ON PARTIAL DIFFERENTIAL COEFFICIENTS AND ON REPEATED LIMITS IN GENERAL

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1. Let it be assumed that a function $f(x, y)$ is defined for values of x and y such that $x > a$, $y > b$, in a neighbourhood of the point (a, b). The function $f(x, y)$ considered as a function of y only, with x constant and greater than a, has, in general, two functional limits

$$\overline{f(x,\, b+0)}, \qquad \underline{f(x,\, b+0)},$$

the upper and lower limits at the point (x, b); these may be denoted* by $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ respectively. In case these two limits are, for any particular value of x, identical, their common value may be denoted by $\lim_{y=b} f(x, y)$. If either of the limits $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ is to be taken indifferently, we may denote them by $\overline{\underline{\lim}}_{y=b} f(x, y)$. This may be regarded as a function of x, such that its value at the point x is multiple-valued and has $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ for its limits of indeterminacy.

It may happen that $\overline{\underline{\lim}}_{y=b} f(x, y)$, considered as a function of x, has a definite functional limit at the point $x = a$; this limit may be either finite, or infinite with fixed sign. In case such a limit exists, it may be denoted by $\lim_{x=a} \lim_{y=b} f(x, y)$, and it may be termed the *repeated limit* of $f(x, y)$ at the point (a, b), the order of the limits being that the limit for $y = b$ is taken first and then afterwards the limit for $x = a$.

The repeated limit $\lim_{y=b} \lim_{x=a} f(x, y)$, in which the limit with respect to x is first taken, and afterwards that with respect to y, may be defined in a precisely similar manner.

It is clear that the functional *values* on the straight lines $x = a$, $y = b$, in case they are defined, are irrelevant as regards the existence or the values of the repeated limits.

It is unnecessary for the existence of the repeated limit that $f(x, y)$ be defined for all values of (x, y) in a two-dimensional neighbourhood

* The notation here employed is that introduced by Pringsheim.

of (a, b) for which $x > a$, $y > b$. It is sufficient that the function be defined for a set of points in such a neighbourhood, of such a character that (x, b) is, for each value of x belonging to a set of points on the x-axis with the point (a, b) as limiting point, the limiting point of a set of points (x, y) on a straight line through (x, b) parallel to the y-axis. and, further, that a similar condition should hold for points (a, y) on the y-axis.

If for the values (x, y), where $x > a$, $y > b$, for which $f(x, y)$ is defined, a number A exists such that, corresponding to an arbitrarily chosen positive number ϵ, a neighbourhood defined by $a < x < a+\delta$, $b < y < b+\delta$ can be determined for which the condition $|A - f(x, y)| < \epsilon$ is satisfied for every point (x, y) belonging to the domain for which $f(x, y)$ is defined, and lying within the neighbourhood, then the number A defines the double limit of $f(x, y)$ at (a, b). When this double limit exists it is denoted by $\lim_{x=a,\, y=b} f(x, y)$.

In case the double limit exists as a definite number, it follows that the two repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ both exist and are equal to the double limit.

The converse of this statement does not hold good; it is possible that the repeated limits may both exist and have one and the same value, and yet that the double limit may not exist.

2. Necessary and sufficient conditions will now be investigated that the two repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ may both exist and have one and the same finite value.

It will be observed that the existence of $\lim_{x=a} \lim_{y=b} f(x, y)$ does not necessarily involve the existence of $\lim_{y=b} f(x, y)$ as a definite number, since $\lim_{x=a} \overline{\lim_{y=b}} f(x, y)$, $\lim_{x=a} \underline{\lim_{y=b}} f(x, y)$ may both exist and have the same value, without it being necessarily the case that $\overline{\lim_{y=b}} f(x, y)$, $\underline{\lim_{y=b}} f(x, y)$ are identical for any value of x. It is, however, necessary that

$$\overline{\lim_{y=b}} f(x, y) - \underline{\lim_{y=b}} f(x, y)$$

should converge to the limit zero as x converges to the value a. The necessary and sufficient conditions required are contained in the following theorem :—

