On some Properties of certain Solutions of a Differential Equation of the Second Order. By Dr. E. J. Routh.

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If we write down Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \quad \text{..................(1)},$$

we have a solution, when n is a positive integer, usually called a Legendre's coefficient, and written $y=P_n$. The chief properties of this function may be briefly summed up. (1) It is an n^{th} differential coefficient. (2) It satisfies a scale of relation by which P_{n+2} is found in terms of P_n and P_{n+1}. (3) All the roots of the equation $P_n=0$ are real, and are confined between certain limits. (4) The integral $\int P_m P_n\,dx$ vanishes between the same limits, while $\int P_n^2\,dx$ has a known value. Also (5) P_n may be generated as the coefficient of t^n by the expansion of a known function of t.

Let us now generalize the differential equation by introducing arbitrary letters into every coefficient. It now becomes

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0 \quad \text{..............(2)},$$

where h is a function of some parameter n, which we shall afterwards take to be a positive integer. Two solutions at least of this generalized equation have been given which are n^{th} differential coefficients, each of which reduces to P_n when the differential equation takes the form (1). Representing either of these by $y=X_n$, we propose to show that the corresponding scale of relation is of the form

$$AX_{n+2}+(B+Cx)X_{n+1}+DX_n=0,$$

where (A, B, C, D) are constants which depend on (a, b, c, f, g, n). Following the same general line of argument as that used in the case of Legendre's coefficients, we shall show that, provided $4-f/a$ in the first solution or f/a in the second is positive, and in each case numerically greater than $\dfrac{2g-bf/a}{\pm(b^2-4ac)^{\frac{1}{2}}}$ (the root being supposed real), the properties of X_n/X_0 follow exactly those just mentioned for P_n.

The differential equation (2) has been discussed by a great many authors, chiefly with the object of finding its general integral. For some of the following references the writer is indebted to the kindness of the referees of this paper.

An equation which reduces to the form (2) by easy transformations is discussed by Euler in his Integral Calculus. He obtains in certain cases solutions in which y is expressed in a finite series of powers of x. As he does not put these into the form of an n^{th} differential coefficient, his solutions do not at present concern us. In 1812 or 1813, Gauss (Collected Works, Vol. III.) expressed the solution as a hypergeometrical series. Liouville, in the thirteenth volume of the *Journal de l'Ecole Polytechnique*, 1832, discusses the equation in a manner which, in the beginning, is the same as that in Art. 3 of this paper. He obtains an integral in the form of an n^{th} differential coefficient, and discusses the meaning of the result when n is not an integer. In 1860, Spitzer treats of that particular case of the equation (2) in which $a=0$, and expresses the solutions as n^{th} differential coefficients. Then, in 1868, H. J. Holmgren (Kongliga Svenska Vetenskaps-Akademiens Handlingar), after giving references to these works of Euler, Liouville, and Spitzer, mentions certain restrictions on their solutions. He takes Liouville's paper as his point of departure, and finds integrals in which these restrictions are removed. The integrals are expressed as n^{th} differentials, and the meaning when n is not a positive integer is given. The solution of the differential equation (2) is also discussed by Boole, under the name of Pfaff's equation, in the latter part of his Differential Equations. In Pfaff's treatise we find a great many cases specified in which the equation has been solved both as n^{th} differential coefficients or otherwise. Probably no case in which n is integral has escaped his search.

After so many writers have obtained solutions, it is not to be supposed that anything new is presented in the two n^{th} differentials found in Arts. 3 and 4. Taking these two forms as the point of departure, we pass on to the scale of relation as the theorem next in order. None of the writers already mentioned seem to have alluded to this result, so that it is possibly new.

Taking next the theorems on the roots of $X_n=0$, we find two papers by Sturm in the first volume of Liouville's *Journal*. In the first of these, he discusses the very general equation

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+gy=0,$$

where k and g are functions of x and some parameter n. Subject to certain specified conditions, he proves theorems concerning the separation of the roots. Afterwards he discusses a theorem equivalent to $\int X_m X_n \phi(x)\,dx=0$. His reasoning is necessarily very different from that given in this paper, for here the scale of relation is the central proposition with which all the others are connected, and this theorem is not alluded to in Sturm's papers. Even therefore, on this point of contact with Sturm's paper, what is here stated presents some novelty.

The value of $\int \phi(x)\, X_n^2\, dx$ has been found between the limits which make $ax^2+bx+c=0$.

The generating function of X_n is also found, but it seems too complicated to be of very great use.

Finally, two special cases, in which exponentials occur in the value of y, are alluded to at the end of the paper.

1. The two solutions of the differential equation

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

which are the subjects of consideration, may be found by two artifices. *First*, we may put $y=\frac{d^{n+1}z}{dx^{n+1}}$; then, if the equation be a perfect n^{th} differential, the value of z can be at once found. *Secondly*, we may seek some factor μ by which the equation (2) can be made a perfect differential.

2. By writing $px+q$ for x, we can always reduce the equation to the form

$$(1-x^2)\frac{d^2y}{dx^2}+(f'x+g')\frac{dy}{dx}+h'y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This form is sometimes more convenient than (2). But, when the factors of ax^2+bx+c are not real, the values of p and q are not real. To avoid these imaginary quantities, we shall use the general form in obtaining the solutions.

