tort, en donnerait 56. Si, au lieu de deux conditions (p, q), on en introduit un plus grand nombre, l'écart entre les deux résultats ira en grandissant. Par exemple, pour cinq conditions (1, 1) le nombre exact est $C^4 = 2376$, tandis que l'ancienne théorie donnerait le nombre 3264, le même que pour celui des coniques qui touchent cinq coniques données.

Note on a Modular Equation for the Transformation of the Third Order. *By* HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read February 13th*, 1879.]

I have given elsewhere* the modular equation for the transformation of the third order between

$$x = f(k^2) = \frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2} \quad \text{and} \quad y = f(\lambda^2) = \frac{(1-\lambda^2+\lambda^4)^3}{\lambda^4(1-\lambda^2)^2};$$

viz.,

$$\begin{aligned} F(x, y) = {} & x\,(x+2^7.3.5^3)^3 + y\,(y+2^7.3.5^3)^3 \\ & -2^{16}x^3y^3 + 2^{11}.3^2.31\,x^2y^2\,(x+y) \\ & -2^2.3^3.9907\,xy\,(x^2+y^2) + 2.3^4.13.193.6367\,x^2y^2 \\ & +2^8.3^5.5^3.4471\,xy\,(x+y) - 2^{15}.5^6.22973\,xy = 0. \end{aligned}$$

The following is the process by which the coefficients were determined.

It follows from the general theory of modular equations that $F(x, y)$ is symmetrical with respect to x and y; and that it is of the order 4 in x and y separately, and of the order 6 in x and y jointly. Hence $F(x, y)$ is of the form

$$\begin{aligned} & A_{3,3}\,x^3y^3 + A_{4,2}\,x^2y^2\,(x^2+y^2) + A_{4,1}\,xy\,(x^3+y^3) \\ & \quad + (x^4+y^4) + A_{3,2}\,x^2y^2\,(x+y) + A_{3,1}\,xy\,(x^2+y^2) \\ & \quad + A_{3,0}\,(x^3+y^3) + A_{2,2}\,x^2y^2 + A_{2,1}\,xy\,(x+y) \\ & \quad + A_{2,0}\,(x^2+y^2) + A_{1,1}\,xy + A_{1,0}\,(x+y) + A_{0,0}. \end{aligned}$$

Several of the coefficients may be conveniently found by employing the method of Sohncke; *i.e.*, by substituting for x and y their expressions as series proceeding by powers of q, and equating the coefficients of the powers of q to zero. We have, by a known formula,

$$k^2(1-k^2) = 2^4 q \prod_{m=1}^{m=\infty} (1+q^{2m-1})^{-24};$$

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* "Proceedings of the London Mathematical Society," Vol. IX., p. 243, note.

calling this quantity z, we have $x = \frac{(1-z)^3}{z^3}$;

we also write $$Z = 2^4 q^3 \prod_{m=1}^{m=\infty} (1+q^{6m-3})^{-24},$$

so that $$y = \frac{(1-Z)^3}{Z^3}.$$

Substituting these values, and equating to zero the coefficients of q^{-28}, q^{-26}, and q^{-24}, we find successively

$$A_{4,2} = 0, \quad A_{4,1} = 0, \quad A_{3,3} = -2^{16}.$$

To determine $A_{3,2}$, we observe that q^{-22} presents itself only in the terms

$$-2^{16} x^3 y^3 + A_{3,2} x^2 y^3$$

Its coefficient in $x^2 y^3$ is 2^{-40}; its coefficient in $x^3 y^3$ is $3^2 . 31 \times 2^{-45}$; viz., this is the coefficient of q^2 in

$$\frac{1}{2^{24}} \left[\frac{1}{2^{24}} (1+q)^{144} - \frac{9}{2^{20}} q (1+q)^{120} + \frac{9}{2^{14}} q^2 \right].$$

Hence $$A_{3,2} \times 2^{-40} + 2^{-45} . 3^2 . 31 . A_{3,3} = 0,$$

or $$A_{3,2} = 2^{11} . 3^2 . 31.$$

Again, to determine $A_{3,1}$ we consider the coefficient of q^{-20}; this power of q presents itself only in the terms

$$y^3 [A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x].$$

It will be found that in the development of y^3 there is no power of q intermediate between q^{-18} and q^{-12}; hence, the coefficient of q^{-2} in

$$A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x,$$

or the coefficient of q^4 in

$$\begin{aligned} A_{3,3} \Big[& \frac{1}{2^4} (1+q)^{144} (1+q^3)^{144} - \frac{3^2}{2^{20}} q (1+q)^{120} (1+q^3)^{120} \\ & + \frac{3^3}{2^{14}} q^2 (1+q)^{100} - \frac{3 . 7}{2^{10}} q^3 (1+q)^{72} - \frac{3^2 . 7}{2^7} q^4 \Big] \\ & + A_{3,2} \left[\frac{1}{2^{16}} q^2 (1+q)^{96} - \frac{3}{2^{11}} q^3 (1+q)^{72} + \frac{3 . 5}{2^8} q^4 \right] \\ & + \frac{A_{3,1}}{2^8} \times q^4 \end{aligned}$$

is equal to zero. Substituting the values of $A_{3,3}$, $A_{3,2}$, and reducing, we find $$A_{3,1} = -2^2 . 3^3 . 9907.$$

