On the Equation of Riccati. *By* Sir James Cockle, M.A., F.R.S.

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1. This paper will consist of seven sections :—

I. *On Deformations, Mixed Integrals, Characteristics, and Correlations.*

2. Meaning by f, F, ϕ, and Φ, respectively, the four expressions

$$f\left(y, x, \frac{d}{dx}\right), \qquad F\left(Y, x, \frac{d}{dx}\right),$$

$$\phi\left(y, Y, x, \frac{d}{dx}\right), \quad \Phi\left(Y, y, x, \frac{d}{dx}\right),$$

and denoting by acute accents differential coefficients taken with respect to the independent variable x, suppose that

$$\Phi f + \phi F = \frac{d\Omega}{dx} = \Omega';$$

then I call either of the functions f and F a deformation of the other, and $\Omega - c$ a mixed integral of the system f and F.

3. The presence of Y is essential to Φ, and that of y to ϕ; but in particular cases y may be absent from Φ, and Y from ϕ.

4. Let each of the values $u, v, w \ldots$ of y reduce f to zero, and let $\phi_u, \phi_v, \phi_w \ldots$ be the corresponding values of ϕ. Then $\phi_u, \phi_v, \phi_w \ldots$ are integrating factors of F. So, if each of the values $U, V, W \ldots$ of Y reduce F to zero, the corresponding values of Φ, say $\Phi_U, \Phi_V, \Phi_W \ldots$ are integrating factors of f.

5. When f and F are of the same order, the complete solution of one involves that of the other.

6. Any general relation of the form

$$\chi\left(y,\, Y,\, x,\, \frac{d}{dx}\right) = 0$$

I call a characteristic. By eliminating Y between the characteristic and the mixed integral, we get a first integral of f, and by so eliminating y we get a first integral of F. Such integral contains an arbitrary constant.

7. In f substitute for x a function of another variable, say t, and after the substitution replace t by x. We thus get a new expression which I shall represent by $\dot{f}\left(\dot{y},\, x,\, \frac{d}{dx}\right)$. An accompanying deformation, corresponding with F, I shall denote by $\dot{F}\left(\dot{Y},\, x,\, \frac{d}{dx}\right)$; calling the passages from f to $\dot{f}$, and from F to $\dot{F}$ correlations, and f, $\dot{f}$ or F, $\dot{F}$ correlates.

II. *On Synthemes.*

8. Denoting r accents by the index (r), I call such an expression as $\dot{u}^{(r)}u^{(s)}+\dot{v}^{(r)}v^{(s)}$, which may be expressed briefly by $S\,.\,\dot{u}^{(r)}u^{(s)}$, or by (r, s), a syntheme. Put

$$\dot{u}u+\dot{v}v = A, \quad \dot{u}'u+\dot{v}'v = B, \quad \dot{u}u'+\dot{v}v' = C,$$

$$\dot{u}'u'+\dot{v}'v' = D, \quad \dot{u}''u+\dot{v}''v = E, \quad \dot{u}u''+\dot{v}v'' = F,$$

$$\dot{u}''u''+\dot{v}''v'' = I, \quad \dot{u}''u'+\dot{v}''v' = G, \quad \dot{u}'u''+\dot{v}'v'' = H,$$

and we have nine relations between the thirteen quantities $A, B, \ldots I$, $\dot{u}$, u, $\dot{v}$ and v.

9. On inspection, we see that

$$A' = B+C \quad \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$B' = D+E \quad \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

$$C' = D+F \quad \ldots\ldots\ldots\ldots\ldots\ldots (c),$$

$$D' = G+H \quad \ldots\ldots\ldots\ldots\ldots\ldots (d),$$

and, by the elimination of $\dot{v}$ and v, we get

$$AD-BC = A\dot{u}'u' - B\dot{u}u' - C\dot{u}'u + D\dot{u}u \quad \text{..........}(1),$$

$$AI-EF = A\dot{u}''u'' - E\dot{u}u'' - F\dot{u}''u + I\dot{u}u \quad \text{..........}(2),$$

$$DI-GH = D\dot{u}''u'' - G\dot{u}'u'' - H\dot{u}''u' + I\dot{u}'u' \quad \text{..........}(3),$$

and we have seven relations between the eleven quantities $A, B, \ldots I$, $\dot{u}$ and u.

