may yet be outside the limits $\pm\frac{1}{2}$. When this is so, then (as the case may be) $x\pm1$ is nearer to mp than x is. Thus, in the expansion of $(4+1)^8$ the first three terms are as 4, 8, 7, so that the second is the greatest; but the third has the exponents (6 and 2) most nearly equal to $6+\frac{2}{5}$ and $2-\frac{2}{5}$, which are 8 times $\frac{4}{5}$ and $\frac{1}{5}$ respectively.

However, as $(m+1)p-x$ is positive and less than unity, $(m+1)p-(x+\frac{1}{2})$ is within the limits $\pm\frac{1}{2}$, so that $x+\frac{1}{2}$ and $x'+\frac{1}{2}$ are as nearly as possible equal to $(m+1)p$ and $(m+1)p'$; therefore it is true that, *when increased by* $\frac{1}{2}$, the exponents of the greatest term of the expansion are as nearly as possible in the ratio of the corresponding terms of the binomial.

On the Transformation of Continued Products into Continued Fractions. By J. W. L. Glaisher, M.A.

[*Read January 8th*, 1874.]

1. The present paper had its origin in a very remarkable continued fraction for π given by Professor Sylvester in the "Philosophical Magazine" for 1869,* viz.,

$$\frac{\pi}{2}=1+\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+}\,\frac{4.5}{1+\&c.}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

which is there shown to be equivalent to Wallis's formula,

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}.$$

It was not, however, by a direct transformation of the product that Professor Sylvester obtained his result, but by means of a complete solution that he had previously found of the equation of finite differences $u_{x+1}=\frac{u_x}{x}+u_{x-1}$; so that a direct proof of (i.) by ordinary algebra seemed likely to be interesting. Further, whenever we obtain an expression for π as a series, product, or continued fraction, it is pretty safe to assume that the formula admits of generalisation; viz., that there exists a corresponding series, &c., involving $\sin^{-1}x$ or some other expression which, for a particular value of the variable, becomes equal to π. In the present instance, as the continued fraction is a transformation of Wallis's formula, it seemed desirable to enquire what was the fraction corresponding to the expression for $\sin x$ as an infinite

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* "Note on a new Continued Fraction applicable to the Quadrature of the Circle," Phil. Mag., S. 4, vol. xxxvii., pp. 373—376 (May, 1869).

product of which Wallis's formula is a particular case. This naturally led to the investigation of a direct rule to transform any product into a continued fraction, and it is convenient to commence with the general formulæ.

2. It is easily seen that

$$\frac{\beta}{a}+\frac{\beta\beta'}{aa'}+\frac{\beta\beta'\beta''}{aa'a''}+\text{\&c.}=\frac{\beta}{a-}\;\frac{a\beta'}{a'+\beta'-}\;\frac{a'\beta''}{a''+\beta''-\text{\&c.}}\;\text{......(ii.)}.$$

This may be regarded as a well-known theorem, as it is used, in one shape or another, in several memoirs in "Crelle"; and the case, when the β's are all equal to unity, and the terms of alternate sign, is proved in Bertrand's "Calcul Différentiel." The demonstration is very simple: $1+\frac{\beta'}{a'}=\cfrac{1}{1-\cfrac{\beta'}{a'+\beta'}}$; and to include the next term we have merely to replace $\frac{1}{a'}$ by $\frac{1}{a'}\left(1+\frac{\beta''}{a''}\right)$, viz., to replace a' by $a'-\frac{a'\beta''}{a''+\beta''}$. It is thus evident that the first n terms of the series are identically equal to the continued fraction taken as far as to the end of the nth quotient.

By a change of sign we have

$$\frac{\beta}{a}-\frac{\beta\beta'}{aa'}+\frac{\beta\beta'\beta''}{aa'a''}-\text{\&c.}=\frac{\beta}{a+}\;\frac{a\beta'}{a'-\beta'+}\;\frac{a'\beta''}{a''-\beta''+\text{\&c.}}\;\text{......(iii.)}.$$

The following are obvious algebraical identities:

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots=1+\frac{b}{a}+\frac{(a+b)b'}{aa'}+\frac{(a+b)(a'+b')b''}{aa'a''}+\ldots\;\text{...(iv.)}$$