3. *The First Solution.*—Writing $y=d^{n+1}z/dx^{n+1}$, and integrating each term $(n+1)$ times by the help of Leibnitz's theorem for negative indices; viz.

$$\delta^{-n}uv=v\,\delta^{-n}u-n\frac{dv}{dx}\delta^{-n-1}u+n\frac{n+1}{2}\frac{d^2v}{dx^2}\delta^{-n-2}u-\&\text{c.},$$

where δ stands for d/dx; we have

$$(ax^2+bx+c)\frac{d^2z}{dx^2}+\{fx+g-(n+1)(2ax+b)\}\frac{dz}{dx}+h'z=0,$$

where $h'=h-(n+1)f+(n+1)(n+2)a$, and the constants of integration have been omitted.

If n be so chosen that $h'=0$, we find, by an easy integration,

$$\frac{dz}{dx} = (ax^2+bx+c)^{n+1}\frac{1}{R},$$

where

$$\frac{dR}{dx} = \frac{fx+g}{ax^2+bx+c}R.$$

Summing up, we may state the theorem thus. Let α and β be the roots of the quadratic

$$(n+1)(n+2)a-(n+1)f+h=0 \quad\text{.....................(4)}.$$

Then a solution of the differential equation (2) is

$$y = A\frac{d^\alpha}{dx^\alpha}\frac{(ax^2+bx+c)^{\alpha+1}}{R}+B\frac{d^\beta}{dx^\beta}\frac{(ax^2+bx+c)^{\beta+1}}{R},$$

where

$$\frac{1}{R}\frac{dR}{dx} = \frac{fx+g}{ax^2+bx+c} \quad\text{..........................(5)}.$$

If either α or β is a positive integer, a solution of the required form has been found.

4. *The Second Solution.*—Let us now multiply the differential equation (2) by an integrating factor μ. The condition of integrability leads to the equation

$$h\mu - \frac{d}{dx}(fx+g)\mu + \frac{d^2}{dx^2}(ax^2+bx+c)\mu = 0.$$

Expanding, this takes the form

$$(ax^2+bx+c)\frac{d^2\mu}{dx^2}+\{(4a-f)x+2b-g\}\frac{d\mu}{dx}+(h-f+2a)\mu = 0.$$

The integral of the given equation is then

$$(ax^2+bx+c)\mu\frac{dy}{dx}+\left\{(fx+g)\mu-\frac{d}{dx}(ax^2+bx+c)\mu\right\}y = C,$$

where C is some constant. Choosing C to be zero, as the integral can be found without its help, we have

$$y = \frac{ax^2+bx+c}{R}\mu,$$

where R has the same meaning as before.

The equation to find μ is of the form already discussed in the first solution. A solution of the required form can be found if either of the values of m given by the quadratic

$$(m+1)(m+2)\,a-(m+1)(4a-f)+h-f+2a=0$$

is an integer. This quadratic can be written in the form

$$m(m-1)\,a+mf+h=0.$$

Summing up, we may state the theorem thus. Let α and β be the roots of the quadratic

$$m(m-1)\,a+mf+h=0 \quad \text{.......................}(6),$$

then a solution of the differential equation (2) is

$$y=\frac{ax^2+bx+c}{R}\left\{A\frac{d^\alpha}{dx^\alpha}(ax^2+bx+c)^{\alpha-1}R+B\frac{d^\beta}{dx^\beta}(ax^2+bx+c)^{\beta-1}R\right\},$$

where R has the same form as before.

If either* of the roots of this quadratic is an integer, a solution of the differential equation as an n^{th} differential coefficient has been found.

5. Comparing the two solutions, we now see that we could derive one from the other (say, the second from the first) by a simple substitution. For, if we write $y=\mu\,(ax^2+bx+c)\,/\,R$, the equation to find μ is of the form solved in the first solution, with $4a-f$ written for f and $2b-g$ for g.

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* If both the roots of either of the fundamental quadratics are positive integers, it might be supposed that each root would give a solution in the form of an n^{th} differential. But the two solutions thus found are really the same. To show this, we shall use the following theorem, which has been obtained by an application of Leibnitz's theorem. If $2k+1$ be an integer, though k may be a fraction,

$$\frac{d^{2k+1}}{dx^{2k+1}}(x^2-1)^k\left(\frac{x-1}{x+1}\right)^l=2^{2k+1}M(x^2-1)^{-k-1}\left(\frac{x-1}{x+1}\right)^l,$$

where $\qquad M=(k+l)(k+l-1)\ldots$ to $2k+1$ factors.

