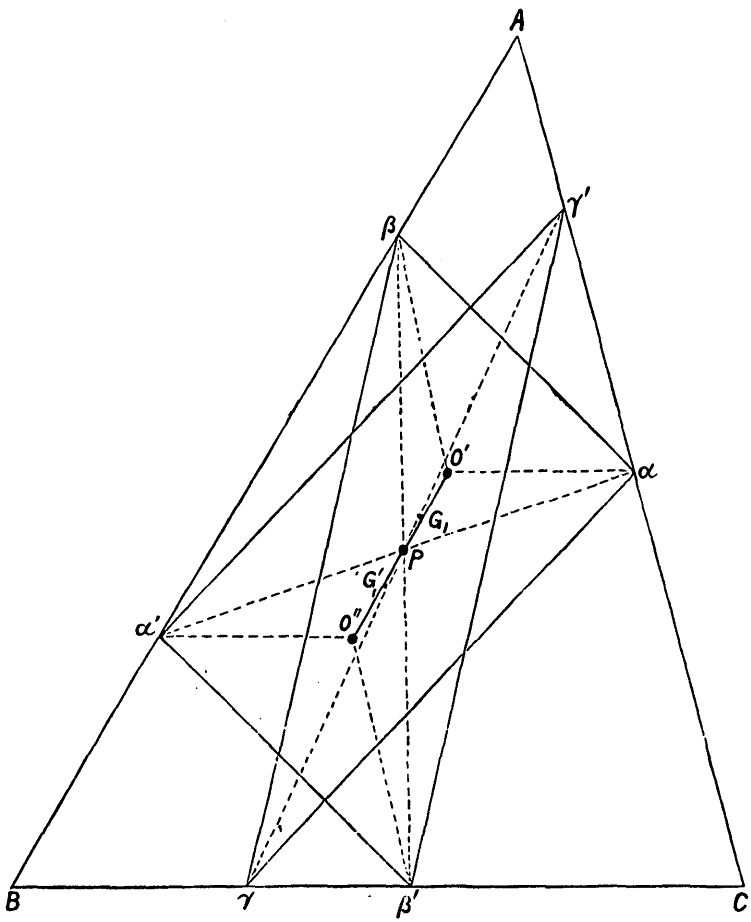
Two In-Triangles which are similar to the Pedal Triangle. *By* R. TUCKER. Received September 7th, 1900. Read November 8th, 1900.

1. If ABC is the primitive triangle, then the angles of the pedal triangle are $\pi-2A, \pi-2B, \pi-2C$. The in-triangles, whose sides are anti-parallels, are triangles which are similar to the pedal triangle.



The in-triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, whose sides are perpendicular to anti-parallels, are another pair of similar triangles.

For we have $A\alpha\beta = \frac{\pi}{2} - B$, $A\alpha\gamma = \frac{\pi}{2} - A + C$;

therefore $\angle\alpha = A\alpha\gamma - A\alpha\beta = \pi - 2A$,

and so for the other angles.

Again, $B\beta'\alpha' = \frac{\pi}{2} - A$, $B\beta'\gamma' = \frac{\pi}{2} - B + C$;

therefore $\angle\beta' = \pi - 2B$,

and so for the other angles.

Hence $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are similar to the pedal triangle.

2. If λ be a constant to be determined, and if, for brevity, we write p, q, r for $\sin 2A$, $\sin 2B$, $\sin 2C$, we have

$$B\gamma \sin B = \lambda . p \cos C \quad \text{and} \quad C\gamma \sin C = \lambda . q \cos (C-A),$$

then $$2a \sin B \sin C = 2\lambda \left[p \sin C \cos C + q \sin B \cos (C-A) \right]$$
$$= \lambda . \Sigma (qr) = 2\lambda \left[\Pi \cos A + \Pi \cos (B-C) \right],$$

and $$\lambda = 4R . \Pi \sin A / \Sigma (qr). \qquad \text{(i.)}$$

If λ' corresponds to λ, then

$$B\beta' \sin B + C\beta' \sin C = \lambda' \left[r \cos (A-B) + p \cos B \right],$$

whence, as before, $\lambda' = 4R . \Pi \sin A / \Sigma (qr) = \lambda$,

i.e., $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are congruent.