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots=1+\frac{b}{a}-\frac{(a+b)b'}{aa'}+\frac{(a+b)(a'-b')b'}{aa'a''}-\ldots\;\text{... (v.)}$$

$$\left(1-\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1-\frac{b'}{a''}\right)\ldots=1-\frac{b}{a}+\frac{(a-b)b'}{aa'}-\frac{(a-b)(a'+b')b''}{aa'a''}+\ldots\;\text{...(vi.)}$$

$$\left(1-\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots=1-\frac{b}{a}-\frac{(a-b)b'}{aa'}-\frac{(a-b)(a'-b')b''}{aa'a''}-\ldots\;\text{...(vii.)}$$

By means of formulæ (ii.)—(vii.) any continued product can be converted into a continued fraction such that n factors of the former shall be equivalent to the nth convergent to the latter [or to the $(n+1)$th convergent, or to the $(n-1)$th, or the $(n-2)$th, &c., as we please].

Thus, from (ii.) and (iv.),

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\left(1+\frac{b'''}{a'''}\right)\ldots$$
$$=1+\frac{\beta}{a-}\;\frac{a\beta'}{a'+\beta'-}\;\frac{a'\beta''}{a''+\beta''-}\;\frac{a''\beta'''}{a'''+\beta'''-\text{\&c.}}\;\text{...(viii.)},$$

where
$$\beta = b,\quad \beta' = (a+b)\,b',\quad \beta'' = (a'+b')\,b'',\quad \beta''' = (a''+b'')\,b''',\ \&c.,$$
$$\alpha = a,\quad \alpha' = a'b,\quad \alpha'' = a''b',\quad \alpha''' = a'''b'',\ \&c.;$$

and from (iii.) and (v.),

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots = 1+\frac{\beta}{\alpha+}\,\frac{\alpha\beta'}{\alpha'-\beta'+}\,\frac{\alpha'\beta''}{\alpha''-\beta''+\&c.}\ldots\text{(ix.)},$$

where
$$\beta = b,\quad \beta' = (a+b)\,b',\quad \beta'' = (a'-b')\,b'',\ \&c.,$$
$$\alpha = a,\quad \alpha' = a'b,\quad \alpha'' = a''b',\ \&c.$$

The last two formulæ [viz., (viii.) and (ix.)] are those which will be chiefly used. The other two that follow from (vi.) and (vii.) are

$$\left(1-\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots = 1-\frac{\beta}{\alpha+}\,\frac{\alpha\beta'}{\alpha'-\beta'+\&c.}\ \ldots\ldots\text{(x.)},$$

where
$$\beta = b,\quad \beta' = (a-b)\,b',\quad \beta'' = (a'+b')\,b'',\ \&c.,$$
$$\alpha = a,\quad \alpha' = a'b,\quad \alpha'' = a''b',\ \&c.;$$

and
$$\left(1-\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots = 1-\frac{\beta}{\alpha-}\,\frac{\alpha\beta'}{\alpha'+\beta'-\&c.}\ \ldots\ldots\text{(xi.)},$$

where
$$\beta = b,\quad \beta' = (a-b)\,b',\quad \beta'' = (a'-b')\,b'',\ \&c.,$$
$$\alpha = a,\quad \alpha' = a'b,\quad \alpha'' = a''b',\ \&c.$$

The special case of $b = b' = b'' = \&c. = x$ deserves notice: thus,

$$\left(1+\frac{x}{a}\right)\left(1+\frac{x}{a'}\right)\left(1+\frac{x}{a''}\right)\ldots = 1+\frac{x}{a}+\frac{x(x+a)}{aa'}+\frac{x(x+a)(x+a')}{aa'a''}+\ldots$$
$$= 1+\frac{x}{a-}\,\frac{a(x+a)}{x+a+a'-}\,\frac{a'(x+a')}{x+a'+a''-\&c.}\ \ldots\ldots\text{(xii.)};$$

and in particular, when $x=1$,

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a'}\right)\left(1+\frac{1}{a''}\right)\ldots = 1+\frac{1}{a-}\,\frac{a(a+1)}{a'+a+1-}\,\frac{a'(a'+1)}{a''+a'+1-\&c.}.$$

4. In all the formulæ (viii.)—(xii.), the first n factors of the product are identically equal to the right-hand side when the continued fraction is replaced by the nth convergent to it. In all cases it is easy to exhibit the right-hand side wholly as a continued fraction, for

$$1+\frac{\beta}{\alpha\pm Q}=\cfrac{1}{1-\cfrac{\beta}{\alpha+\beta\pm Q}},$$

and

$$1-\frac{\beta}{\alpha\pm Q}=\cfrac{1}{1+\cfrac{\beta}{\alpha-\beta\pm Q}}.$$

Thus, for example, from (viii.)