To shorten the algebraical processes, let us take the differential equation in its simplified form (3). The second solution, after substitution for R, becomes

$$y\frac{R}{1-x^2}=A\left(\frac{d}{dx}\right)^\alpha(1-x^2)^{\frac{1}{2}(\alpha-\beta-1)}Q^{\frac{1}{2}g}+B\left(\frac{d}{dx}\right)^\beta(1-x^2)^{\frac{1}{2}(\beta-\alpha-1)}Q^{\frac{1}{2}g},$$

where $\quad Q=\dfrac{1+x}{1-x}.$

But, by the theorem just mentioned,

$$\left(\frac{d}{dx}\right)^{\alpha-\beta}(x^2-1)^{\frac{1}{2}(\alpha-\beta-1)}Q^{\frac{1}{2}g}=2^{\alpha-\beta}M(x^2-1)^{\frac{1}{2}(\beta-\alpha-1)}Q^{\frac{1}{2}g}.$$

Differentiating this β times, we see that the two integrals are the same, or (since M may vanish) one of them is zero identically. The other solution may be treated in the same way.

Thus any general theorem proved for one solution will have a corresponding theorem for the other solution, provided we make the proper changes in the quantities f and g.

6. Since the roots of the two fundamental quadratics (4) and (6) are such that $m+n=-1$, we may notice that, whenever either value of n is an integer, one of the values of m is an integer and of an opposite sign. Thus the two solutions are complementary for our present purpose. In general, if one root of either quadratic be an integer, a solution as an n^{th} differential coefficient has been found.

7. In the first quadratic (4) the sum of the two roots is $f/a-3$, and in the second quadratic (6) the sum is $-f/a+1$. We shall presently require the first or second of these to be negative or zero, according as we are using the first or second solution. It is clear, therefore, that the roots of either quadratic cannot be both positive integers except zero. In all the uses made of these two solutions when we speak of the positive integral value of n, there will be but one such value.

8. As the function R plays an important part in the solution of the equation, it will be convenient for the sake of reference to express its value here. In all cases we have

$$\frac{1}{R}\frac{dR}{dx}=\frac{fx+g}{ax^2+bx+c}.$$

If b^2-4ac be positive, the factors of ax^2+bx+c are real, let them be $a(x-\lambda)(x-\mu)$. We then have

$$\left.\begin{array}{c} R=(x-\lambda)^p(x-\mu)^q \\ p=\dfrac{f}{2a}+\dfrac{2ag-bf}{2a^2(\lambda-\mu)},\quad q=\dfrac{f}{2a}-\dfrac{2ag-bf}{2a^2(\lambda-\mu)} \end{array}\right\}$$

In the simple case in which (Art. 2) $a=-1$, $b=0$, $c=1$, this takes the form

$$R=(1-x^2)^{-\frac{1}{2}f}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}g}.$$

If b^2-4ac be negative, the value of R assumes a very different form. We have

$$\log R=\frac{f}{2a}\log(ax^2+bx+c)+\frac{2ag-bf}{a(4ac-b^2)^{\frac{1}{2}}}\tan^{-1}\frac{2ax+b}{(4ac-b^2)^{\frac{1}{2}}}.$$

If $b^2-4ac=0$, we have

$$R=\left(\frac{2ax+b}{2a}\right)^{\frac{f}{a}}e^{-\frac{(2ag-bf)}{a(2ax+b)}}.$$

Other important cases occur when $a = 0$, or $a = 0$ and $b = 0$; but these do not need special reference at present.

The Scale of Relation and Analysis of the Two Solutions.

9. The two solutions of the differential equations may be written in the form
$$y = P\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n\pm 1}Q,$$
where P and Q are known functions of x which are independent of n. By giving n all positive integer values from $n = 0$ onwards, we can arrange these values of y in two series. There will be a scale of relation between the terms of either of these series. This will be of great use in finding the forms of the function y. In order to discover this functional equation of n, we shall premise the following lemma.

10. *Lemma.*—Let
$$\left.\begin{aligned} X_n &= \frac{d^n}{dx^n}(x^2-1)^n Q \\ (x^2-1)\frac{dQ}{dx} &= (Mx+N)Q \end{aligned}\right\},$$
where M and N are given constants independent of n, then will
$$AX_{n+2}+(B+Cx)X_{n+1}+DX_n = 0.$$
Here A, B, C, D are constants independent of x, as follows:
$$\begin{aligned} A &= -(M+n+2)(M+2n+2), \\ B &= M(M+2n+3)N, \\ C &= (M+2n+2)(M+2n+3)(M+2n+4), \\ D &= (n+1)(M+2n+4)\{N^2-(M+2n+2)^2\}. \end{aligned}$$

The simplest proof which we can give of this equation is to substitute for X_n, &c., and integrate the result n times. It will therefore be sufficient to prove
$$A\frac{d^2}{dx^2}(P^{n+2}Q)+(B+Cx)\frac{d}{dx}(P^{n+1}Q)+(D-CnP)P^nQ = 0,$$
where $P = x^2-1$. After effecting these differentiations, we divide by P^n and substitute for dQ/dx. We thus obtain an equation of the form
$$\alpha P+\beta x+\gamma = 0,$$

where α, β, γ are three functions of the coefficients A, B, C, D. Equating these three to zero, and (to avoid fractions as much as possible) choosing the value of A as indicated above, we find the values of B, C, D.