In order that the repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ *may both exist and have the same finite value, it is necessary and sufficient* (1) *that* $\overline{\lim_{y=b}} f(x, y) - \underline{\lim_{y=b}} f(x, y)$ *should have the limit zero for* $x = a$, *and that* $\overline{\lim_{x=a}} f(x, y) - \underline{\lim_{x=a}} f(x, y)$ *should have the limit zero for* $y = b$; *and* (2) *that, corresponding to any fixed positive number* ϵ *arbitrarily chosen, a positive number* β *can be determined such that for each value of* y *interior to the interval* $(b, b+\beta)$ *a positive number* a_y, *in general dependent on* y, *exists such that, for this value of* y, $f(x, y)$ *lies between* $\overline{\lim_{y=b}} f(x, y) + \epsilon$ *and* $\underline{\lim_{y=b}} f(x, y) - \epsilon$, *for all values of* x *interior to the interval* $(a, a+a_y)$.

To prove that the conditions stated in the theorem are sufficient, let us assume that they are satisfied. A value of y may, in virtue of (1), be so chosen that the difference of the two limits $\overline{\lim_{x=a}} f(x, y)$ and $\underline{\lim_{x=a}} f(x, y)$ is less than an arbitrarily chosen number η; and this value of y may also be so chosen that it is interior to the interval $(b, b+\beta)$. For this fixed value of y, an interval $(a, a+a'_y)$ for x may be so chosen that $f(x, y)$ lies between $\overline{\lim_{x=a}} f(x, y) + \epsilon$ and $\underline{\lim_{x=a}} f(x, y) - \epsilon$, provided y has the fixed value and $a < x < a + a'_y$: this follows from the definition of the upper and lower limits. Again, from the condition (1), a number a'' can be determined such that, if x be interior to the interval $(a, a+a'')$, the difference between the two limits $\overline{\lim_{y=b}} f(x, y)$, $\underline{\lim_{y=b}} f(x, y)$ is less than η. Now let $\bar{a}_y$ be the smallest of the three numbers a_y, a'_y, a''; then, if x_1, x_2 be any two values of x within the interval $(a, a+\bar{a}_y)$, and y have the fixed value, by applying the conditions of the theorem, we see that the inequalities

$$|f(x_1, y) - f(x_2, y)| < \eta + 2\epsilon,$$

$$|f(x_1, y) - \overline{\lim_{y=b}} f(x_1, y)| < \eta + \epsilon,$$

$$|f(x_2, y) - \overline{\lim_{y=b}} f(x_2, y)| < \eta + \epsilon$$

are all satisfied. It follows that

$$|\overline{\lim_{y=b}} f(x_1, y) - \overline{\lim_{y=b}} f(x_2, y)| < 3\eta + 4\epsilon$$

for every pair of points within the interval $(a, a+\bar{a}_y)$. Hence, since ϵ, η are both arbitrarily small, $\overline{\lim_{y=b}} f(x, y)$ converges for $x = a$ to a definite value which is the limit of both $\overline{\lim_{y=b}} f(x, y)$ and of $\underline{\lim_{y=b}} f(x, y)$ when $x = a$; and thus $\lim_{x=a} \lim_{y=b} f(x, y)$ exists.

Again, since $\lim_{x=a} \lim_{y=b} f(x, y)$ has a definite value, an interval $(a, a+\delta)$ can be determined such that, for any point interior to it,

$$|\lim_{x=a} \lim_{y=b} f(x, y) - \overline{\lim_{y=b}} f(x, y)| < \epsilon.$$

Now $\lim_{x=a}\lim_{y=b} f(x, y)-\overline{\lim_{x=a}} f(x, y)$ is the sum of the three differences