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\dots=\frac{1}{1-}\,\frac{\beta}{\alpha+\beta-}\,\frac{\alpha\beta'}{\alpha'+\beta'-}\,\frac{\alpha'\beta''}{\alpha''+\beta''-\&c.}.$$

This form is rather more elegant in the case of (viii.) and (x.), but it violates the regularity of the signs in (ix.) and (xi.); thus (ix.) gives

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b}{a}\right)\left(1+\frac{b''}{a''}\right)\dots=\frac{1}{1-}\,\frac{\beta}{\alpha+\beta+}\,\frac{\alpha\beta}{\alpha'-\beta'+\&c.};$$

the form adopted seems therefore preferable, viz., that in which the results are of the form $1\pm$continued fraction.

5. I now proceed to apply the formulæ obtained to the case of the infinite products for the sine and cosine; we have

$$\frac{\sin\pi x}{\pi x}=(1+x)(1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\left(1-\frac{x}{3}\right)\dots,$$

whence

$$\frac{\pi}{n\sin\frac{\pi}{n}}=\left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{2n-1}\right)\left(1-\frac{1}{2n+1}\right)\dots$$

$$=1+\frac{1}{n-1+}\,\frac{(n-1)\,n}{1+}\,\frac{n\,(n+1)}{n-1+}\,\frac{(2n-1)\,2n}{1+}\,\frac{2n\,(2n+1)}{n-1+\&c.}\dots\text{(xiii.)},$$

the denominators being alternately unity and $n-1$. Sylvester's fraction (i.), viz. $\frac{\pi}{2}=1+\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+\&c.}$ is the particular case when $n=2$, and (xiii.) is the general formula for the sine referred to in § 1. We see at once that Sylvester's fraction is the only one deducible from (xiii.), in which the denominators are all identical; but by giving different values to n we obtain a set of continued fractions with alternating denominators; thus, putting $n=3$, 4, and 6, we have

$$\frac{2\pi}{3\sqrt{3}}=1+\frac{1}{2+}\,\frac{2.3}{1+}\,\frac{3.4}{2+}\,\frac{5.6}{1+}\,\frac{6.7}{2+}\,\frac{8.9}{1+}\,\frac{9.10}{2+\&c.},$$

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3+}\,\frac{3.4}{1+}\,\frac{4.5}{3+}\,\frac{7.8}{1+}\,\frac{8.9}{3+}\,\frac{11.12}{1+}\,\frac{12.13}{3+\&c.},$$

$$\frac{\pi}{3}=1+\frac{1}{5+}\,\frac{5.6}{1+}\,\frac{6.7}{5+}\,\frac{11.12}{1+}\,\frac{12.13}{5+}\,\frac{17.18}{1+}\,\frac{18.19}{5+\&c.}.$$

From the cosine series we have

$$\sec\frac{\pi}{2n} = \left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{3n-1}\right)\left(1-\frac{1}{3n+1}\right)\dots$$

$$= 1+\frac{1}{n-1+}\;\frac{(n-1)\,n}{1+}\;\frac{n\,(n+1)}{2n-1+}\;\frac{(3n-1)\,3n}{1+}\;\frac{3n\,(3n+1)}{2n-1+\&c.}\;\dots \text{(xiv.)}$$

Take $n=2$ and 3, and there result

$$\sqrt{2} = 1+\frac{1}{1+}\;\frac{1.2}{1+}\;\frac{2.3}{3+}\;\frac{5.6}{1+}\;\frac{6.7}{3+}\;\frac{9.10}{1+}\;\frac{10.11}{3+\&c.},$$

$$\frac{2}{\sqrt{3}} = 1+\frac{1}{2+}\;\frac{2.3}{1+}\;\frac{3.4}{5+}\;\frac{8.9}{1+}\;\frac{9.10}{5+}\;\frac{14.15}{1+}\;\frac{15.16}{5+\&c.}.$$

If we put $n=1$ in (xiv.) as it stands, we merely obtain $\infty = \infty$, but if we take $n = 1+\epsilon$ (ϵ being infinitesimal), replace $\cos\frac{1}{2}\pi\,(1-\epsilon)$ by $\frac{1}{2}\pi\epsilon$, and divide out by ϵ, we deduce Sylvester's formula over again. We may also obtain results by giving n fractional values, *e.g.*, take $n=\frac{3}{2}$.