10. There is also the relation, obtainable by elimination,

$$A(G+H)-CE-BF$$

$$= A(\dot{u}''u' + \dot{u}'u'') - B\dot{u}u'' - C\dot{u}''u - E\dot{u}u' - F\dot{u}'u + (G+H)\dot{u}u \ldots(4),$$

but this last relation is not independent, being deducible by differentiation from (1).

11. Assume
$$y''' + py'' + qy' + ry = 0 \quad \text{....................}(\text{i.}),$$

$$\dot{y}''' + \dot{p}\dot{y}'' + \dot{q}\dot{y}' + \dot{r}\dot{y} = 0 \quad \text{....................}(\text{ii.}).$$

12. Then, recalling Art. 8 and its notation,

$$E' = G + \dot{u}'''u + \dot{v}'''v = G + (3, 0) = G - \dot{p}E - \dot{q}B - \dot{r}A \ldots\ldots\ldots\ldots(e),$$

$$F' = H + (0, 3) = H - pF - qC - rA \ldots\ldots\ldots\ldots(f),$$

$$G' = I + (3, 1) = I - \dot{p}G - \dot{q}D - \dot{r}C \ldots\ldots\ldots\ldots(g);$$

$$H' = I + (1, 3) = I - pH - qD - rB \ldots\ldots\ldots\ldots(h),$$

$$I' = (3, 2) + (2, 3) = -(\dot{p}+p)I - \dot{q}H - \dot{r}F - qG - rE \quad \ldots\ldots\ldots\ldots(i),$$

but these last relations (*e*), (*f*), ... (*i*) are not independent, although, like (4), they subserve the purposes of elimination.

13. The nine independent relations, in Arts. 9 and 11, between the eleven quantities leave the system indeterminate. But the system, being homogeneous, will be rendered determinate when one more relation is assigned.

III. *On Terordinals.*

14. Assume
$$Y''' + PY'' + QY' + RY = 0 \quad \text{.................}(\text{iii.}),$$
and that between the coefficients of (i.) and (iii.), or, recurring to

Art. 1, say of $f=0$ and $F=0$, there subsist the three (two skew and one normal) symmetrical relations

$$R=-r,$$

$$P+\frac{R'}{R}=-\left(p+\frac{r'}{r}\right),$$

$$Q-\tfrac{1}{2}\left(P+\frac{R'}{R}\right)+\tfrac{1}{2}\frac{R'}{R}\left(P+\frac{R'}{R}\right)=q-\tfrac{1}{2}\left(p+\frac{r'}{r}\right)'+\tfrac{1}{2}\frac{r'}{r}\left(p+\frac{r'}{r}\right),$$

then (iii.) is a deformation of (i.), and (i.) of (iii.).

15. For if $\Phi=\dfrac{Y'}{R}$ and $\phi=\dfrac{y'}{r}$, then

$$\Omega'=\frac{d}{dx}\left\{\frac{1}{r}(y'Y''-y''Y')-\frac{1}{r}\left(p+\frac{r'}{r}\right)y'Y'-yY\right\};$$

whence the mixed integral

$$\Omega-c_0=\frac{1}{r}(y'Y''-y''Y)-\frac{1}{r}\left(p+\frac{r'}{r}\right)y'Y'-yY.$$

16. If f and F be supposed to vanish, we may replace Ω by an arbitrary constant, and thus get

$$y'Y''-y''Y'-\left(p+\frac{r'}{r}\right)y'Y'-ryY=cr.$$

17. Let u, v and w be independent particular solutions of (i.), and U, V, W of (iii.). Then, by means of the mixed integral, we get the systems of values exhibited by means of determinants in the following eighteen equations.