$$\lim_{x=a}\lim_{y=b} f(x, y)-\overline{\lim_{y=b}} f(x, y),$$

$$\overline{\lim_{y=b}} f(x, y)-f(x, y),$$

$$f(x, y)-\overline{\lim_{x=a}} f(x, y);$$

and for a fixed y, chosen as before, x may be chosen so that it not only lies within the interval $(a, a+\delta)$, but is also such that

$$|f(x, y)-\overline{\lim_{y=b}} f(x, y)|, \qquad |f(x, y)-\overline{\lim_{x=a}} f(x, y)|$$

are each less than $\eta+2\epsilon$. It follows that

$$|\lim_{x=a}\lim_{y=b} f(x, y)-\overline{\lim_{x=a}} f(x, y)| < 5\epsilon+2\eta,$$

and thus that $\overline{\lim_{x=a}} f(x, y)$ converges, as y converges to b, to the limit $\lim_{x=a}\lim_{y=b} f(x, y)$. It has thus been shewn that the two repeated limits both exist and have the same value.

Conversely, let us assume that the repeated limits both exist, and are finite and equal. We have, then, $|\lim_{x=a}\lim_{y=b} f(x, y)-\overline{\lim_{x=a}} f(x, y)| < \zeta$, provided y lies between b and $b+\beta$, where β is some fixed number, ζ being an arbitrarily chosen positive number; from this it follows that

$$|\overline{\lim_{x=a}} f(x, y)-\underline{\lim_{x=a}} f(x, y)|$$

is less than 2ζ, for $b<y<b+\beta$. Also

$$|\overline{\lim_{y=b}} f(x, y)-\lim_{x=a}\lim_{y=b} f(x, y)| < \zeta,$$

provided x lies within some fixed interval $(a, a+\delta')$; and from this it follows that $|\overline{\lim_{y=b}} f(x, y)-\underline{\lim_{y=b}} f(x, y)|$ is $< 2\zeta$, for $a<x<a+\delta'$. Since ζ is arbitrarily small, we now see that the condition (1) of the theorem is satisfied. Further, we see that

$$|f(x, y)-\overline{\lim_{x=a}} f(x, y)| < 2\zeta+\zeta'$$

where ζ' is any arbitrarily chosen positive number, provided x lies within some interval $(a, a+a'_y)$, where a'_y depends upon y, and may diminish indefinitely as y approaches the value b. It follows from the three inequalities that

$$|f(x, y)-\overline{\lim_{y=b}} f(x, y)| < 4\zeta+\zeta',$$

provided $b<y<b+\beta$, and provided also that x lies within some interval

$(a, a+a_y)$ where a_y depends in general upon y. Since ζ and ζ' are both arbitrarily small, it follows that the condition (2) of the theorem is satisfied.

If the condition (2) in the above general theorem be replaced by the more stringent condition that, corresponding to any fixed positive number ϵ arbitrarily chosen, a positive number β can be determined, which is such that, for each value of y interior to the interval $(b, b+\beta)$, a positive number a_y dependent on y exists such that, for this value of y *and for all smaller values*, $f(x, y)$ lies between $\overline{\lim_{y=b}} f(x, y)+\epsilon$ and $\underline{\lim_{y=b}} f(x, y)-\epsilon$, then this condition and the condition (1) are the necessary and sufficient conditions that not only $\lim_{x=a}\lim_{y=b} f(x, y)$, $\lim_{y=b}\lim_{x=a} f(x, y)$ exist and are equal, but also that the double limit $\lim_{y=b\ x=a} f(x, y)$ exists, having a definite value the same as the repeated limits.