11. If we substitute $(x^2-1)^q Q$ for Q, we may easily find a useful extension of this theorem. We then have

$$\left.\begin{aligned} X_n &= \frac{d^n}{dx^n}(x^2-1)^{n+q}Q \\ (x^2-1)\frac{dQ}{dx} &= \{(M-2q)x+N\}Q \end{aligned}\right\},$$

and it follows, as before, that

$$AX_{n+2}+(B+Cx)X_{n+1}+DX_n=0,$$

where A, B, C, D have the same values as before, and q is any integer, positive or negative.

12. If we substitute $(2ax+b)/(b^2-4ac)^{\frac{1}{2}}$ for x, and make some slight changes in the constants in order to simplify the result, we have the following theorem.

Let $$X_n=\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n+q}Q,$$

$$(ax^2+bx+c)\frac{dQ}{dx}=\left\{(M-2q)\frac{2ax+b}{2}+N\right\}Q,$$

then will $$AX_{n+2}+(B+Cx)X_{n+1}+DX_n=0,$$

where $A=-(M+n+2)(M+2n+2)$,

$$B=M(M+2n+3)\frac{N}{a}+\frac{b}{2a}C,$$

$$C=(M+2n+2)(M+2n+3)(M+2n+4),$$

$$D=(n+1)(M+2n+4)\left\{\left(\frac{N}{a}\right)^2-(M+2n+2)^2\frac{b^2-4ac}{4a^2}\right\}.$$

In the *first solution*, viz., that of Art. 3, we have $q=1$, $Q=1/R$,

so that $$-\frac{1}{Q}\frac{dQ}{dx}=\frac{1}{R}\frac{dR}{dx}=\frac{fx+g}{ax^2+bx+c},$$

therefore $$M=-\frac{f}{a}+2,\quad N=\frac{bf}{2a}-g.$$

In the *second solution*, viz., that of Art. 4, we have $q=-1$, $Q=R$,

so that $$M=\frac{f}{a}-2,\quad N=-\frac{bf}{2a}+g.$$

Substituting these values of M and N, we have the scale of relation for each solution.

13. Let us apply the scale of relation to analyse the first solution, which we may conveniently write in the form

$$y=\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n+1}\frac{1}{R}=X_n.$$

When $n=0$, $X_0=\dfrac{ax^2+bx+c}{aR}$;

when $n=1$, $X_1=\dfrac{ax^2+bx+c}{aR}\left\{\left(4-\dfrac{f}{a}\right)x+\dfrac{2b-g}{a}\right\}.$

Thus it appears that X_0 and X_1 have a common factor; hence, by the scale of relation, all the succeeding functions have the same factor. Also, since x enters into the scale of relation only in one coefficient, and in the first power, we see that the general term must be of the

form $$X_n=\frac{ax^2+bx+c}{R}\{G_nx^n+G_{n-1}x^{n-1}+\ldots+G_1x+G_0\},$$

where G_0, G_1, &c. are all constants. We may conveniently write this solution in the form

$$y=\frac{ax^2+bx+c}{R}\,G_n(x).$$

14. The scale of relation will fail to give X_{n+2} when X_{n+1} and X_n are known, if either $M+n+2=0$ or $M+2n+2=0$. Since $M=-f/a+2$, this would require either n or $2n$ to be equal to $f/a-4$. But it will be immediately seen that we shall take $4-f/a$ to be positive, so this case will not occur.

We may also notice that X_n/X_0 might be of less dimensions than n if the quantity represented by C in the scale of relation could vanish. But this cannot occur if $4-f/a$ is positive.

It follows also from the same supposition that A and C must always have opposite signs; thus the coefficients of the highest powers of x in the series 1, X_1/X_0, X_2/X_0, &c., are all of the same sign, and therefore positive.

No two consecutive functions in this series can vanish for the same value of x; for, if so, every function would vanish. This cannot occur, because the first function in the series is unity.

15. The constants A and D in the scale of relation will have the same sign if their product is positive. Remembering that

$$M = -f/a+2, \quad N = bf/2a-g,$$

we see that this will happen for all positive values of n, provided—

(1) $(-f/a+4)$ is positive, greater than zero, and numerically greater than $\dfrac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}$;

(2) The factors of ax^2+bx+c are real.

If these two conditions are satisfied, the series of functions 1, X_1/X_0, &c., are such that, when any one of them, as X_{n+1}/X_0, vanishes the two on each side, viz., X_n/X_0 and X_{n+2}/X_0, have opposite signs. They therefore resemble Sturm's functions, and may be used like those functions.

If we continue the proof as in Sturm's theorem, using the results of the last article, we see that the roots of the equations

$$\frac{X_1}{X_0}=0, \quad \frac{X_2}{X_0}=0, \quad \frac{X_3}{X_0}=0, \quad \text{\&c.},$$

are all real, and that the roots of each separate or lie between the roots of the next in order.

16. The necessary conditions that this should happen are that A and D should have the same sign when $n=0$, and that $4-f/a$ should be positive. The first of these conditions implies that X_2/X_0 should be negative when $X_1/X_0=0$. We therefore infer, *conversely, that if* $4-f/a$ *be positive, the one root of* $X_1/X_0=0$ *cannot lie between those of* $X_2/X_0=0$ *unless* $4-f/a$ *be also numerically greater than*

$$\frac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}.$$

17. Again, by referring to Art. 8, we see that both conditions may be expressed by making the exponents p and q of the two factors of R algebraically less than 2. Now, when this happens, the expression $\dfrac{(ax^2+bx+c)^{n+1}}{R}$ is finite for all values of x when n is any positive integer except zero. It therefore follows that the roots of all its differential coefficients up to the n^{th} inclusive, *i.e.*, the *roots of* $X_n=0$, *are not only real, but lie between the roots of* $ax^2+bx+c=0$.