3. Now $\beta'\gamma = B\beta' - B\gamma = \lambda \left[r \cos (A-B) - p \cos C \right] / \sin B$
$= \lambda \cos C . q / \sin B = 2\lambda \cos B \cos C \propto \sec A.$

4. Using trilinear coordinates, we have

$$\left.\begin{array}{llll}
(\alpha) & q \cos A & 0 & r \cos (A-B) \\
(\beta) & p \cos (B-C) & r \cos B & 0 \\
(\gamma) & 0 & q \cos (C-A) & p \cos C \\
(\alpha') & r \cos A & q \cos (C-A) & 0 \\
(\beta') & 0 & p \cos B & r \cos (A-B) \\
(\gamma') & p \cos (B-C) & 0 & q \cos C
\end{array}\right\} . \qquad \text{(ii.)}$$

5. The lines $\alpha\alpha'$, $\beta\beta'$ are

$$-qr\cos(A-B)\cos(C-A)\,\alpha+r^2\cos A\cos(A-B)\,\beta$$
$$+q^2\cos A\cos(C-A)\,\gamma=0,$$

$$r^2\cos B\cos(A-B)\,\alpha-rp\cos(B-C)\cos(A-B)\,\beta$$
$$+p^2\cos B\cos(B-C)\,\gamma=0;$$

hence $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ cointersect in P, the centre of similitude* of the triangles, given by

$$a/p\cos(B-C)=\ldots=\ldots. \qquad \text{(iii.)}$$

6. The equations to $\alpha\beta$, $\gamma'\alpha'$ are respectively

$$-r\cos B\cos(A-B)\,\alpha+p\cos(A-B)\cos(B-C)\,\beta+q\cos A\cos B\,.\,\gamma=0, \qquad \text{(iv.)}$$

$$-q\cos C\cos(C-A)\,\alpha+r\cos A\cos C\beta+p\cos(B-C)\cos(C-A)\,\gamma=0. \qquad \text{(v.)}$$

Let their point of intersection be P' (Q', R' for the analogous pairs); then AP', BQ', CR' meet in

$$a/\cos(B-C),\ \ldots,\ \ldots,\ \textit{i.e.}, \text{ in the nine-point centre.} \qquad \text{(vi.)}$$

7. The equations to $\alpha'\beta'$, $\gamma\alpha$ are

$$q\cos(A-B)\cos(C-A)\,\alpha-r\cos A\cos(A-B)\,\beta+p\cos A\cos B\,.\,\gamma=0, \qquad \text{(vii.)}$$

$$r\cos(C-A)\cos(A-B)\,\alpha+p\cos C\cos A\,.\,\beta-q\cos A\cos(C-A)\,\gamma=0. \qquad \text{(viii.)}$$

Let their point of intersection be P'' (Q'', R'' for the analogous pairs); then AP'', BQ'', CR'' meet in

$$a/\cos A=\ldots=\ldots,\ \textit{i.e.}, \text{ in the circumcentre.} \qquad \text{(ix.)}$$

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* [Many of the geometrical results follow at once from this fact, but the equations to the lines and points are given, as they may suggest other properties. Further, P is the P'' of my paper "On a Group of Triangles inscribed in a given Triangle ABC, &c.," Vol. xxiv., pp. 131–142, whence other properties can be derived than those given in the present paper.]

8. The centroids of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ respectively are given by

$$\left.\begin{array}{ll} & \alpha/\cos A\,(2q+r) = \ldots = \ldots\ (G_1) \\ \text{and} & \alpha/\cos A\,(q+2r) = \ldots = \ldots\ (G_1') \end{array}\right\}; \qquad \text{(x.)}$$

therefore their join is given by the equation

$$\Sigma \cos B \cos C\,(qr-p^2)\,\alpha = 0, \qquad \text{(xi.)}$$

and this passes through $(q+r)\cos A$, ..., ..., *i.e.*, through $p\cos(B-C)$, ..., ..., *i.e.*, the point P [*cf.* (iii.)].

9. If O', O'' are the in-centres of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ respectively, then, since

$$\angle\, C\alpha O' = C\alpha\gamma + \gamma\alpha O' = \left(\frac{\pi}{2}-C+A\right)+\left(\frac{\pi}{2}-A\right) = \pi - C,$$

$\alpha O'$ is parallel to BC, and similarly $\beta O'$ is parallel to CA, and $\gamma O'$ is parallel to AB.