6. For comparison with the above continued fractions for the sine and cosine, it is interesting to express in the same way several other series and products which also are equivalent to these functions; thus

$$\frac{\sin\pi x}{\pi x} = (1+x)(1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\left(1-\frac{x}{3}\right)\dots$$

$$= 1+\frac{x}{1-}\;\frac{1+x}{x-}\;\frac{1-x}{1+x-}\;\frac{2\,(2+x)}{x-}\;\frac{2\,(2-x)}{1+x-\&c.}\;\dots\dots\dots\dots\dots \text{(xv.)},$$

whence $$\frac{2}{\pi} = 1+\frac{1}{2-}\;\frac{2.3}{1-}\;\frac{2.1}{3-}\;\frac{4.5}{1-}\;\frac{4.3}{3-}\;\frac{6.7}{1-}\;\frac{6.5}{3-\&c.}.$$

Similarly $$\cos\frac{\pi x}{2} = 1+\frac{x}{1-}\;\frac{1+x}{x-}\;\frac{1-x}{2+x-}\;\frac{3\,(3+x)}{x-\&c.}\;\dots\dots\dots \text{(xvi.)}$$

Again, we have

$$\frac{\sin\pi x}{\pi x} = (1-x^2)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\dots$$

$$= 1-\frac{x^2}{1^2}-\frac{x^2(1-x^2)}{1^2.\,2^2}-\frac{x^2(1-x^2)\,(2^2-x^2)}{1^2.\,2^2.\,3^2}-\dots$$

$$= 1-\frac{x^2}{1-}\;\frac{1-x^2}{1+2.2-x^2-}\;\frac{2^2\,(2^2-x^2)}{1+3.4-x^2-}\;\frac{3^2\,(3^2-x^2)}{1+4.6-x^2-\&c.}\dots\text{(xvii.)},$$

the denominators being of the form $1+n\,.\,2(n-1)-x^2$.

If herein we take $x = 1+\epsilon$, ϵ being infinitesimal, expand the left-hand side, and divide out by ϵ, we obtain, after some reductions,

$$2 = \frac{2^2.\,1\,.\,3}{3\,.\,4-}\;\frac{3^2.\,2\,.\,4}{6\,.\,4-}\;\frac{4^2.\,3\,.\,5}{8\,.\,5-\&c.},$$

which at once reduces to $1 = \frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-\&c.}$, and serves as a verification.

Similarly, from the cosine series we have

$$\cos\tfrac{1}{2}\pi x = 1 - \frac{x^2}{1-}\,\frac{1-x^2}{8.1^2+2-x^2-}\,\frac{3^2(3^2-x^2)}{8.2^2+2-x^2-}\,\frac{5^2(5^2-x^2)}{8.3^2+2-x^2-\&c.},$$

the denominators being of the form $8n^2+2-x^2$.

Also, from $\sin\tfrac{1}{2}\pi x = x - \frac{x(x^2-1^2)}{1.2.3} + \frac{x(x^2-1^2)(x^2-3^2)}{1.2.3.4.5} - \dots$
we deduce

$$\frac{\sin\frac{1}{2}\pi x}{x} = 1 + \frac{1-x^2}{2.3-}\,\frac{2.3\,(3^2-x^2)}{3^2+4.5-x^2-}\,\frac{4.5\,(5^2-x^2)}{5^2+6.7-x^2-\&c.}.$$

Taking herein $x = 1+e$ as before, there results

$$1 = \frac{1}{3-}\,\frac{2.3}{7-}\,\frac{4.5}{11-}\,\frac{6.7}{15-}\,\frac{8.9}{19-\&c.} \quad\dots\dots\dots\dots\text{(xviii.)}$$