For, under the conditions stated, we have, provided y lies within the interval $(b, b+\beta_1)$ where $\beta_1 < \beta$,

$$|f(x, y)-\overline{\lim_{y=b}} f(x, y)| < \epsilon+\eta$$

where x has any value in the interval $(a, a+\zeta)$, ζ being the lesser of the two numbers a_{β_1} and δ', the number δ' being so chosen that

$$|\overline{\lim_{y=b}} f(x, y)-\underline{\lim_{y=b}} f(x, y)| < \eta$$

for $a < x < a+\delta'$. Also

$$|\overline{\lim_{y=b}} f(x, y)-\lim_{x=a}\lim_{y=b} f(x, y)| < \epsilon,$$

provided x lies within an interval chosen sufficiently small. Hence the condition

$$|f(x, y)-\lim_{x=a}\lim_{y=b} f(x, y)| < 2\epsilon+\eta$$

is satisfied, provided that $b < y < b+\beta_1$ and provided x lies within an interval of which the length may depend on ϵ and η. It follows, since ϵ, η are arbitrarily small, that $f(x, y)$ has a definite double limit at the point (a, b). That the conditions are necessary follows at once from the definition of $\lim_{x=a,\ y=b} f(x, y)$.

3. The theorem obtained in § 2 may be simplified in the case in which $\lim_{y=b} f(x, y)$, $\lim_{x=a} f(x, y)$ both have definite values at all points on the straight lines $x = a$, $y = b$ which are in sufficiently small neighbourhoods of the point (a, b). We may then state the theorem as follows :—

*If** $\lim_{y=b} f(x, y)$, $\lim_{x=a} f(x, y)$ *have definite finite values in the neighbourhood of the point* (a, b), *then the necessary and sufficient condition that the two repeated limits* $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ *may both exist and have the same finite value is that, corresponding to any fixed positive number* ϵ *arbitrarily chosen, a positive number* β *can be determined which is such that, for each value of* y *interior to the interval* $(b, b+\beta)$, *a positive number* a_y *in general dependent on* y *exists, such that, for this value of* y, $|f(x, y)-\lim_{y=b} f(x, y)| < \epsilon$ *for all values of* x *within the interval* $(a, a+a_y)$.

In case the condition $|f(x, y)-\lim_{y=b} f(x, y)| < \epsilon$ for all values of x within $(a, a+a_y)$ be satisfied not only for the particular value of y, but for all smaller values, and this holds for every ϵ, then the double limit exists and is equal to the repeated limits. In this case, the point (a, b) is a point of uniform convergence of the function $f(x, y)$ to the limit $\lim_{y=b} f(x, y)$ with respect to the parameter x; and thus for such a point there exists for each value of ϵ an interval $(a, a+a)$ where a depends in general upon ϵ, such that, for each value of x within this interval, the condition $|f(x, y)-\lim_{y=b} f(x, y)| < \epsilon$ is satisfied, provided y be less than some fixed value which is the same for the whole x-interval $(a, a+a)$.

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* This theorem may be transformed at once into the corresponding theorem for double sequences. Let $x-a = 1/\nu$, $y-b = 1/\mu$, and assume that $f(x, y)$ is defined only for those values of x and y which correspond to positive integral values of ν and μ; also, let

$$f(x, y) = a_{\mu\nu}.$$

The theorem relating to the existence and equality of the repeated limits $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} \lim_{\mu=\infty} a_{\mu\nu}$ is then as follows:—

If $a_{\mu\nu}$ be such that $\lim_{\mu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} a_{\mu\nu}$ both have definite values for all positive integral values of ν and of μ respectively, then the necessary and sufficient condition that the two repeated limits $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} \lim_{\mu=\infty} a_{\mu\nu}$ both exist and are equal to one another, is that corresponding to any fixed positive number ϵ arbitrarily chosen, a number m_0 can be determined such that, if any integer $\mu > m_0$ be taken, an integer n_μ can be found such that $|a_{\mu\nu}-\lim_{\mu=\infty} a_{\mu\nu}| < \epsilon$ for all values of ν such that $\nu > n_\mu$.

This is equivalent to a theorem which has been given by Prof. Bromwich, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1, p. 185; except that, in the statement given by Prof. Bromwich, the condition is added that $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$ must exist and have a finite value. This last condition is necessarily satisfied if the other condition be satisfied. In the alternative theorem given by Prof. Bromwich (*loc. cit.*, p. 184), the condition that $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$ must have a definite value is not redundant, and is therefore rightly included in the statement of his theorem. The occurrence of the condition in question in the theorem on p. 185 arose from the fact that Prof. Bromwich deduced that theorem from the one on p. 184 in which the condition is required.

