd $k \geq 3$	32 classes	{0,1,9,64,73,81}	Theorem B — main theorem

2. Preliminaries

2.1 Modular Periodicity

The following observation reduces the problem from all primes to a finite computation:

Lemma 0 (Modular Periodicity) — S(p,k) mod N = S(p mod N, k) mod N for any positive integer N.

Proof: $S(p,k) = \sum_{i=1}^{p-1} i^k$. Adding N complete residue classes {0,...,N−1} contributes $\sum_{i=0}^{N-1} i^k \equiv 0$ mod N (for any k, by pairing i with N−i for N even, or using the fact that the sum over a complete residue system is 0 mod N for $k \geq 1$). Therefore S(p,k) mod N depends only on p mod N. □

This reduces the problem to analyzing S(m,k) mod 144 for $m \in \{1,...,144\}$.

2.2 Chinese Remainder Theorem Decomposition

Since 144 = 16 × 9 with gcd(16, 9) = 1, the Chinese Remainder Theorem gives:

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S(m,k) mod 144  is determined by  S(m,k) mod 16  and  S(m,k) mod 9
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Specifically, S(m,k) mod 144 is constant for all odd k if and only if both S(m,k) mod 9 and S(m,k) mod 16 are individually constant for all odd k. We analyze each component separately.

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3. The Two Lemmas

3.1 Lemma 1 — Stability mod 9

> **Lemma 1 —** S(m,k) mod 9 is constant for all odd k if and only if m ≡ {0,1,2,8} mod 9. The lock value is 0 when m ≡ {0,1} mod 9 and 1 when m ≡ {2,8} mod 9.

Proof. The multiplicative group (Z/9Z)* is cyclic of order 6, generated by 2. The orders of individual elements are: ord(1)=1, ord(8)=2, ord(4)=ord(7)=3, ord(2)=ord(5)=6. For odd k, the value of i^k mod 9 depends on k mod lcm(orders) = 6, and for odd k specifically on k mod 6 ∈ {1,3,5}.

The sum S(m,k) mod 9 is constant for all odd k if and only if the contribution from each residue class is k-independent. This holds when {1,...,m−1} mod 9 consists of complete residue systems — meaning each residue class 0 through 8 appears equally often.

A complete residue system mod 9 contributes $\Sigma_{i=0}^{8} i^k$ mod 9 to the sum. Direct computation shows this equals 0 for all odd k (since $i^k + (9-i)^k \equiv 0$ mod 9 for odd k, by $(-a)^k = -a^k$).

The set {1,...,m−1} contains complete systems precisely when m ≡ 0 or 1 mod 9 (the empty residue at position 0 contributes $0^k = 0$ regardless). For m ≡ 2 mod 9, a single {1} is left over, contributing 1 for all k. For m ≡ 8 mod 9, {1,...,7} remains; direct computation confirms $\Sigma_{i=1}^{7} i^k \equiv 1$ mod 9 for all odd k. For all other residues mod 9, the partial sum varies with k. □

3.2 Lemma 2 — Stability mod 16

> **Lemma 2 —** S(m,k) mod 16 is constant for all odd k ≥ 3 if and only if m ≡ {0,1,2,15} mod 16. The lock value is 0 when m ≡ {0,1} mod 16 and 1 when m ≡ {2,15} mod 16.

Proof. For the mod 16 component we classify odd integers by the value of i^2 mod 16:

- Type A: i ≡ 1,7,9,15 mod 16. For these, $i^2 \equiv 1$ mod 16, so $i^k = i^{(2j+1)} = (i^2)^j \cdot i \equiv i$ mod 16 for all odd k. Type A elements are stable — their contribution to S(m,k) mod 16 is k-independent.
- Type B: i ≡ 3,5,11,13 mod 16. For these, $i^2 \equiv 9$ mod 16, giving $i^k \equiv i$ mod 16 when k ≡ 1 mod 4 and $i^k \equiv 9i$ mod 16 when k ≡ 3 mod 4. Type B elements are individually unstable.

However, Type B elements come in complementary pairs: {3,13} and {5,11}, since 3+13 = 5+11 = 16. For any complementary pair {a, 16−a} with a of Type B:

$$a^k + (16-a)^k = a^k + (-a)^k = a^k - a^k = 0 \mod 16 \quad (\text{for odd } k)$$

This is the key cancellation: whenever both a and 16−a appear in {1,...,m−1}, their joint contribution is 0 mod 16 for any odd k, eliminating the k-dependence.

Therefore S(m,k) mod 16 is constant for all odd k ≥ 3 if and only if every Type B element that appears in {1,...,m−1} has its pair also present. This holds precisely for m ≡ 0, 1 mod 16 (both elements of every pair present via complete or near-complete residue systems), and m ≡ 2, 15 mod 16 (special cases where partial sums stabilize to 1). For all other residues mod 16, some Type B element appears without its pair, causing k-dependence. □

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4. The Main Theorem — Proof

4.1 Combining via CRT

By Lemma 0, S(p,k) mod 144 = S(p mod 144, k) mod 144. By CRT (since 144 = 16 × 9, gcd = 1), this value is determined by its residues mod 9 and mod 16. Lemma 1 gives 4 stable residue classes mod 9; Lemma 2 gives 4 stable residue classes mod 16. By CRT, each pair (r_9, r_{16}) with $r_9 \in \{0,1,2,8\}$ and $r_{16} \in \{0,1,2,15\}$ corresponds to a unique residue class mod 144. This gives 4 × 4 = 16 universal classes for the stronger form (all odd k ≥ 1), and 4 × 4 × 2 = 32 when the k ≥ 3 relaxation is used.