In like manner, $\alpha'O''$, $\beta'O''$, $\gamma'O''$ are respectively parallel to BC, CA, AB. Hence their coordinates are given by

$$\left.\begin{array}{ll} & q\cos A,\ \ r\cos B,\ \ p\cos C \\ \text{and} & r\cos A,\ \ p\cos B,\ \ q\cos C \end{array}\right\}. \qquad \text{(xii.)}$$

Hence the equation to $O'O''$ is

$$\Sigma\alpha \cos B \cos C\,(qr-p^2) = 0,\ \ldots\ [cf.\ \text{(xi.)}],$$

and the mid-point of $O'O''$ is P (iii.).

10. The symmedian line through α [*cf.* (iv.) and (viii.)] is

$$-r\cos B\cos(A-B)\,\alpha + p\cos(A-B)\cos(B-C)\,\beta + q\cos A\cos B\,.\,\gamma$$
$$= \lambda\left[r\cos(C-A)\cos(A-B)\,\alpha + p\cos C\cos A\,.\,\beta - q\cos A\cos(C-A)\,\gamma\right].$$

If, for the moment, this line cuts $\beta\gamma$ in D, then

$$\beta D : D\gamma = r^2 : q^2;$$

hence, from (ii.), we get the coordinates of D to be proportional to

$$pq^2\cos(B-C),\quad q^2r\cos B + qr^2\cos(C-A),\quad r^2p\cos C.$$

Substituting in the above equation to αD, we get, after dividing by $S\,[\equiv \Sigma\,(qr)]$,

$$\lambda r = -q;$$

and the equation to αD becomes

$$\alpha r \cos(A-B)\left[r\cos(C-A)+q\cos B\right]$$
$$+\beta p\left[r\cos C\cos A-q\cos(A-B)\cos(B-C)\right]$$
$$-\gamma q\cos A\left[q\cos B+r\cos(C-A)\right]=0. \qquad \text{(xiii.)}$$

The symmedian through β then is

$$-\alpha r\cos B\left[r\cos C+p\cos(A-B)\right]$$
$$+\beta p\cos(B-C)\left[p\cos(A-B)+r\cos C\right]$$
$$+\gamma q\left[p\cos A\cos B-r\cos(B-C)\cos(C-A)\right]=0.$$

Hence the symmedian point of $\alpha\beta\gamma$ is

$$pq\left[p\cos A+q\cos(B-C)\right],\ \ldots,\ \ldots; \qquad \text{(xiv.)}$$

and similarly of $\alpha'\beta'\gamma'$ is

$$rp\left[p\cos A+r\cos(B-C)\right],\ \ldots,\ \ldots.$$

11. Drawing αX, perpendicular to $\beta\gamma$, to meet it in X (*i.e.*, parallel to an anti-parallel), we get X given by

$$q\cos 2C\cos(B-C), \quad -qr\sin B, \quad r\cos 2B\cos 2C;$$

and from α and X we can find the coordinates of H_1 (the orthocentre of $\alpha\beta\gamma$) to be

$$a\cos 2C, \quad b\cos 2A, \quad c\cos 2B;$$

similarly H_2 (for $\alpha'\beta'\gamma'$) is given by

$$a\cos 2B, \quad b\cos 2C, \quad c\cos 2A. \qquad \text{(xv.)}$$

Hence the equation to H_1H_2 is

$$\Sigma bc\alpha\,(\cos^2 2A-\cos 2B\cos 2C)=0, \qquad \text{(xvi.)}$$

a line which passes through P.

12. The circles $\alpha\beta\gamma$, $\alpha'\beta'\gamma$ are given by

$$\left.\begin{aligned} S^2.\Sigma a\beta\gamma &= 2\Sigma a\alpha\left[\Sigma qr\sin C\cos(A-B)\sin(2C-A)\,\alpha\right] \\ S^2.\Sigma a\beta\gamma &= 2\Sigma a\alpha\left[\Sigma qr\sin B\cos(C-A)\sin(2B-A)\,\alpha\right] \\ &\qquad (cf.\ \S\,10) \end{aligned}\right\}. \qquad \text{(xvii.)}$$

Their radical axis is

$$\Sigma\left[\alpha qr\sin A\sin(B-C)(\cos 3A-2\cos B\cos C)\right]=0, \qquad \text{(xviii.)}$$

and it passes through the circumcentre.