18. Let us apply the scale of relation to analyse the second solution (Art. 4), which we may write

$$y = \frac{ax^2+bx+c}{aR}\,\frac{d^n}{dx^n}\left(\frac{ax^2+bx+c}{a}\right)^{n-1} R = X_n.$$

When $n = 0$, $\qquad X_0 = 1$,

$n = 1$, $\qquad X_1 = \frac{fx+g}{a}$.

By continual substitution in the scale of relation, we find X_2, &c. It appears at once that the general form of the solution is given by

$$X_n = H_n x^n + H_{n-1} x^{n-1} + \dots + H_0,$$

where the H's are all constants.

The same remarks apply here as in the first solution, Art. 14. Since $M = f/a - 2$, it follows that A and C cannot vanish, provided f/a is positive. Thus X_n is an integral rational function of n* dimensions, and the coefficient of the highest power is positive.

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* Conversely, we may enquire what the condition is that an integral solution should exist. Taking the differential equation in its simplified form (Art. 2), viz.,

$$(1-x^2)\frac{d^2y}{dx^2} + (fx+g)\frac{dy}{dx} + hy = 0 \quad \dots\dots\dots\dots(3),$$

let us suppose such a solution to be

$$y = A_m x^m + A_{m-1} x^{m-1} + \dots + A_p x^p + \dots \quad \dots\dots\dots\dots(\text{I.}).$$

Substituting, and taking only the highest powers of x which enter, we have

$$m(m-1-f) - h = 0 \quad \dots\dots\dots\dots(\text{II.}).$$

This quadratic is the same as that used in the second solution when $a = -1$ (Art. 4). We see that there can be no highest power in the series unless this quadratic has a positive integral root. Thus the differential equation has no solution in integral positive powers of x except in those cases in which we may apply the method given in the text.

The determination of this integral solution is not further connected with the subject of this paper. But one or two points may be noticed in passing. Let α and β be the roots of the quadratic (II.), and let $m = \alpha$ be the root, supposed integral, which we take as the highest power of x. Returning to the substitution from (I.) in (3), we find

$$-(\alpha-1-\beta)A_{\alpha-1} = g\alpha A_\alpha, \quad (p-\alpha)(p-\beta)A_p = g(p+1)A_{p+1} + (p+1)(p+2)A_{p+2}$$
$$\dots\dots\dots\dots(\text{III.}).$$

Thus each coefficient of the series for y is found in terms of the preceding ones. Now $A_{-1} = 0$ because its $p+1 = 0$, and $A_{-2} = 0$ because its $p+2 = 0$ and $A_{-1} = 0$. The series found therefore terminates before any negative powers of x are introduced. It is clear that none of these values of A_p can be infinite unless the quadratic (II.) has two integral positive roots, and then only if $\alpha > \beta$. Even in this case, if it should happen that

$$gA_{\beta+1} + (\beta+2)A_{\beta+2} = 0 \quad \dots\dots\dots\dots(\text{IV.}),$$

19. The constants A and D in the scale of relation will have the same sign if their product is positive. This will be the case provided—

(1) f/a is positive, greater than zero, and numerically greater than $\frac{bf/a-2g}{(b^2-4ac)^{\frac{1}{2}}}$;

(2) The factors of ax^2+bx+c are real.

Following the same reasoning as before, we see that the functions X_0, X_1, X_2, &c. resemble Sturm's functions. *The roots of each are therefore all real and separate, or lie between the roots of the function next in order.*

The conditions may also be expressed by making both the exponents (viz., p and q) of the value of R given in Art. 8 to be positive. From this we infer, as in Art. 16, that the roots of $X_n=0$ *are not only real, but lie between the roots of* $ax^2+bx+c=0$.

These theorems concerning the second solution may be deduced from those for the first by using the transformation given in Art. 4.

On the value of $\int \phi(x)\, X_m X_n\, dx$.

20. We next propose to examine under what circumstances the equation $\int \phi(x)\, X_m X_n\, dx = 0$ may be true, and to find the value of $\int \phi(x)\, X_n^2\, dx$, where $\phi(x)$ is some known function of x.

The differential equation considered is

$$(ax^2+bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0,$$

then A_n becomes arbitrary instead of infinite. Putting $A_n=0$, or giving it any value we please, we then find A_{n-1}, $A_{n-2}\ldots A_0$ by the formulæ (IV.). When this condition happens to be satisfied, the differential equation (3) has two solutions in positive integral powers of x.

As an example, consider the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+2(1-f)\,y=0.$$

Here the quadratic leads to $m=2$ or $m=f-1$; choosing the former, we find, without difficulty, $\qquad y=A\,[(f-1)(f-2)\,x^2+2g\,(f-1)\,x+f-2+g^2]$.