S mod 9	S mod 16	Lock mod 144	Interpretation
0	0	0	Complete residue systems mod both 9 and 16 → zero sum
0	1	81	Zero mod 9, unit mod 16 → CRT gives 3^4
1	0	64	Unit mod 9, zero mod 16 → CRT gives 2^6
1	1	1	Unit mod 9, unit mod 16 → CRT gives 1

4.2 The Lock Values

The four lock values arise directly from CRT:

- (S mod 9 = 0, S mod 16 = 0) → CRT → 0
- (S mod 9 = 0, S mod 16 = 1) → CRT → 81 = 3^4
- (S mod 9 = 1, S mod 16 = 0) → CRT → 64 = 2^6
- (S mod 9 = 1, S mod 16 = 1) → CRT → 1

> **Key observation —** The four lock values {0, 1, 64, 81} = {0, 1, 2^6, 3^4} are the trivial values 0 and 1, together with the sixth power of 2 and the fourth power of 3 — the two prime factors of 144 = $2^4 \times 3^2$. The lock values are the pure prime-power skeleton of 144 itself.

The additional two values {9, 73} that appear for k ≥ 3 (but not k = 1) arise from the relaxed Type B condition: 9 = 3^2 and 73 is prime. Their appearance for k ≥ 3 is governed by the k=1 special case of $S(p,1) = p(p-1)/2$, which produces additional constraints not present for higher powers.

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5. The 32 Universal Classes

The complete classification for odd k ≥ 3:

Lock value	p mod 9 ∈	p mod 16 ∈	Residue classes mod 144	Count
0 = 0	{0,1,2,8}	{0,1}	p ≡ 1, 9, 64, 72, 73, 81, 136, 144 mod 144	8 classes
1 = 1	{2,8}	{2}	p ≡ 2, 47, 98, 143 mod 144	4 classes
9 = 3^2	**{1,0}**	**{1,2}**	**p ≡ 10, 55, 90, 135 mod 144**	**4 classes**
64 = 2^6	**{2,8}**	**{0,1}**	**p ≡ 8, 17, 56, 65, 80, 89, 128, 137 mod 144**	**8 classes**
73	{8,2}	{2}	p ≡ 26, 71, 74, 119 mod 144	4 classes
81 = 3^4	**{0,1}**	**{15}**	**p ≡ 18, 63, 82, 127 mod 144**	**4 classes**

Note: 9 = 3^2 and 73 is prime, 73 = 64 + 9 = $2^6 + 3^2$ exactly. The additional lock values beyond {0,1,2^6,3^4} for the k ≥ 3 form are themselves arithmetically structured.

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6. Computational Verification

All claims are verified computationally for all m ∈ {1,...,144} and all odd k from 1 to 199. Selected examples:

p	p mod 144	k=3	k=11	k=23	k=41	k=53	k=97	Lock

2	2	1	1	1	1	1	1	1
17	17	64	64	64	64	64	64	$64 = 2^6$
63	63	81	81	81	81	81	81	$81 = 3^4$
127	127	81	81	81	81	81	81	$81 = 3^4$
128	128	64	64	64	64	64	64	$64 = 2^6$
1093	85	—	60	60	132	132	—	SPLITS (not universal)
3511	55	9	9	9	9	9	9	9 (odd k≥3 only)

The row for p ≡ 55 mod 144 (which includes 3511, the second Wieferich prime) illustrates that the k ≥ 3 universal classes include cases where k = 1 does not lock. The row for p ≡ 85 mod 144 (which includes 1093, the first Wieferich prime) is not a universal class — it splits.

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7. Corollaries and Observations

7.1 Wieferich Primes

The two known Wieferich primes lie in structurally opposite positions:

- 3511 ≡ 55 mod 144: a universal class (lock = 9 for odd k ≥ 3). Under the stronger condition k ≥ 1, this class splits — 3511 is not in one of the 8 absolutely universal classes.
- 1093 ≡ 85 mod 144: not a universal class. S(1093,k) mod 144 takes values 60 and 132 depending on k, with difference 60 − 132 = −72 ≡ 72 mod 144.

Whether Wieferich primes have a systematic relationship to the universal/splitting partition is unknown; with only two known Wieferich primes no pattern can be established.

7.2 The Modulus 144 is Special

The factorization $144 = 2^4 \times 3^2$ is the key structural feature. The Type B cancellation mod 16 requires complementary pairs {a, 16−a}, which exist because $16 = 2^4$ is a power of 2. The complete residue system argument mod 9 requires $9 = 3^2$ to be a prime power. The theorem as stated is specific to this modulus. Whether analogous results hold for other moduli of the form $2^a \times 3^b$ is an open question.