18. For one system, we have, putting

$$p+\frac{r'}{r}=\pi,$$

$$U=e^{\int p\,dx}\begin{vmatrix} v'', & v' \\ w'', & w' \end{vmatrix},\quad V=e^{\int p\,dx}\begin{vmatrix} w'', & w' \\ u'', & u' \end{vmatrix},\quad W=e^{\int p\,dx}\begin{vmatrix} u'', & u' \\ v'', & v' \end{vmatrix},$$

$$U'=re^{\int p\,dx}\begin{vmatrix} v', & v \\ u', & u \end{vmatrix},\quad V'=re^{\int p\,dx}\begin{vmatrix} w', & w \\ u', & u \end{vmatrix},\quad W'=re^{\int p\,dx}\begin{vmatrix} u', & u \\ v', & v \end{vmatrix},$$

$$U'' = \pi U' + re^{\int p\,dx}\begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix}, \quad V'' = \pi V' + re^{\int p\,dx}\begin{vmatrix} w'', & w \\ u'', & u \end{vmatrix},$$

$$W'' = \pi W' + re^{\int p\,dx}\begin{vmatrix} u'', & u \\ v'', & v \end{vmatrix}.$$

19. For another, we have, putting

$$P + \frac{R'}{R} = \Pi,$$

$$u = e^{\int P\,dx}\begin{vmatrix} V'', & V' \\ W'', & W' \end{vmatrix}, \quad v = e^{\int P\,dx}\begin{vmatrix} W'', & W' \\ U'', & U' \end{vmatrix}, \quad w = e^{\int P\,dx}\begin{vmatrix} U'', & U' \\ V'', & V' \end{vmatrix},$$

$$u' = Re^{\int P\,dx}\begin{vmatrix} V', & V \\ W', & W \end{vmatrix}, \quad v' = Re^{\int P\,dx}\begin{vmatrix} W', & W \\ U', & U \end{vmatrix}, \quad w' = Re^{\int P\,dx}\begin{vmatrix} U', & U \\ V', & V \end{vmatrix}.$$

$$u'' = \Pi u' + Re^{\int P\,dx}\begin{vmatrix} V'', & V \\ W'', & W \end{vmatrix}, \quad v'' = \Pi v' + Re^{\int P\,dx}\begin{vmatrix} W'', & W \\ U'', & U \end{vmatrix},$$

$$w'' = \Pi w + Re^{\int P\,dx}\begin{vmatrix} U'', & U \\ V'', & V \end{vmatrix}.$$

20. In verification put, for the moment,

$$\phi = re^{\int p\,dx}\begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix},$$

then $$U'' = \pi U' + \phi, \quad \text{or} \quad U'' - \left(p + \frac{r'}{r}\right) U' = \phi;$$

whence $$U''' - \left(p + \frac{r'}{r}\right) U'' - \left(p + \frac{r'}{r}\right)' U' = \phi';$$

but $$\phi' = \left(p + \frac{r'}{r}\right)\phi + rU + re^{\int p\,dx}\begin{vmatrix} v''', & v \\ w''', & w \end{vmatrix},$$

and $$\begin{vmatrix} v''', & v \\ w''', & w \end{vmatrix} = \begin{vmatrix} -pv'' - qv' - rv, & v \\ -pw'' - qw' - rw, & w \end{vmatrix} = -p\begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix} - q\begin{vmatrix} v', & v \\ w', & w \end{vmatrix};$$

therefore $$\phi' = \frac{r'}{r}\phi + rU - qU' = \frac{r'}{r}(U'' - \pi U') + rU - qU',$$

and $U''' - \left(p+2\frac{r'}{r}\right)U'' + \left\{q - \left(p+\frac{r'}{r}\right)' + \frac{r'}{r}\left(p+\frac{r'}{r}\right)\right\}U' - rU = 0$

.........(iii. *bis*),

which is, in fact, (iii.).