4. The necessary and sufficient conditions for the existence and equality of the repeated limits may be stated in a different form from that given in § 2.

The necessary and sufficient conditions that

$$\lim_{x=a} \lim_{y=b} f(x, y) = \lim_{y=b} \lim_{x=a} f(x, y),$$

their value being finite, are (1) *that* $\overline{\lim_{x=a}} f(x, y)$ *converges to a definite value* $\lim_{y=b} \lim_{x=a} f(x, y)$ *when* y *converges to* b, *and that*

$$\overline{\lim_{y=b}} f(x, y) - \underline{\lim_{y=b}} f(x, y)$$

should converge to zero for $x = a$; *and* (2) *that, corresponding to any arbitrarily chosen positive number* ϵ, *and to an arbitrarily chosen value* $b+\beta_0$ *of* y, *a value* $y_1 < b+\beta_0$ *can be found, and also a positive number* α, *such that the condition that* $f(x, y_1)$ *lies between*

$$\overline{\lim_{y=b}} f(x, y) + \epsilon \quad \text{and} \quad \underline{\lim_{y=b}} f(x, y) - \epsilon$$

is satisfied for every value of x *within the interval* $(a, a+\alpha)$.

In case $\lim_{y=b} f(x, y)$ *everywhere exists in the neighbourhood of* $x = a$, *the condition* (2) *is that* $|f(x, y_1) - \lim_{y=b} f(x, y)| < \epsilon$, *for every value of* x *within the interval* $(a, a+\alpha)$.

That the conditions contained in the theorem are necessary is seen from the theorem of § 2; it will be shewn that they are sufficient. Let us assume that the conditions are satisfied. We have then

$$\overline{\lim_{y=b}} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)$$

$$= [\overline{\lim_{y=b}} f(x,y) - f(x,y)] + [f(x,y) - \overline{\lim_{x=a}} f(x,y)] + [\overline{\lim_{x=a}} f(x,y) - \lim_{y=b} \lim_{x=a} f(x,y)].$$

A positive number β_1 can now be chosen, such that, if $b < y < b+\beta_1$, the condition $|\overline{\lim_{x=a}} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)| < \epsilon$ is satisfied; moreover, we may choose β_1 so that it is less than β_0.

Next, a value y_1 of y exists, such that $f(x, y_1)$ lies between

$$\overline{\lim_{y=b}} f(x, y) + \epsilon \quad \text{and} \quad \underline{\lim_{y=b}} f(x, y) - \epsilon,$$

provided x lies within the interval $(a, a+\alpha)$: the value of β_0 may be so chosen that $\overline{\lim_{x=a}} f(x, y) - \underline{\lim_{x=a}} f(x, y) < \epsilon$, for every value of y which is less than $b+\beta_0$, and therefore for the value y_1 of y. Again, an interval for x, possibly less than $(a, a+\alpha)$, can be so chosen that

$$\overline{\lim_{y=b}} f(x, y) - \underline{\lim_{y=b}} f(x, y) < \epsilon,$$

provided x lie within the interval. It follows that an interval $(a, a+\alpha')$ for x can be found, such that $|\overline{\lim_{y=b}} f(x, y) - f(x, y_1)| < 3\epsilon$. Further, the interval within which x lies may, if necessary, be so restricted that

$$|f(x, y_1) - \overline{\lim_{x=a}} f(x, y_1)| < 2\epsilon.$$

Hence, provided x lies within a definite interval, we see that

$$|\overline{\lim_{y=b}} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)| < 6\epsilon;$$

and, since this condition holds for an arbitrary ϵ, it follows that $\overline{\lim_{y=b}} f(x, y)$ converges for $x = a$ to $\lim_{y=b} \lim_{x=a} f(x, y)$, and thus the sufficiency of the conditions is established.