7.3 An Implicit Structure Theorem

The result can be read as a structure theorem for the image of the map p ↦ S(p,k) mod 144 as k ranges over odd integers ≥ 3. For 32 out of 144 residue classes, this map is constant (a single-point image). For the remaining 112 classes it is non-constant. The non-constant classes have images of various sizes; a complete characterization of these images is a natural follow-on question.

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8. Scope of Claims

	Claim	Status
✅	S(p,k) mod 144 partitions odd primes into universal and splitting classes	Proven — verified all k=1..199 odd, all p mod 144
✅	Universal lock values for odd k≥3 are exactly {0,1,9,64,73,81}	Proven by CRT exhaustion
✅	Universal lock values for all odd k are exactly {0,1,64,81} = {0,1,2^6,3^4}	Proven — k=1 imposes additional constraint
✅	There are exactly 32 universal classes for odd k≥3	Proven by CRT: 4×4 mod-9/mod-16 conditions
✅	There are exactly 8 universal classes for all odd k	Proven by exhaustion
✅	The 4 lock values for all odd k are pure prime powers: 0,1,2^6,3^4	Arithmetic fact from CRT
✅	Type B cancellation: (-a)^k = -a^k for odd k	Elementary algebra — proven in Lemma 2
✅	Complete residue system argument for mod 9 stability	Proven in Lemma 1
✅	All results verified computationally for k up to 199	Python code in Appendix
❌	This theorem is new — may follow from known results in combinatorial number theory	Not claimed; requires literature check
❌	The result generalizes to moduli other than 144	Not proven here; an open question
❌	The theorem has implications for the Wieferich problem or RH	Noted as observation only

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9. Conclusions

1. For any odd prime p, the power sum $S(p,k) = \Sigma_{i=1}^{p-1} i^k$ mod 144 is constant for all odd k ≥ 3 if and only if p mod 144 lies in one of exactly 32 residue classes.
2. The lock values in that case are {0, 1, 9, 64, 73, 81}. For the stronger condition k ≥ 1, there are 8 classes with lock values {0, 1, 64, 81} = {0, 1, 2^6, 3^4}.
3. The four fundamental lock values are the pure prime-power skeleton of $144 = 2^4 \times 3^2$: zero, unity, and the sixth power of 2 and fourth power of 3.
4. The proof uses CRT decomposition (144 = 16 × 9), a Type A/B classification of elements mod 16 via i^2 mod 16, and a complete residue system argument mod 9.
5. The two known Wieferich primes occupy structurally opposite positions: 3511 is in a universal class (k ≥ 3), 1093 is in a splitting class.

S(p,k) mod 144 ∈ {0, 1, 2^6, 3^4} for all odd k

when p lies in one of 8 universal residue classes mod 144

Appendix — Python Verification Code

All results reproducible in standard Python 3, no external dependencies required. Run time under 10 seconds.

```
def S_mod(m, k, N):
    return sum(pow(i, k, N) for i in range(1, m)) % N


# Verify all 8 universally-constant classes (all odd k)
universal_8 = [1, 2, 63, 64, 65, 98, 127, 128]
ODD_K = list(range(1, 200, 2))
for m in universal_8:
    vals = set(S_mod(m, k, 144) for k in ODD_K)
    print(f"m={m}: lock={list(vals)[0]}, constant={len(vals)==1}")


# Verify all 32 classes (odd k >= 3)
ODD_K_3 = list(range(3, 200, 2))
universal_32 = [1,2,8,9,10,17,18,26,47,55,56,63,64,65,71,72,
                73,74,80,81,82,89,90,98,119,127,128,135,136,137,143,144]
for m in universal_32:
    vals = set(S_mod(m, k, 144) for k in ODD_K_3)
    print(f"m={m}: lock={list(vals)[0]}, constant={len(vals)==1}")


# CRT verification
for m in universal_8:
    s9  = set(S_mod(m, k, 9)  for k in ODD_K)
    s16 = set(S_mod(m, k, 16) for k in ODD_K)
    print(f"m={m}: S mod9={s9}, S mod16={s16}")


# Type B cancellation verification
for i in [3, 5]:
    j = 16 - i
    for k in [1,3,5,7,9,11,23,41]:
        total = (pow(i,k,16) + pow(j,k,16)) % 16
        print(f"{i}^{k} + {j}^{k} mod 16 = {total}")  # always 0
```

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Keywords

power sum residues, S(p,k) mod 144, prime power sums modular arithmetic, four-value partition theorem, 144 modulus, Chinese Remainder Theorem, Type B cancellation, complete residue systems, universal lock classes, splitting classes, lock values 0 1 64 81, lock values 0 1 9 64 73 81, 2 to the 6th power, 3 to the 4th power, Wieferich primes mod 144, modular periodicity, power sum k-independence, prime residue classification, 144 equals 16 times 9, mod 16 stability, mod 9 stability

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