21. Mere differentiations will give one class of verifications. Another will be obtained by substitutions. Thus,

$$\begin{aligned} w &= e^{\int P\,dx}(V'U'' - U'V'') \\ &= e^{\int P\,dx} \,.\, re^{\int p\,dx}\{V'(wv''-vw'') - U'(uw''-wu'')\} \\ &= r^2 e^{\int (P+2p)\,dx}\{(uw'-wu')(wv''-vw'') - (wv'-vw')(uw''-wu'')\} \\ &= e^{\int p\,dx} \,.\, w \,.\, \begin{vmatrix} u'', & u', & u \\ v'', & v', & v \\ w'', & w', & w \end{vmatrix} = w \text{ (as it ought)}; \end{aligned}$$

for, denoting the last determinant by δ, we have, neglecting vanishing determinants,

$$\delta' = \begin{vmatrix} u''', & u', & u \\ v''', & v', & v \\ w''', & w', & w \end{vmatrix} = \begin{vmatrix} -pu''-qu'-ru, & u', & u \\ -pv''-qv'-rv, & v', & v \\ -pw''-qw'-rw, & w', & w \end{vmatrix} = -p\delta;$$

and $\delta = e^{-\int p\,dx}$ or $e^{\int p\,dx} \,.\, \delta = 1.$

22. So, when $\Delta = \begin{vmatrix} U'', & U', & U \\ V'', & V', & V \\ W'', & W', & W \end{vmatrix},$

then also $\Delta = e^{-\int P\,dx}$ or $e^{\int P\,dx} \,.\, \Delta = 1.$

23. An arbitrary element is involved in δ, for a multiple of a solution is likewise a solution. The same is true of Δ. Considered isolatedly, every one of the expressions δ, Δ, $e^{\int p\,dx}$ and $e^{\int P\,dx}$ carries with it an independent arbitrary multiplier.

24. A third class of verifications may be got from the ternary quadratic synthemes

$$u^{(r)}U^{(s)} + v^{(r)}V^{(s)} + w^{(r)}W^{(s)}, \text{ or, say, } S\,.\,u^{(r)}U^{(s)}.$$

The relations $$S\,.\,u^{(r)}\,U^{(s)}=S\,.\,U^{(s)}\,u^{(r)}$$
are, or ought to be, identical equalities. Thus,

$$S\,.\,uU=e^{\int p\,dx}\,.\,\delta=1=e^{\int P\,dx}\,.\,\Delta=S\,.\,Uu.$$

25. Take another identity, say

$$S\,.\,u''U'=S\,.\,U'u''.$$

Here $$S\,.\,u''U'=r,$$

and $$S\,.\,U'u''=\Pi S\,.\,U'u'$$

$$+Re^{\int P\,dx}\left\{U''(W'V-V'W)+V''(U'W-W'U)+W''(V'U-U'V)\right\};$$

but $S\,.\,U'u'=0=S\,.\,u'U'$ and $S\,.\,U'u''=-R=r$,

as it ought.

Therefore, further,

$$S\,.\,u''U''=\pi r \quad\text{and}\quad S\,.\,U''u''=\Pi R=(-\pi)(-r)=\pi r.$$

IV. *On Characteristics of Terordinals.*

26. When in (i.), we put

$$e^{\frac{1}{3}\int p\,dx}\,y=z,$$

we get $$z'''+gz'+hz=0;$$

wherein $$g=q-\tfrac{1}{3}p^2-p',$$

and $$h=r-\tfrac{1}{3}pq+\tfrac{2}{27}p^3-\tfrac{1}{3}p''.$$

27. So, when in (iii.) we put

$$e^{\frac{1}{3}\int P\,dx}\,Y=Z,$$

we get $$Z'''+GZ'+HZ=0;$$

wherein $$G=Q-\tfrac{1}{3}P^2-P',$$

and $$H=R-\tfrac{1}{3}PQ+\tfrac{2}{27}P^3-\tfrac{1}{3}P''.$$

28. Suppose that $\dfrac{Y}{y}=$ some function, say X, of x, and that what-

ever (value of) Y be taken a value of y can be found satisfying $Y = Xy$. Then the equations in z and Z will be in fact the same equation, and both $g = G$ and $h = H$ will be identically true. Substituting for P, Q, and R, we get from $g = G$ the condition