13. The circles $C\beta'\gamma'$, $A\gamma'\alpha'$ have for equations

$$S.\Sigma a\beta\gamma = 2\Sigma a\alpha\left[p\sin B\alpha+(p+q)\sin A\beta\right]\cos C, \qquad \text{(xix.)}$$

$$S.\Sigma a\beta\gamma = 2\Sigma a\alpha\left[q\sin C\beta+(q+r)\sin B\gamma\right]\cos A; \qquad \text{(xx.)}$$

and the circles $B\beta\gamma$, $C\gamma\alpha$ are given by

$$S.\Sigma a\beta\gamma = 2\Sigma a\alpha\left[r\sin C\alpha+(r+p)\sin A\gamma\right]\cos B, \qquad \text{(xxi.)}$$

$$S.\Sigma a\beta\gamma = 2\Sigma a\alpha\left[p\sin A\beta+(p+q)\sin B\alpha\right]\cos C. \qquad \text{(xxii.)}$$

Hence the radical axis of $C\beta'\gamma'$ and $C\gamma\alpha$ is

$$\alpha/a = \beta/b;$$

and therefore it, and the analogous radical axes, pass through K, the symmedian point of ABC.

14. From the above we see that the radical axis of the circles $B\beta\gamma$, $C\beta'\gamma'$ is

$$\alpha\left[p\sin B\cos C - r\sin C\cos B\right] + \left[\beta(p+q)\cos C-\gamma(r+p)\cos B\right]\sin A = 0; \qquad \text{(xxiii.)}$$

hence, if it cuts BC in L, and the analogous radical axes cut CA, AB in M, N respectively, then these axes meet in P.

15. The circles $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$ have their equations

$$S.\Sigma a\beta\gamma = \Sigma a\alpha\,(cr\beta+bq\gamma)\cos A, \qquad \text{(xxiv.)}$$

$$S.\Sigma a\beta\gamma = \Sigma a\alpha\,(ap\gamma+cr\alpha)\cos B, \qquad \text{(xxv.)}$$

$$S.\Sigma a\beta\gamma = \Sigma a\alpha\,(bq\alpha+ap\beta)\cos C. \qquad \text{(xxvi.)}$$

The radical axis of (xxiv.) and (xxv.) is

$$cr(\alpha\cos B-\beta\cos A)+(ap\cos B-bq\cos A)\,\gamma = 0.$$

If this cuts AB in N' (and L', M' are analogous points), then AL', BM', CN' pass through the circumcentre.

16. The equation to the conic through the six points is

$$\Sigma\left[qr\alpha^2/\cos A\cos(B-C)\right] = \Sigma\left[(S+2p^2)\beta\gamma/\cos(C-A)\cos(A-B)\right]. \qquad \text{(xxvii.)}$$

17. It may be noted that

$$\left.\begin{aligned} \angle A\alpha\beta &= \frac{\pi}{2}-B = C\gamma'\beta' \\ \angle B\beta\gamma &= \frac{\pi}{2}-C = A\alpha'\gamma' \\ \angle C\gamma\alpha &= \frac{\pi}{2}-A = B\beta'\alpha' \end{aligned}\right\}.$$

18. To construct the figure, let DEF be the pedal triangle; then its sides are Rp, Rq, Rr.

If DK, EL, FM are the perpendiculars of DEF, then

$$DK = Rqr, \quad EL = Rrp, \quad FM = Rpq. \qquad (a)$$

Now
$$\lambda = 4R\Pi\,(\sin A)/\Sigma\,(qr),$$
$$= R\,(p+q+r)/\Sigma\,(qr),$$
$$= R\,\frac{DE+EF+FD}{DK+EL+FM}.$$

Hence the sides of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ (*i.e.*, $\lambda . Rp$, $\lambda . Rq$, $\lambda . Rr$) are known.

[I am indebted to a referee for a suggestion which enables me to considerably simplify the construction, viz.,

$$B\gamma' : \gamma\beta' : \beta'C = \text{cosec}\,2B : \text{cosec}\,2A : \text{cosec}\,2C,$$

i.e., by (a),
$$= EL : FM : DK.]$$

On Quantitative Substitutional Analysis. *By* A. YOUNG. Communicated November 8th, 1900. Received November 9th, 1900. Received, in revised form, January 12th, 1901.

From any function P of n variables may be obtained $n!$ functions, not necessarily all different, by permuting the variables in P in all possible ways; or, what is the same thing, by operating on P with each of the $n!$ substitutions of the symmetric group of the variables. It frequently happens that between these functions linear relations with constant coefficients exist; such may be written

$$(\lambda_1+\lambda_2 s_2+\lambda_3 s_3+\ldots)\,P = 0,$$