All the coefficients may successively be determined by this method, but the work becomes very laborious for the later coefficients. They may be more easily obtained by the consideration that, if ξ, η are any two corresponding values of k^2 and λ^2, $f(\xi)$ and $f(\eta)$ are corresponding values of x and y. Availing ourselves of this principle, we find

$$\text{(i.)}\ F(0, y) = y\,(y+2^7 . 3 . 5^3)^3,$$

$$\text{(ii.)}\ F\left(\frac{27}{4}, y\right) = \left(y^2 - \frac{3^3}{2}\times 133283y - \frac{3^8 . 11^3 . 23^3}{2^4}\right)^2;$$

$$\text{(iii.)}\ F(y, y) = -2^{16}y\left(y-\frac{3^3 . 5^3}{16}\right)\left(y-\frac{5^3}{4}\right)^2(y+2^7)^2.$$

From (1) we infer that $F(x, y)$ is of the form

$$\text{(A)}\ \ldots\ldots \begin{cases} y\,(y+2^7 . 3 . 5^3)^3 + x\,(x+2^7 . 3 . 5^3)^3 - 2^{16}x^3y^3 \\ \qquad +2^{11} . 3^2 . 31x^2y^2(x+y) - 2^2 . 3^3 . 9907xy\,(x^2+y^2) \\ \qquad +A_{2,2}\,x^2y^2 + A_{2,1}xy\,(x+y) + A_{1,1}xy = 0. \end{cases}$$

From (iii.) we find

$$\begin{aligned} A_{1,1} &= -2^{15} . 5^6 . 22973, \\ A_{2,1} &= 2^8 . 3^5 . 5^3 . 4471, \\ A_{2,2} &= 2^{11} . 1262587 + 2^3 . 3^3 . 9907 - 2 \\ &= 2 . 3^4 . 15974803 \\ &= 2 . 3^4 . 13 . 193 . 6367. \end{aligned}$$

This completes the determination of the coefficients; a verification is supplied by the equation (ii.); and it only remains to show how the equations (i.), (ii.), (iii.) are inferred from the modular equation between $\xi = k^2$ and $\eta = \lambda^2$; viz.,

$$\Phi(\xi, \eta) = (\xi^2+6\xi\eta+\eta^2)^2 - 16\xi\eta\,(4\xi\eta-3\xi-3\eta+4)^2 = 0.$$

(i.) Let ρ denote either root of the equation $\rho^2-\rho+1=0$; if $k^2=\rho$, we have $x = f(\rho) = 0$; we also find

$$\Phi(\rho, \eta) = (\eta-\rho)\,(\eta^3 - [128-253\rho]\,\eta^2 - (128+253\rho^2)\,\eta + 1).$$

If $\omega_1, \omega_2, \omega_3$ are the roots of the cubic factor, the four roots of $F(0, y)$ are

$$f(\rho) = 0,\quad f(\omega_1),\quad f(\omega_2),\quad f(\omega_3).$$

But the cubic factor is one of the two conjugate factors of the expression

$$(\eta^2-\eta+1)^3 + 2^7 . 3 . 5^3 . \eta^2(\eta-1)^2.$$

Hence
$$f(\omega_1) = f(\omega_2) = f(\omega_3) = -2^7 . 3 . 5^3,$$

and
$$F(0, y) = y\,(y+2^7 . 3 . 5^3)^3.$$

(ii.) Let $k^2 = \frac{1}{2}$, so that $f(k^2) = \frac{27}{4}$; we find

$$\Phi\left(\tfrac{1}{2}, y\right) = (\eta-\eta^2)^2 - 48\tfrac{1}{2}\,(\eta-\eta^2) + \tfrac{1}{16}.$$

If $\eta_1, \eta_2, \eta_3, \eta_4$ are the roots of $\Phi\left(\frac{1}{2}, \eta\right)$, the four roots of $F\left(\frac{27}{4}, y\right)$ are $f(\eta_1), f(\eta_2), f(\eta_3), f(\eta_4)$; *i.e.* if z_1, z_2 are the roots of $z^2-48\frac{1}{2}z+\frac{1}{16}$, these four roots are $z_1^{-2}(1-z_1)^3$, $z_2^{-2}(1-z_2)^3$, each taken twice. Hence

$$F\left(\tfrac{27}{4}, y\right) = [y-z_1^{-2}(1-z_1)^3]^2[y-z_2^{-2}(1-z_2)^3]^2;$$

and by the ordinary methods we find

$$[y-z_1^{-2}(1-z_1)^3]\,[y-z_2^{-2}(1-z_2)^3]^2$$
$$= y^2-\frac{3^2}{2}\times 133283y-\frac{3^3\,.\,11^3\,.\,23^3}{2^4}.$$