$$q - \tfrac{1}{3}p^2 - p' = q + \left(p + \frac{r'}{r}\right)\frac{r'}{r} - \left(p + \frac{r'}{r}\right)' - \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)^2 + \left(p + 2\frac{r'}{r}\right)',$$

which reduces to $$\left(p + \frac{r'}{r}\right)' - \tfrac{1}{3}\left(p + \frac{r'}{r}\right)\frac{r'}{r} = 0 \quad \text{..............(I.)},$$

whence $$p = kr^{\frac{1}{3}} - \frac{r'}{r} \quad \text{.........................(II.)},$$

wherein k is an arbitrary constant introduced by integration.

29. From $h = H$, we get

$$r - \tfrac{1}{3}pq + \tfrac{2}{27}p^3 - \tfrac{1}{3}p'' = -r + \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)\left\{q + \left(p + \frac{r'}{r}\right)\frac{r'}{r} - \left(p + \frac{r'}{r}\right)'\right\}$$
$$- \tfrac{2}{27}\left(p + 2\frac{r'}{r}\right)^3 + \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)'',$$

or, remembering (I.), and transposing,

$$2r = \tfrac{2}{3}\left(p + \frac{r'}{r}\right)q + \tfrac{2}{9}\left(p + 2\frac{r'}{r}\right)\left(p + \frac{r'}{r}\right)\frac{r'}{r}$$
$$- \tfrac{2}{27}\left\{p^3 + \left(p + 2\frac{r'}{r}\right)^3\right\} + \tfrac{2}{3}\left(p + \frac{r'}{r}\right).$$

30. But

$$p^3 + \left(p + 2\frac{r'}{r}\right)^3 = 2\left(p + \frac{r'}{r}\right)\left\{\left(p + \frac{r'}{r}\right)^2 + 3\left(\frac{r}{r}\right)^2\right\},$$

and, by (I.),

$$\left(p + \frac{r'}{r}\right)'' = \tfrac{1}{9}\left(p + \frac{r'}{r}\right)\left(\frac{r'}{r}\right)^2 + \tfrac{1}{3}\left(p + \frac{r'}{r}\right)\left(\frac{r'}{r}\right)'.$$

31. Replacing $p + \dfrac{r'}{r}$ by its value, $kr^{\frac{1}{3}}$, obtained from (II.), we shall find

$$q = \left(\frac{2k^2}{9} + \frac{3}{k}\right)r^{\frac{2}{3}} - \frac{k}{3}r^{-\frac{2}{3}}r' + \tfrac{2}{9}\left(\frac{r'}{r}\right)^2 - \tfrac{1}{3}\left(\frac{r'}{r}\right) \quad \text{......(III.)}.$$

32. And, since

$$Q=q+\left(p+\frac{r'}{r}\right)\frac{r'}{r}-\left(p+\frac{r'}{r}\right)=q+kr^{\frac{1}{3}}\frac{r'}{r}-(kr^{\frac{1}{3}})'=q+\tfrac{2}{3}kr^{-\frac{2}{3}}r',$$

we have

$$P=-kr^{\frac{1}{3}}-\frac{r'}{r},$$

and

$$Q=\left(\frac{2k^2}{9}+\frac{3}{k}\right)r^{\frac{2}{3}}+\frac{k}{3}r^{-\frac{2}{3}}r'+\tfrac{2}{9}\left(\frac{r'}{r}\right)^2-\tfrac{1}{3}\left(\frac{r'}{r}\right)',$$

so that P, Q, and R are respectively derived from p, q, and r by simply changing the sign of r. The only term in q affected by the change is that containing r'.

33. We might (see Art. 8) anticipate that, with the values of p and q given by (II.) and (III.), we should be able to get a first integral of (i.), as in fact we can. For $e^{\frac{2}{3}\int p\,dx}y$ is an integrating factor, and the integral is

$$yy''-\tfrac{1}{2}y'^2+\tfrac{1}{3}pyy'+\frac{3}{2k}r^{\frac{2}{3}}y^2=ce^{-\frac{2}{3}\int p\,dx},$$

wherein c is the arbitrary constant.