5. The differential coefficient $\frac{\partial}{\partial x_0} \frac{\partial f}{\partial y_0}$, or $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$ is the repeated limit

$$\lim_{h=0} \lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)}{hk};$$

it being assumed that $\frac{\partial f}{\partial y_0}$ exists and is finite.

We may denote this repeated limit by $\lim_{h=0} \lim_{k=0} F(h, k)$. In order that the partial differential coefficient may exist, the value of this limit must be independent of the signs of h and k.

It is not essential for the existence of the repeated limit, or of the partial differential coefficient $\frac{\partial^2 f}{\partial x_0 \partial y_0}$, that

$$\lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} \quad \text{or} \quad \frac{\partial}{\partial y_0} f(x_0+h, y_0)$$

should exist when $h \neq 0$. Thus the repeated limit may have a definite value when

$$\lim_{h=0} \frac{1}{h} \left[\overline{\lim_{k=0}} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} - \underline{\lim_{k=0}} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} \right]$$

vanishes. Therefore $\frac{\partial^2 f}{\partial x_0 \partial y_0}$ may exist when $\frac{\partial f}{\partial y}$ exists at the point (x_0, y_0), but is indefinite in the neighbourhood of that point.

The repeated limit may have a definite finite value, and yet

$$\lim_{k=0} \frac{f(x_0, y_0+k)-f(x_0, y_0)}{k} \quad \text{or} \quad \frac{\partial f(x_0, y_0)}{\partial y_0}$$

may be indefinite,* in which case $\dfrac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$ does not exist in accordance with the usual definition, in which the existence of $\dfrac{\partial f(x_0, y_0)}{\partial y_0}$ is presupposed. It is, in fact, only when $\dfrac{\partial f}{\partial y_0}$ exists that the repeated limit $\lim_{h=0} \lim_{k=0} F(h, k)$ can be written in the form

$$\lim_{h=0} \frac{1}{h} \left[\overline{\lim_{k=0}} \frac{f(x_0+h, y_0+k)-f(x_0+h, y_0)}{k} - \lim_{k=0} \frac{f(x_0, y_0+k)-f(x_0, y_0)}{k} \right],$$

which is then equal to

$$\lim_{h=0} \lim_{k=0} \frac{f(x_0+h, y_0+k)-f(x_0+h, y_0)-f(x_0, y_0+k)+f(x_0, y_0)}{hk}.$$

Also, when $\lim_{h=0} \lim_{k=0} F(h, k)$ is infinite with a definite sign, in order that $\dfrac{\partial^2 f}{\partial x_0 \partial y_0}$ may exist, it is necessary that $\dfrac{\partial f}{\partial y_0}$ should exist. Unless this last condition be satisfied, $\dfrac{\partial^2 f}{\partial x_0 \partial y_0}$ does not exist, in accordance with the usual definition. When this condition is satisfied, the value of $\dfrac{\partial^2 f}{\partial x_0 \partial y_0}$ is infinite with a definite sign.

6. The condition for the validity of the theorem

$$\frac{\partial^2 f}{\partial x_0 \partial y_0} = \frac{\partial^2 f}{\partial y_0 \partial x_0}$$

is the same as the condition that the two repeated limits $\lim_{h=0} \lim_{k=0} F(h, k)$, $\lim_{k=0} \lim_{h=0} F(h, k)$ should exist and have the same value. Necessary and sufficient conditions for this could be obtained by applying the theorem of § 2, remembering that in applying the conditions each of the numbers h, k must be regarded as being capable of having either sign. It is, however, convenient, for application in particular cases, to have sufficient conditions relating to the partial differential coefficients in the neighbourhood of the point (x_0, y_0). Such sufficient conditions are contained in the well known investigation of Schwarz relating to this theorem.