From $\cos\tfrac{1}{2}\pi x = 1 - \frac{x^2}{1.2} + \frac{x^2(x^2-2^2)}{1.2.3.4} - \frac{x^2(x^2-2^2)(x^2-4^2)}{1.2.3.4.5.6} + \dots$
we deduce

$$\cos\tfrac{1}{2}\pi x = 1 - \frac{x^2}{1.2-}\,\frac{1.2\,(2^2-x^2)}{2^2+3.4-x^2-}\,\frac{3.4\,(4^2-x^2)}{4^2+5.6-x^2-\&c.};$$

where, putting $x=1$,

$$1 = \frac{1.2}{5-}\,\frac{3.4}{9-}\,\frac{5.6}{13-}\,\frac{7.8}{17-\&c.}.$$

7. The formulæ

$$\cos\tfrac{1}{3}\pi x = 1 - \frac{x^2}{1.2} + \frac{x^2(x^2-1^2)}{1.2.3.4} - \frac{x^2(x^2-1^2)(x^2-2^2)}{1.2.3.4.5.6} + \dots$$

$$\sin\tfrac{1}{3}\pi x = \tfrac{1}{2}\sqrt{3}\left\{x - \frac{x(x^2-1^2)}{1.2.3} + \frac{x(x^2-1^2)(x^2-2^2)}{1.2.3.4.5} - \dots\right\}$$

yield similar continued fractions to those written down in § 6.

If in the first we put $x = 1+e$ as before, we find

$$\frac{\pi}{\sqrt{3}} = 2 - \frac{1}{6-}\,\frac{1.6}{11-}\,\frac{2.10}{16-}\,\frac{3.14}{21-}\,\frac{4.18}{26-\&c.}.$$

The ordinary series for $\sin x$ and $\cos x$, viz. $x - \frac{x^3}{1.2.3} + \dots$ and $1 - \frac{x^2}{1.2} + \dots$, when transformed into continued fractions, give

$$\frac{\sin x}{x} = 1 - \frac{x^2}{2.3+}\,\frac{2.3x^2}{4.5-x^2+}\,\frac{4.5x^2}{6.7-x^2+\&c.},$$

and $$\cos x = 1 - \frac{x^2}{1.2+}\,\frac{1.2x^2}{3.4-x^2+}\,\frac{3.4x^2}{5.6-x^2+\&c.},$$

and possess no special point of interest.

A number of series to which the methods used in this paper may be applied will be found in two papers, "On the Deduction of Series from Infinite Products," and "Remarks on certain Series occurring in a paper 'On the Deduction of Series from Infinite Products,'" "Messenger of Mathematics," New Series, vol. ii., pp. 138-142 and 153–157; but it is not worth while to give these applications here, as they present no novelty, and are generally less interesting than those already noticed.

8. It is perhaps worth remarking that every equation connecting a product and a continued fraction gives a numerical identity; *e.g.*, from (xvii.), if n be any integer,

$$1-\frac{n^2}{1-}\,\frac{1-n^2}{1+2.2-n^2-}\,\frac{2^2(2^2-n^2)}{1+3.4-n^2-}\cdots\cdots\frac{(n-1)^2\{(n-1)^2-n^2\}}{1+n.2(n-1)-n^2}=0.$$

If the factors of a product be taken in a different order, the equivalent continued fraction is altered in a corresponding manner; but the results, of course, take the most elegant form when the factors are so arranged that the denominators are in order of magnitude. If in (xiii.), (xiv.), (xv.), and (xvi.) the order of every pair of factors be inverted, the effect is merely to change the sign of n and x.

From (xii.) we have the identity

$$\left(1-\frac{x}{a-}\,\frac{a(a-x)}{a+a'-x-}\,\frac{a'(a'-x)}{a'+a''-x-\&c.}\right)\left(1+\frac{x}{a-}\,\frac{a(a+x)}{a+a'+x-}\,\frac{a'(a'+x)}{a'+a''+x-\&c.}\right)$$

$$=1-\frac{x^2}{a^2-}\,\frac{a^2(a^2-x^2)}{a^2+a'^2-x^2-}\,\frac{a'^2(a'^2-x^2)}{a'^2+a''^2-x^2-\&c.},$$

which is perhaps worth notice.

One chief use of expressing series &c. as continued fractions is that sometimes we are enabled thereby to infer their irrationality by Lambert's principle; but it will be seen that Lambert's method generally gives no result when applied to the cases discussed in this paper.