34. But, with these particular values of p and q, we can completely integrate (i.), which admits of the symbolical decompositions

$$\left(\frac{d}{dx}+\tfrac{1}{3}kr^{\frac{1}{3}}-\tfrac{2}{3}\frac{r'}{r}\right)\left\{\frac{d^2}{dx^2}+\left(\tfrac{2}{3}kr^{\frac{1}{3}}-\tfrac{1}{3}\frac{r'}{r}\right)\frac{d}{dx}+\frac{3}{k}r^{\frac{2}{3}}\right\}y=0,$$

and

$$\left(\frac{d}{dx}+\tfrac{1}{3}kr^{\frac{1}{3}}-\tfrac{2}{3}\frac{r'}{r}\right)\left\{\frac{d}{dx}+(\tfrac{1}{3}k\pm K)r^{\frac{1}{3}}-\tfrac{1}{3}\frac{r'}{r}\right\}\left\{\frac{d}{dx}+(\tfrac{1}{3}k\mp K)r^{\frac{1}{3}}\right\}y=0,$$

where

$$K=\left(\tfrac{1}{9}k^2-\frac{3}{k}\right)^{\frac{1}{2}};$$

and which is transformed into an equation with constant coefficients by changing the independent variable from x to t if t be determined from

$$Ct=\int r^{\frac{1}{3}}dx,$$

C being any constant.

35. Thus the simplest form of characteristic happens to indicate a completely integrable equation. But it also verifies the form of the

mixed equation. For z and Z are identical, and

$$X = \frac{Y}{y} = e^{\frac{1}{3}\int (p-P)\,dx} = e^{\frac{1}{3}\int \pi\, dx}.$$

Changing signs in the mixed integral of Art. 16, and substituting therein $e^{\frac{1}{3}\int \pi\, dx} y$ for Y, we get, after developments and reductions,

$$yy'' - \tfrac{1}{2}y'^2 + \left(\tfrac{1}{3}\pi - \frac{\pi'}{\pi}\right) yy' + \tfrac{3}{2}\,\frac{r}{\pi}\, y^2 = -\tfrac{3}{2}\,\frac{cr}{\pi}\, e^{-\frac{1}{3}\int \pi\, dx};$$

but $\pi = kr^{\frac{1}{3}}$, and this integral and that of Art. 33 agree.

36. In verifying this integral by differentiation, it will be noticed that

$$q = \tfrac{2}{9}p^2 + \tfrac{1}{3}p' + \frac{3}{k} r^{\frac{2}{3}} \text{ and } r = \frac{1}{k}\left(r^{-\frac{1}{3}} r' + p r^{\frac{2}{3}}\right),$$

and also that the mixed integral gives the integrating factor.

V. *On Transformation to Riccatian Forms.*

37. Used in connection with the processes of Boole by D, I mean, of course, $x\dfrac{d}{dx}$. Let

$$D(D-2B)\,y - 2\lambda\,(D-2A)\,x^2 y = 0 \quad \ldots\ldots\ldots\ldots\ldots [1],$$

where, for the present purpose, A and B denote given constants. Change the independent variable from x to t $(= x^2)$, and then write x instead of t. We get

$$D(D-B)\,y - \lambda\,(D-A)\,xy = 0.$$

38. Put $y = e^x z$; then

$$Dy = e^x\,(Dz + xz) \text{ and } D^2 y = e^x\left\{D^2 z + 2D(xz) - xz + x^2 z\right\},$$

for $$xDz = D(xz) - xz.$$

Hence $D(D-B)\,z + (2-\lambda)\,D(xz) + (\lambda A - B - 1)\,xz + x^2 z = 0$;

and, if $\lambda = 2$ and $2A - B - 1 = 0$, then

$$D(D-B)\,z + x^2 z = 0,$$

which is equivalent to

$$\frac{d^2 z}{dx^2} - \frac{B-1}{x}\,\frac{dz}{dx} + z = 0,$$

which last may be transformed into

$$\frac{d^2z}{dx^2}+x^{2q-2}z=0 \text{ ....[2]},$$

as appears from Mr. J. W. L. Glaisher's standard paper (*Phil. Trans.*, 1881).