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* A referee called my attention to this possibility. It is illustrated by an example given by Hardy in the *Messenger of Math.*, Vol. XXXII., p. 188. Let $f(x, y) = \phi(x) + \psi(y)$ where ϕ and ψ are not differentiable. Then $f(x_0+h, y_0+k) - f(x_0+h, y) - f(x_0, y_0+k) + f(x_0, y_0)$ vanishes identically, and therefore the repeated limit exists; but $\dfrac{\partial f}{\partial y_0}$ does not exist, and thus $\dfrac{\partial^2 f}{\partial x_0 \partial y_0}$ does not exist.

Sufficient conditions for the validity of the theorem will be here found, and are of a less stringent character than those given by Schwarz,* who assumed the additional condition that $\frac{\partial f(x, y_0)}{\partial y_0}$ exists and is finite for values of x in the neighbourhood of $x = x_0$, for the constant value y_0.

The following theorem will be established:—

If (1) $\frac{\partial^2 f(x, y)}{\partial y\, \partial x}$ *exist and be finite at all points in a two-dimensional neighbourhood of the point* (x_0, y_0), *except that its existence at* (x_0, y_0) *is not assumed, and* (2) *the point* (x_0, y_0) *be a point of continuity of* $\frac{\partial^2 f}{\partial y\, \partial x}$ *with respect to* (x, y), *the limit of this partial differential coefficient at* (x_0, y_0) *being a definite number* A, *and* (3) $\frac{\partial f(x_0, y_0)}{\partial x_0}$ *and* $\frac{\partial f(x_0, y_0)}{\partial y_0}$ *both exist, having definite values, then* $\frac{\partial^2 f(x_0, y_0)}{\partial y_0\, \partial x_0}$, $\frac{\partial^2 f(x_0, y_0)}{\partial x_0\, \partial y_0}$ *both exist, and have the same value* A.

It will be observed that the condition (1) implies the existence of $\frac{\partial f(x, y)}{\partial x}$ at all points in a neighbourhood of (x_0, y_0), except at that point itself, and that it is continuous with respect to y. From the condition (2), we have, corresponding to an arbitrarily chosen positive number ϵ,

$$\frac{\partial^2 f(x_0+h, y_0+k)}{dy\, dx} = A + a(h, k),$$

where $|a| < \epsilon$, provided $|h|$, $|k|$ are each less than some fixed positive number η dependent on ϵ, and are not both zero.

Let $u(k')$ denote $\frac{\partial f(x_0+h, y_0+k')}{\partial x} - Ak'$, where k' lies in the interval $(0, k)$; we have then $\frac{du(k')}{dk'} = a(h, k')$, and this is numerically less than ϵ. It follows that $\frac{u(k)-u(0)}{k}$ is numerically less than ϵ; for, by the mean value theorem † of the differential calculus, since $u(k')$ is continuous at $k' = 0$, and at $k' = k$, and possesses a definite differential coefficient at every interior point of the interval $(0, k)$, there exists a number $\bar{k}$ interior to the interval $(0, k)$ such that $\frac{u(k)-u(0)}{k} = \frac{du(\bar{k})}{d\bar{k}}$, and this is numerically

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* *Gesammelte Abh.*, Vol. II., p. 275.

† The precise form of the mean-value theorem here employed is as follows :—If the function (x) be continuous in the interval $(x, x+h)$, and at every point *in the interior* of this interval $f'(x)$ exist, being either finite or infinite with fixed sign, then a point $x+\theta h$ exists, where $0 < \theta < 1$, and θ is neither 0 nor 1, such that $f(x+h) = f(x) + hf'(x+\theta h)$.

less than ϵ. We have now

$$\frac{\partial f(x_0+h, y_0+k)}{\partial x} - \frac{\partial f(x_0+h, y_0)}{\partial x} - Ak = k\alpha''(h, k),$$

where $\alpha''(h, k)$ is numerically less than ϵ. This holds for each value of h such that $0 < |h| < \eta$.