If $g=0$ and $f=2$, this value of y becomes $y=B\,(x^2+1)$ or $y=Cx$, according as we absorb the $f-2$ or the g into the arbitrary multiplier. If $g=0$ and $f=4$, the value of y becomes $y=2A\,(3x^2+1)$. All these are easily seen to be solutions.

Holmgren (to whose paper the writer had a reference) has shown that the equation (3) admits of two integral solutions when (in our notation) α, β, and $\frac{1}{2}(\alpha-\beta-1\pm g)$ are all positive integers, α being $>\beta$. In this case, the solution derived from the greater root may be written as an n^{th} differential in the form

$$y=(x+1)^{\frac{1}{2}(\alpha+\beta-g+1)}\left(\frac{d}{dx}\right)^{\frac{1}{2}(\alpha-\beta-g-1)}\left[\frac{(x-1)^{\alpha}}{(x+1)^{\beta+1}}\right].$$

where h is a function of n. Writing m for n, let z be the corresponding value of y. We then have

$$(ax^2+bx+c)\frac{d^2z}{dx^2}+(fx+g)\frac{dz}{dx}+h'z=0.$$

We easily find (as Laplace does in treating of his functions) that

$$(h'-h)\,yz=(ax^2+bx+c)\left(y\frac{d^2z}{dx^2}-z\frac{d^2y}{dx^2}\right)+(fx+g)\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right).$$

We now multiply both sides by the integrating factor of the right-hand side. This factor is easily seen to be $\frac{R}{ax^2+bx+c}$, where R has the same value as before (Art. 8). We then have

$$(h'-h)\int yz\,\frac{R\,dx}{ax^2+bx+c}=R\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right),$$

where both sides are to be taken between the same limits.

21. If we adopt as our values of y and z those given by the first solution as analysed in Art. 13, we write for y and z their values

$$y=\frac{ax^2+bx+c}{R}\,G_n(x),\quad z=\frac{ax^2+bx+c}{R}\,G_m(x).$$

The right-hand side of the equation will now become

$$\frac{(ax^2+bx+c)^2}{R}\left\{G_n(x)\frac{dG_m(x)}{dx}-G_m(x)\frac{dG_n(x)}{dx}\right\}.$$

Since the G-functions are all integral and rational when m and n are positive integers, we see that this will vanish when the roots of $ax^2+bx+c=0$ are real, and the limits are chosen to be those roots. It is, however, also necessary that the exponents of the two factors of R should not be so large as those of the corresponding factors in $(ax^2+bx+c)^2$. This requires that each of the exponents, viz., p and q, which occur in R (see Art. 8), should be algebraically less than 2. These conditions are the same as those given in Art. 17 as the conditions that the roots of the function $G(x)=0$ should all be real.

The proof requires that h and h' should not be equal except when $n=m$. Referring to Art. 3, we see that in the first solution

$$h=(n+1)\{f-(n+2)\,a\}.$$

Writing m for n to obtain h', it follows that $h'=h$ only when $n=m$

and $n+m = f/a-3$. But this latter case cannot occur for any unequal positive integral values of m and n, because we have already assumed in Art. 15 that $4-f/a$ is positive and greater than zero. A similar remark applies when we use the second solution in the next article.

22. If we adopt as our values of y and z those given by the second solution as analysed in Art. 18, we see that the right-hand side of the equation arrived at in Art. 20 becomes

$$= R\left(X_n \frac{dX_m}{dx} - X_m \frac{dX_n}{dx}\right).$$

Now the functions X_m and X_n are integral rational functions of x when m and n are positive integers. Hence the right-hand side is zero if the roots of $ax^2+bx+c=0$ are real and those roots are taken as the limits. It is necessary that the exponents of the corresponding factors in the value of R should be positive. This will be the case whenever the conditions that the roots of $X_n=0$ should be all real (given in Art. 19) are satisfied.

23. *Summing up*, we see that, if y_m and y_n be two values of y corresponding to two positive integral values of the parameter in either the first or the second solution (Arts. 3 and 4), then

$$\int y_m y_n \frac{R\,dx}{ax^2+bx+c} = 0,$$

if the limits are the roots of $ax^2+bx+c=0$, provided those roots are real. It is also necessary that the indices of the factors of R should be such that, when we use the first solution, $\dfrac{(ax^2+bx+c)^2}{R}$ vanishes at each limit; and, when we use the second solution, R vanishes at each limit.*

* We have seen that the conditions necessary that $\int \phi(x)\, X_m X_n\, dx = 0$ are the same as those that all the functions X_n should have their roots real. The reason for this identity will be better seen by the help of the following theorem, which may be used for functions more general than those to which it is here applied.

Theorem.—*If X_n be an integral rational function of x of n dimensions, such that $\int \phi(x)\, X_n X_m\, dx = 0$ for all positive integral values of m less than n, then the roots of the equation $X_n = 0$ are all real and lie between the limits of integration.* It is supposed that the limits of integration are fixed, and that $\phi(x)$ is finite and keeps one sign between those limits.