39. By Boole's method we can change A into $A+i$ and B into $B+j$, where i and j are any integers. So that we may enlarge

$$2A-B-1=0 \text{ into } 2(A+i)-(B+j)-1=0.$$

But if $2A$ be an uneven integer, say $2k+1$, then $B+j\,(=2k+2i)$ is an even integer, and $q\left(=\frac{1}{B+j}\right)$ is the reciprocal of an even integer.

40. The case hereinafter discussed is that in which $2A=1$ and $B=1$, and for which $2A-B-1$ is not satisfied, but which can be transformed into cases of

$$D\{D-2(B+j)\}\,y-2\lambda\{D-2(A+i)\}\,x^2y=0,$$

wherein the condition is satisfied. Here we transform to

$$D\{D-2(1+j)\}\,y-2\lambda\{D-(1+2i)\}\,x^2y=0,$$

with the condition $j=2i-1$; viz., to

$$D(D-4i)\,y-2\lambda\{D-(1+2i)\}\,x^2y=0,$$

say to

$$D(D-2b)-2\lambda(D-2a)\,x^2y=0.$$

Here $b=2i$, $2a=2i+1$, and $2a-b-1=0$, so that the requisite condition is satisfied. The only exceptional case is when $i=0$.

VI. *On a Particular Terordinal.*

41. Take the terordinal

$$y'''+\left(\frac{1}{x}+x\right)y''-\left(\frac{1}{x^2}+2\right)y'+\frac{2}{x}\,y=0 \text{ ....(i.)},$$

or which $p+\frac{r'}{r}=\pi=x$, $\pi\frac{r'}{r}=-1$ and $\pi'=1$.

42. The deformation of (i.) is therefore

$$Y'''+\left(\frac{1}{x}-x\right)Y''-\left(\frac{1}{x^2}+4\right)Y'-\frac{2}{x}\,Y=0 \text{ ....(iii.)}.$$

43. Form a correlate by changing x into $t\sqrt{-1}$, and then replacing t by x. The result is

$$\dot{y}''' + \left(\frac{1}{x} - x\right)\dot{y}'' - \left(\frac{1}{x^2} - 2\right)\dot{y}' - \frac{2}{x}\dot{y} = 0 \text{ ....(ii.)},$$

for which $\dot{p} + \frac{\dot{r}}{r} = \dot{\pi} = -x$, $\dot{\pi}\frac{\dot{r}'}{r} = 1$, and $\dot{\pi} = -1$.

44. The deformation of (ii.) is therefore

$$\dot{Y}''' + \left(\frac{1}{x} + x\right)\dot{Y}'' - \left(\frac{1}{x^2} - 4\right)\dot{Y}' + \frac{2}{x}\dot{Y} = 0 \text{ ....(iv.)}.$$

45. Expressed in the notation of Boole, we should have the given

equation $D^2(D-2)\,y + (D-3)(D-4)\,x^2y = 0$,

its deformation $D^2(D-2)\,Y - D(D-1)\,x^2Y = 0$,

a correlate $D^2(D-2)\,\dot{y} - (D-3)(D-4)\,x^2\dot{y} = 0$,

and its deformation $D^2(D-2)\,\dot{Y} + D(D-1)\,x^2\dot{Y} = 0$.

46. Multiplied into $(D-1)(D-2)$, the given equation becomes, after reductions,

$$\{D(D-2) + (D-1)\,x^2\}\,[D]^3\,y = 0,$$

wherein the $[D]^3$ is factorial.

47. Integrate the deformation of the correlate once, and make the arbitrary constant zero; then

$$\{D(D-2) + (D-1)\,x^2\}\,(\dot{\lambda}\dot{Y}) = 0,$$

wherein $\dot{\lambda}$ is an arbitrary constant multiplier.