It is not unlikely that by simple transformations, such as

$$a-\frac{1}{b}=a-1+\cfrac{1}{1+\cfrac{1}{b-1}},$$

most of the numerical results [*e.g.*, (xviii.)] may be reduced to forms in which their truth is evident; but I have not investigated the question.

9. There is a remarkable fact connected with the continued fraction

$$\frac{\pi}{2}=1+\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+\&c.},$$

to which Professor Sylvester draws attention in the following words:—

"At first sight it might seem as if the above-stated continued fraction were incapable of teaching anything that cannot be got direct out of the Wallisian representation itself, that has become transformed into it...... But I think a substantial difference does arise in favour of the continued fraction form, inasmuch as it indicates a certain obvious correction to be applied in order that the convergence may become more exact. For if we call

$$\frac{n(n+1)}{1+}\ \frac{(n+1)(n+2)}{1+}\dots \text{ ad infinitum } u_n,$$

we have $u_n = \frac{n^2+n}{1+u_{n+1}}$. This shows than u_n cannot remain finite when n becomes infinite; for then u_{n+1} would also be finite, and consequently u_n would be a finite fraction of infinity, which is a contradiction in terms.

"Hence ultimately $u_n u_{n+1} = n^2+n$, *i. e.*, $u_n = n$. .... Thus we may write, when n is very large,

$$\frac{\pi}{2} = 1+\frac{1}{1+}\ \frac{2}{1+}\ \frac{6}{1+}\dots\frac{n^2-n}{1+n}."$$

Taking $n = 4$ and 5, Professor Sylvester shows the advantage of taking account of the value of u_n.

To examine the nature of this correction, we recur to (v.) and (iii.), and find

$$u = \frac{2.2.4.4.6.6\dots\ 2m\ .\ 2m}{1.3.3.5.5.7\dots(2m-1)(2m+1)}$$

$$= (1+1)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\dots\dots$$

$$\dots\dots\left(1+\frac{1}{2m-1}\right)\left(1-\frac{1}{2m+1}\right)$$

$$= 1+1-\frac{2}{3}+\frac{2^2}{3^2}-\frac{2^2.4}{3^2.5}+\frac{2^2.4^2}{3^2.5^2}-\dots\dots$$

$$\dots\dots+\frac{2^2.4^2\dots(2m-2)^2}{3^2.5^2\dots(2m-1)^2}-\frac{2^2\dots(2m-2)^2\,2m}{3^2.5^2\dots(2m-1)^2\,(2m+1)}$$

$$= 1+\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{1+\dots}\ \frac{(n-1)\,n}{1},$$

if we take $n = 2m$. (We might, of course, just as well have included $2m+1$ terms in u, and taken $n = 2m+1$.)

Professor Sylvester's correction consists in taking the last quotient in the continued fraction just written to be $\frac{(n-1)\,n}{1+n}$ instead of $\frac{(n-1)\,n}{1}$; viz., it consists in taking the last factor in the denominator of the last term of the series to be $4m+1$ instead of $2m+1$, and therefore in taking the last factor in the product u to be $\frac{4m}{4m+1}$ instead of $\frac{2m}{2m+1}$.

In order to include the remaining terms of the series, we must replace the factor $\frac{1}{2m+1}$ in the last term by

$$\frac{1}{2m+1}\left\{1-\frac{2m}{2m+1}+\frac{2m(2m+2)}{(2m+1)(2m+3)}-\dots\right\},$$

and we see that we make a step towards including these terms by replacing the quantity within brackets by the geometrical progression

$$1-\frac{2m}{2m+1}+\left(\frac{2m}{2m+1}\right)^2-\dots,$$

viz., by replacing $\frac{1}{2m+1}$ by $\frac{1}{2m+1}\cdot\frac{2m+1}{4m+1}$, that is, by $\frac{1}{4m+1}$.

If we call the quantity within the brackets v_{2m}, we have

$$v_{2m}=1-\frac{2m}{2m+1}\,v_{2m+1};$$

and if we assume that, when m is large, we may take $v_{2m}=v_{2m+1}$, we get $v_{2m}=\frac{2m+1}{4m+1}$, as before, and thence the same result. We therefore see that we do obtain the required correction from the series, though in not so satisfactory a manner as from the continued fraction.