For, if possible, let $X_n = 0$ have a pair of imaginary roots, or a real root which does not lie between the limits of integration. In either case, X_n has a factor which keeps one sign between the limits. Let $f(x)$ be the remaining factors, then it is obvious that $f(x)$ can be expanded in a series

$$f(x) = a_0 X_0 + a_1 X_1 + \ldots + a_{n-1} X_{n-1}.$$

24. *To find by a general method the value of* $\int \phi(x)\, X_n^2\, dx$ *for any limits which make* $\int \phi(x)\, X_m X_n\, dx = 0$.

We have, by the scale of relation as given in Arts. 10 and 12,

$$A_n X_{n+2} + (B_n + C_n x) X_{n+1} + D_n X_n = 0,$$

where A_n, B_n, C_n, D_n are functions of n, but not of x. Multiply this equation by $X_n \phi(x)$ and integrate; we find

$$C_n \int x \phi(x)\, X_n X_{n+1}\, dx + D_n \int \phi(x)\, X_n^2\, dx = 0.$$

Now write $(n-1)$ for n in the scale of relation, and multiply by $\phi(x)\, X_{n+1}$. We find, after integration,

$$A_{n-1} \int \phi(x)\, X_{n+1}^2\, dx + C_{n-1} \int x \phi(x)\, X_n X_{n+1}\, dx = 0.$$

We immediately deduce

$$\frac{C_n}{D_n} \int \phi(x)\, X_{n+1}^2\, dx = \frac{C_{n-1}}{A_{n-1}} \int \phi(x)\, X_n\, dx.$$

From this, by continued reduction, we have

$$\int \phi(x)\, X_n^2\, dx = \frac{C_0}{C_{n-1}} \cdot \frac{D_{n-1} D_{n-2} \dots D_1}{A_{n-2} A_{n-3} \dots A_0} \int \phi(x)\, X_1^2\, dx.$$

25. Taking the second solution first, the value of $\int \phi(x)\, X_1^2\, dx$ may be found in gamma integrals. Thus, putting $P = ax^2 + bx + c$, we have, by Art. 18,

$$\int \phi(x)\, X_1^2\, dx = \int \frac{R}{P} \left(\frac{fx+g}{a}\right)^2 dx = \int \frac{fx+g}{a^2}\, dR,$$

by the definition of R given in Art. 8. Now, substituting for R from that article and remembering that p and q are positive (Art. 22), we get, after an integration by parts,

$$\int_\mu^\lambda \phi(x)\, X_1^2\, dx = -\frac{f}{a^2} (\lambda - \mu)^{1+f/a} \int_0^1 (z-1)^p z^q\, dz.$$

In the same way we find for the first solution

$$\int_\mu^\lambda \phi(x)\, X_1^2\, dx = -\frac{4a-f}{a^2} (\lambda - \mu)^{5-f/a} \int_0^1 (z-1)^{2-p} z^{2-q}\, dz.$$

Here (as in Art. 21) we have supposed that p and q are less than 2.

Now the product $\phi(x) f(x)\, X_n$ keeps one sign between the limits; hence every term of the integral $\int \phi(x) f(x)\, X_n\, dx$ has the same sign. But, by hypothesis, if we substitute this series for $f(x)$ we see that the integral vanishes, which is impossible.

The Generating Function.

26. We have, by Lagrange's theorem in the differential calculus,

$$u = x + t\phi(u),$$

$$f(u) = f(x) + \&c. + \frac{t^{n+1}}{L(n+1)} \frac{d^n}{dx^n} (\phi x)^{n+1} f'(x) + \&c.$$

Differentiating both sides with regard to t and substituting for du/dt, we have

$$\frac{f'(u)\,\phi(u)}{1 - t\phi'(u)} = \&c. + \frac{t^n}{Ln} \frac{d^n}{dx^n} (\phi x)^{n+1} f'(x) + \&c.$$

In the *first solution* we put $f'(x) = 1/R$ and $a\phi(u) = au^2 + bu + c$. Hence, if $R(u)$ be the same function of u that R is of x in Art. 8, we have

$$\frac{au^2 + bu + c}{R(u)} \frac{1}{a - t(2au + b)} = \&c. + \frac{t^n}{Ln} \frac{d^n}{dx^n} \left(\frac{ax^2 + bx + c}{a}\right)^{n+1} \frac{1}{R} + \&c.$$

In the *second solution* we give $\phi(u)$ the same meaning as before, and put $f'(x) = R(ax^2 + bx + c)^{-2}$. We find

$$\frac{R(u)}{au^2 + bu + c} \frac{1}{a - t(2au + b)} = \&c. + \frac{t^n}{Ln} \frac{d^n}{dx^n} \left(\frac{ax^2 + bx + c}{a}\right)^{n-1} R + \&c.$$

The coefficients of $\frac{t^n}{Ln}$ on the right-hand sides of these equations are the values of X_n given in Arts. 13 and 18 for the first and second solutions. We must, however, not forget to multiply both sides of the equation by the factor $\frac{ax^2 + bx + c}{aR}$ in the case of the second solution. The left-hand sides of these equations are therefore known functions of t which by expansion give the two values of X_n.