(iii.) Since the only roots of the equation

$$f(\eta) = f(\xi)$$

are
$$\xi,\ 1-\xi,\ \frac{1}{\xi},\ \frac{1}{1-\xi},\ \frac{\xi-1}{\xi},\ \frac{\xi}{\xi-1},$$

the roots of the equation $F(y, y) = 0$ are all of the form $f(\theta)$, where θ is a root of one of the six equations—

(1) .... $\Phi(\theta, \theta) = \theta^2(\theta-1)^2(\theta^2-\theta+1) = 0,$

(2) ...... $\Phi(\theta, 1-\theta) = (16\theta^2-16\theta+1)(4\theta^2-4\theta-1)^2 = 0,$

(3) ...... $\theta^4\Phi\left(\theta, \frac{1}{\theta}\right) = (\theta-1)^2(\theta^2-6\theta+1)^2(\theta^2+14\theta+1) = 0,$

(4) ...... $(1-\theta)^4\Phi\left(\theta, \frac{1}{1-\theta}\right)$
$$= (\theta^2-\theta+1)\times[(1-\theta+\theta^2)^3+128\theta^2(1-\theta)^2] = 0,$$

(5) ...... $\theta^4\Phi\left(\theta, \frac{\theta-1}{\theta}\right)$
$$= (\theta^2-\theta+1)\times[(\theta^2-\theta+1)^3+128\theta^2(\theta-1)^2] = 0,$$

(6) ...... $(\theta-1)^4\Phi\left(\theta, \frac{\theta}{\theta-1}\right) = \theta^2(\theta^2-16\theta+16)(\theta^2+4\theta-4)^2 = 0.$

These equations are of order 8, the first and second having each two infinite roots. The 48 roots give, in all, five distinct values for $f(\theta)$ according to the following scheme:—

A. $f(\theta) = \infty$; 12 roots, viz.—

$\theta = \infty$, 2 roots in (1) and in (2);
$\theta = 0$, 2 roots in (1) and in (6);
$\theta = 1$, 2 roots in (1) and in (3).

B. $f(\theta) = 0$; 6 roots, viz,

$\theta^2-\theta+1 = 0$, in (1), (4), and (5).

C. $f(\theta) = \dfrac{3^3 \cdot 5^3}{2^4}$; 6 roots, viz.—

$$16\theta^2 - 16\theta + 1 = 0, \quad \text{in (2)};$$
$$\theta^2 + 14\theta + 1 = 0, \quad \text{in (3)};$$
$$\theta^2 - 16\theta + 16 = 0, \quad \text{in (6)}.$$

D. $f(\theta) = -128$; 12 roots, viz.—

$$(\theta^2 - \theta + 1)^3 + 128\theta^2(\theta - 1)^2 = 0, \text{ in (4) and (5)}.$$

E. $f(\theta) = \dfrac{5^3}{2^2}$; 12 roots, viz.—

$$(4\theta^2 - 4\theta - 1)^2 = 0, \quad \text{in (2)};$$
$$(\theta^2 - 6\theta + 1)^2 = 0, \quad \text{in (3)};$$
$$(\theta^2 + 4\theta - 4)^2 = 0, \quad \text{in (6)}.$$

Hence the roots of $F(y, y) = 0$ are $0, \dfrac{3^3 \cdot 5^3}{2^4}$, each once; ∞, -2^7, $\dfrac{5^3}{2^2}$, each twice; *i.e.*—

$$F(y, y) = -2^{16} y \left(y - \frac{3^3 \cdot 5^3}{2^4}\right) \left(y - \frac{5^3}{2^2}\right)^2 (y + 2^7)^2.$$

The multiplicity of the roots of $F(y, y) = 0$ may be otherwise, and more simply, determined as follows. The form of $F(x, y)$ (see the equation A, *supra*) shows that $F(y, y)$ has one root equal to zero and two roots equal to infinity; the multiplicities of the finite roots are determined by the equation

$$2^{16}\left[\frac{3^3 \cdot 5^3}{2^4} + 2 \times \frac{5^3}{2^2} - 2 \times 2^7\right] = 2A_{3,2} = 2^{13} \cdot 3^2 \cdot 31.$$

Note on the Formula for the Multiplication of Four Theta Functions. By HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read February 13th*, 1879.]

The normal formula for the multiplication of four Theta Functions is ("Proceedings of the London Mathematical Society," Vol. I., Part viii., p. 4)

$$(1) \ldots \left\{ \begin{aligned} & 2\theta_{\mu_1, \mu_1'}(x_1)\, \theta_{\mu_2, \mu_2'}(x_2)\, \theta_{\mu_3, \mu_3'}(x_3)\, \theta_{\mu_4, \mu_4'}(x_4) \\ & = \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j, \sigma'-\mu_j'}(s - x_j) + \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j, \sigma'-\mu_j'+1}(s - x_j) \\ & + (-1)^{\sigma'} \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j+1, \sigma'-\mu_j'}(s - x_j) + (-1)^{\sigma'+1} \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j+1, \sigma'-\mu_j'+1}(s - x_j). \end{aligned} \right.$$