The correction, when applied to the product, amounts to replacing $\frac{2m}{2m+1}\cdot\frac{2m+2}{2m+1}\cdot\frac{2m+2}{2m+3}\cdot\frac{2m+4}{2m+3}\dots$ by $\frac{4m}{4m+1}$. That this quantity is a first approximation to the value of the product just written, may be shown as follows. Let q denote the product; then

$$\begin{aligned}\log q&=\log\left(1-\frac{1}{2m+1}\right)+\log\left(1+\frac{1}{2m+1}\right)+\log\left(1-\frac{1}{2m+3}\right)+\dots\\&=\log\left\{1-\frac{1}{(2m+1)^2}\right\}+\log\left\{1-\frac{1}{(2m+3)^2}\right\}+\dots\\&=-\frac{1}{(2m+1)^2}-\frac{1}{(2m+3)^2}-\dots.\end{aligned}$$

Now

$$\begin{aligned}\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\dots&=-\int\frac{dx}{x^2}+\frac{1}{2x^2}-\frac{1}{12}\frac{d}{dx}\frac{1}{x^2}+\dots\\&=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+\dots,\end{aligned}$$

whence, rejecting terms of the order m^{-2} and higher orders,

$$\frac{1}{(2m)^2}+\frac{1}{(2m+1)^2}+\dots=\frac{1}{2m},$$

and
$$\log q=-\frac{1}{2m}+\frac{1}{4m}=-\frac{1}{4m}=-\log\left(1+\frac{1}{4m}\right);$$

so that
$$q=\frac{4m}{4m+1}.$$

It thus appears that the continued fraction puts in evidence an important correction, which cannot be derived with facility from the product, nor with such rigour from the series.

10. It is interesting to investigate the resulting error in calculating π from the first n factors of Wallis's product (which correspond to the nth convergent to the continued fraction), and to examine in what ratio the error is reduced by Sylvester's correction.

Let
$$v = \frac{2.2.4.4.6.6 \dots \quad 2m}{1.3.3.5.5.7 \dots (2m-1)},$$
$$u = \frac{2.2.4.4.6.6 \dots \quad 2m \quad . \quad 2m}{1.3.3.5.5.7 \dots (2m-1)(2m+1)},$$
and
$$u' = \frac{2.2.4.4.6.6 \dots \quad 2m \quad . \quad 4m}{1.3.3.5.5.7 \dots (2m-1)(4m+1)};$$
then
$$v = \frac{(2^2.4^2.6^2 \dots 4m^2)^2}{2m\,(1.2.3 \dots 2m)}.$$

Reducing this by means of the formula
$$1.2.3 \dots x = \surd(2\pi x)\, x^x e^{-x} \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots\right\},$$
and rejecting terms above m^{-2}, we find
$$v = \frac{\pi}{2}\left\{1 + \frac{1}{12m} + \frac{1}{288m^2}\right\}^4 \left\{1 + \frac{1}{24m} + \frac{1}{1152m^2}\right\}^{-2}$$
$$= \frac{\pi}{2}\left\{1 + \frac{1}{4m} + \frac{1}{32m^2}\right\}.$$

Multiply by $\left(1 + \frac{1}{2m}\right)^{-1}$, viz., by $1 - \frac{1}{2m} + \frac{1}{4m^2}$, and we obtain
$$u = \frac{\pi}{2}\left\{1 - \frac{1}{4m} + \frac{5}{32m^2}\right\}.$$

If we apply the correction, and multiply v by $\left(1 + \frac{1}{4m}\right)^{-1}$, we obtain
$$u' = \frac{\pi}{2}\left\{1 + \frac{1}{32m^2}\right\},$$
so that the replacing $\frac{2m}{2m+1}$ by $\frac{4m}{4m+1}$ destroys the term in $\frac{1}{m}$.

Thus, n being large, the error in the value of π obtained from the first n terms of Wallis's formula (uncorrected) is nearly $\frac{\pi}{2n}$, and from the first n terms (corrected) is nearly $\frac{\pi}{8n^2}$; so that the error in the latter case is about the $\left(\frac{1}{4n}\right)$th part of what it is in the former. This exactly agrees with what Professor Sylvester found, the concluding sentence of his paper being—"The approximation being thus more than fifteen and twenty-one times bettered for the fourth and fifth convergents respectively by aid of the correction."