27. If we substitute for u its value given by the quadratic at the beginning of this article, we have an expression for the generating function which is free from all integrations, and which takes different forms when we substitute for $R(u)$ the various forms given in Art. 8.

To take a simple illustration, let the equation be given in the form (Art. 2)

$$(1 - x^2)\frac{d^2y}{dx^2} + (fx + g)\frac{dy}{dx} + hy = 0,$$

and let us adopt the second solution, $y = X_n$. Then X_n is the coefficient of t^n / Ln in the expansion of

$$\left[\frac{(1 - 4xt + 4t^2)^{\frac{1}{2}} - (1 - 2xt)}{2t^2(1 - x^2)}\right]^{-U-1} \left[\frac{x - 2t + (1 - 4xt + 4t^2)^{\frac{1}{2}}}{1 + x}\right]^{\frac{1}{2}v} \frac{1}{(1 - 4xt + 4t^2)^{\frac{1}{2}}}.$$

Special Cases.

28. *The special case* in which $a = 0$ presents some interesting forms, partly because the expression for R now contains exponentials, and partly because the values of M and N in the scale of relation (Art. 12) are both infinite. The differential equation is

$$(bx+c)\frac{d^2y}{dx^2}+(fx+g)\frac{dy}{dx}+hy=0.$$

Here $$R=(bx+c)^p e^{fx/b}, \text{ where } p=\frac{bg-cf}{b^2}.$$

We shall suppose that f and b are not zero.

In the *first solution* (Art. 3) we have $h=(n+1)f$,

$$y=\frac{d^n}{dx^n}\frac{(bx+c)^{n+1}}{R}=e^{-fx/b}\left(-\frac{f}{b}+\frac{d}{dx}\right)^n(bx+c)^{n+1-p}.$$

Putting X_n for this value of y, the scale of relation is

$$X_{n+2}+\{fx+g-2(n+2)b\}X_{n+1}+b^2(n+1)(n+2-p)X_n=0.$$

In the *second solution* (Art. 4) we have $h=-nf$,

$$y=\frac{bx+c}{R}\frac{d^n}{dx^n}(bx+c)^{n-1}R=(bx+c)^{1-p}\left(\frac{f}{b}+\frac{d}{dx}\right)^n(bx+c)^{n-1+p}.$$

Putting X_n for this value of y, the scale of relation is

$$X_{n+2}-\{fx+g+2(n+1)b\}X_{n+1}+b^2(n+1)(n+p)X_n=0.$$

From these scales of relation the other properties follow without difficulty; provided in the first case $p<2$, and in the second case p is positive.

These scales of relation may be deduced from the general form given in Art. 12 by expanding the coefficients in powers of f/a. They may be obtained independently by the process explained in Art. 10.

29. Another interesting special case occurs when $a=0$, $b=0$, $c=1$. Taking the first solution, we have

$$R=e^{\frac{1}{2}fx^2+gx},\quad y=\frac{d^n}{dx^n}e^{-\frac{1}{2}fx^2-gx}=X_n.$$

The scale of relation is

$$X_{n+2}+(fx+g)X_{n+1}+(n+1)fX_n=0.$$

Theorems similar to those proved in the general case follow from this scale of relation.

Other special cases occur in which the function R has a peculiar form,—such, for instance, as that in which $a=1$, $b=0$, $c=0$.

May 14th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

B. Hanumanta Rau, B.A., Acting Head-master, Government Normal School, Madras, was elected a member.

Papers were read by Rev. T. C. Simmons: An Application of Determinants to the Solution of Certain Types of Simultaneous Equations; and by H. M. Jeffery, F.R.S.: On Binodal Quartics (the reading was illustrated by several diagrams).

Mr. Tucker communicated a paper by Professor J. Larmor: On the Flow of Electricity in a System of Linear Conductors.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxviii., No. 236.

"Educational Times" for May.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Parts 1, 2, 3, 1885.

"Transactions of the Cambridge Philosophical Society," Vol. xiv., Part 1, 1885.

"Proceedings of the Canadian Institute, Toronto," Third Series, Vol. iii., Fasc. 1, March, 1885.

"Sitzungsberichte der k. Preussischen Akademie der Wissenschaften zu Berlin," xl. to liv., 23 October 1884 to 11 December.

"Journal de l'Ecole Polytechnique," 54th Cahier; Paris 1884.

"Bulletin des Sciences Mathématiques," T. ix., April and May, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 4; Leipzig, 1885.

"Atti della R. Accademia dei Lincei," Rendiconti., Vol. i., F. 8, 9, 10; Roma, 1885.

"Über die Theorie der aufeinander abwickelbaren Oberflächen," von J. Weingarten, 4to; Berlin, 1884. (Separatabdruck aus der Festschrift der Königl. Technischen Hochschule zu Berlin.)

On the Flow of Electricity in a System of Linear Conductors.

By Professor J. Larmor.

[*Read May 14th,* 1885.]

1. The analytical determination of the currents that are set up by given steady electro-motive forces in a system of linear conducting bodies has been treated by Kirchhoff,* who takes a separate variable to represent the current flowing in each branch of the system.

* *Pogg. Annal.*, Bd. 72, 1847; *Gesammelte Abhandlungen*, pp. 22—23.