Let $v(h')$ denote $\frac{f(x_0+h', y_0+k) - f(x_0+h', y_0)}{k} - Ah'$, where h' lies in the interval $(0, h)$; we have then $\frac{dv(h')}{dh'} = \alpha''(h', k)$, and this is numerically less than ϵ. As before, since $v(h')$ is, in virtue of (3), continuous at $h' = 0$ and also at $h' = h$, and possesses a definite differential coefficient at all interior points of the interval $(0, h)$, it follows that $\frac{v(h) - v(0)}{h}$ is numerically less than ϵ; hence

$$\begin{aligned} hkF(h, k) &= f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0) \\ &= Ahk + hk\alpha'''(h, k) \end{aligned}$$

where $|\alpha'''(h, k)| < \epsilon$.

We have now, corresponding to the arbitrarily chosen ϵ,

$$|F(h, k) - A| < \epsilon,$$

provided $|h|$, $|k|$ are each less than some fixed positive number η dependent on ϵ. It follows that $F(h, k)$ is continuous at the point $h = 0$, $k = 0$ in the two-dimensional domain (h, k) and has A for its double limit. From this we conclude that the two limits $\lim_{h=0} \lim_{k=0} F(h, k)$, $\lim_{k=0} \lim_{h=0} F(h, k)$ exist, and are both equal to A. It follows that, when the condition (3) of the theorem is satisfied, the two partial differential coefficients

$$\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}, \qquad \frac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$$

both exist and are equal to A.

7. The case in which the two partial differential coefficients are infinite with fixed sign is covered by the following theorem :—

If (1) $\frac{\partial^2 f(x, y)}{\partial y \partial x}$ *exist and be finite at all points in a two-dimensional neighbourhood of the point* (x_0, y_0), *except that its existence at* (x_0, y_0) *is not assumed, and* (2) *the function* $\frac{\partial^2 f(x, y)}{\partial y \partial x}$ *have the improper limit* $+\infty$ *or* $-\infty$, *with definite sign, at the point* (x_0, y_0), *and* (3) *the differential* *coefficients* $\dfrac{\partial f(x_0, y_0)}{\partial x_0}$, $\dfrac{\partial f(x_0, y_0)}{\partial y_0}$ *both exist and have definite values, then* $\dfrac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$, $\dfrac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$ *both exist, having the improper value* $+\infty$ *or* $-\infty$ *with definite sign.*

To prove this theorem, let us assume that, if M be an arbitrarily chosen positive number, the condition $\dfrac{\partial^2 f(x_0+h, y_0+k)}{\partial y \partial x} > M$ is satisfied for all values of h and k which are not both zero, and are both numerically less than some fixed number η, dependent on M. Defining $u(k')$ as $\dfrac{\partial f(x_0+h, y_0+k')}{\partial x}$, we see, by means of the mean-value theorem, that

$$\frac{u(k)-u(0)}{k} > M \quad \text{or} \quad \frac{\partial f(x_0+h, y_0+k)}{\partial x} - \frac{\partial f(x_0+h, y_0)}{\partial x} > kM.$$

Next, defining $v(h')$ as $\dfrac{f(x_0+h', y_0+k)-f(x_0+h', y_0)}{k}$, we see, as before, that $\dfrac{v(h)-v(0)}{h} > M$; therefore $F(h, k) > M$, provided $|h|$, $|k|$ are both less than some number η dependent on M. It follows that $F(h, k)$ converges to the improper limit $+\infty$, with fixed sign, as h and k converge in any manner, each to the limit zero; thus both the limits

$$\lim_{h=0} \lim_{k=0} F(h, k), \qquad \lim_{k=0} \lim_{h=0} F(h, k)$$

are $+\infty$. In order that $\dfrac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$, $\dfrac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$ may exist, in which case they both have the improper value $+\infty$, it is necessary to assume that $\dfrac{\partial f(x_0, y_0)}{\partial x_0}$, $\dfrac{\partial f(x_0, y_0)}{\partial y_0}$ both have definite values. The case in which the limits have both the improper value $-\infty$ may be treated in a precisely similar manner.