Of course the product (or continued fraction), even when corrected, is very unsuitable for the calculation of π. The number of decimal places that the first n factors of Wallis's product afford is equal to the integer next below $\log_{10}\frac{2n}{\pi}$, viz., next below $\log_{10} n - \cdot 1961\ldots$, or, roughly, next below $\log_{10} n$; while, if the correction is applied to the nth factor, the number of places is equal to the integer next below $\log\frac{8n^2}{\pi}$, viz., next below $2\log_{10} n + \cdot 4059\ldots$, or, roughly, $2\log_{10} n$; so that the application of the correction about doubles the number of decimal places afforded by the first n terms of the product. When the correction is made, 1000 factors of the product yield but six decimal places of π; and if the correction is not made, only two.

Postscript.—*Added June* 27, 1874.—Since this paper was read, Mr. Muir has called my attention to a paper by Stern in the tenth volume of "Crelle" (1833), in which Sylvester's continued fraction for π is given. On p. 267, Stern converts the product $\frac{d_1 d_2 \ldots d_n}{e_1 e_2 \ldots e_n}$ into a continued fraction by a process very similar in principle to that used in this paper, and thence derives Sylvester's fraction from Wallis's formula. He does not, however, obtain the transformations of $\sin\frac{\pi}{n}$ and $\cos\frac{\pi}{n}$, or of the other products discussed in this paper, so that I have not thought it worth while to make any alteration in consequence.

Stern attributes Sylvester's fraction to Euler (Comm. Acad. Petropol., t. xi., p. 48 (1739); and I find that it is there obtained from the consideration of the ratio of the integrals $\int_0^1 \frac{y\,dy}{\sqrt{(1-y^2)}}$ and $\int_0^1 \frac{dy}{\sqrt{(1-y^2)}}$, which are to one another as 2 to π. In the Phil. Mag. for May, 1874, Mr. Muir has also deduced Sylvester's fraction by means of an integral.

June 11*th*, 1874.

Dr. HIRST, F.R.S., President, and subsequently Prof. CAYLEY, F.R.S., V.P., in the Chair.

The Rev. A. J. Stevens, M.A., and Mr. W. Ritchie, were elected Members of the Society; Dr. Casey was proposed for election; and Mr. G. S. Carr was admitted into the Society.

Before vacating the Chair, the President made a statement to the effect that he had much pleasure in announcing to the Members present that it was Lord Rayleigh's intention to present to the Society the sum of £1000, to assist in the publication of the Proceedings and for the purchase of Mathematical Journals.

Mr. Roberts proceeded to give a sketch of his paper "On the Parallels of Developables and of Curves of Double Curvature."

Lord Rayleigh next brought forward a "Note on the Numerical Calculation of the Roots of Fluctuating Functions."

The Secretary then read parts of papers by Mr. J. Griffiths "On a remarkable relation between the difference of two Fagnanian arcs of an ellipse of eccentricity e and that of two corresponding arcs of a hyperbola of eccentricity $\frac{1}{e}$;" by Mr. Routh, F.R.S., "On Stability of a Dynamical System with two Independent Motions," "On Rocking Stones," and "Small Oscillations to any Degree of Approximation."

The following presents were received:—

"Euclidian Geometry," by F. Cuthbertson, M.A.: from the Author.

"Journal of the Institute of Actuaries and Assurance Magazine," vol. xviii., Pt. ii., No. xciv., Jan. 1874.

"Bulletin des Sciences Mathématiques et Astronomiques," tome sixième, Mai 1874.

"Proceedings of Royal Society," vol. xxii., Nos. 150, 151.

"Demonstration géométrique de quelques théorèmes au moyen de la considération d'une rotation infiniment petite," and "Construction directe du rayon de courbure de la courbe de contour apparent d'une surface qu'on projette orthogonalement sur un plan;" par M. A. Mannheim: from the Author.

"Astronomical and Meteorological Observations made during the year 1871 at the United States Naval Observatory:" Rear-Admiral B. F. Sands, Washington, 1873.

"A new short Treatise of Algebra, with the Geometrical Construction of Equations as far as the Fourth Power or Dimension, together with a Specimen of the Nature and Algorithm of Fluxions," by John Harris, D.D.: London, 1714.

"Traité des progressions par addition ou des séries algébriques, terminé par de nouvelles vues sur la quadrature du cercle et précédé par un discours sur la nécessité d'un nouveau système de calcul," troisième edition: Paris, 1795.

The last two presented by Mr. Drach.