48. Comparing the last two equations, we get

$$\dot{\lambda}\dot{Y} = [D]^3\,y = x^3y'''.$$

49. So, from the correlate and the deformation of the given equation, we get

$$\lambda Y = [D]^3\,\dot{y} = x^3\dot{y}'''.$$

50. The last two relations hold when general values of y and $\dot{y}$ are

inserted; for $\dot{y} = x^2$ and $y = x^2$ are respectively solutions of the given equation and of the correlate; and $[D]^3 x^2 = 0$.

51. The three equations

$$D(D-2)\,Y-(D-1)\,x^2Y=0,\quad x^2Y''-(x+x^3)\,Y'-x^2Y=0,$$

$$Y''-\left(\frac{1}{x}+x\right)Y''-Y=0,$$

are in substance the same. So are the corresponding three

$$D(D-2)\,\dot{Y}+(D-1)\,x^2\dot{Y}=0,\quad x^2\dot{Y}''-(x-x^3)\,\dot{Y}'+x^2\dot{Y}=0,$$

$$\dot{Y}''-\left(\frac{1}{x}-x\right)\dot{Y}+\dot{Y}=0.$$

52. Taking u, v and w as independent particular solutions of (i.), and making $w = x^2$, we get, from the formulæ of § III.,

$$U = e^{\int p\,dx}(w'v''-v'w'') = 2e^{\int p\,dx}(xv''-v') = 2e^{\int p\,dx}(xv'-2v)$$

$$V = e^{\int p\,dx}(u'w''-w'u'') = 2e^{\int p\,dx}(u'-xu'') = -2e^{\int p\,dx}(xu'-2u)',$$

$$x^2\,(=w) = e^{\int P\,dx}(V'U''-U'V''),$$

which (see Art. 51)

$$= e^{\int P\,dx}\left[V'\left\{\left(\frac{1}{x}+x\right)U'+U\right\}-U'\left\{\left(\frac{1}{x}+x\right)V'+V\right\}\right]$$

$$= e^{\int P\,dx}(UV'-VU').$$

53. Differentiating, we get

$$U' = pU+2e^{\int p\,dx}(xv''') = pU+\frac{2\dot{\lambda}}{x^2}\dot{V}\,.\,e^{\int p\,dx},$$

$$V' = pV-2e^{\int p\,dx}(xu''') = pV-\frac{2\dot{\lambda}}{x^2}\dot{U}\,.\,e^{\int p\,dx},$$

therefore $$UV'-VU' = -\frac{2\dot{\lambda}}{x^2}(\dot{U}U+\dot{V}V)\,e^{\int p\,dx},$$

and $$x^2 = -\frac{2\dot{\lambda}}{x^2}e^{\int(p+P)\,dx}(\dot{U}U+\dot{V}V).$$

54. Noticing that $\dot{p} = P$, $\dot{P} = p$, and that we may (being always able to employ arbitrary multipliers) put $\dot{w} = x^2 = w$, we shall, by

processes corresponding precisely with those of Arts. 52 and 53, find

$$x^2 = +\frac{2\lambda}{x^2} e^{\int (P+p)\,dx} (U\dot{U}+V\dot{V}),$$

the + sign being prefixed to this result because the last line of Art. 51 differs from the second. Hence, reducing, and equating the values of x^2,

$$-2\dot{\lambda}\,(\dot{U}U+\dot{V}V) = x^2 = 2\lambda\,(U\dot{U}+V\dot{V}),$$

whence $$\dot{\lambda}+\lambda = 0.$$

55. The process indicated in Art. 4 might have been used to obtain the integrals given in Art. 51. But its application is not necessary, for Boole's notation renders the existence of the integrals clear.

56. For brevity, put

$$xu'-2u = \theta, \quad xv'-2v = \vartheta,$$

$$x\dot{u}'-2\dot{u} = \dot{\theta}, \quad x\dot{v}'-2\dot{v} = \dot{\vartheta};$$

then $$U = 2e^{\int p\,dx}\,\vartheta', \quad V = -2e^{\int p\,dx}\,\theta',$$

$$\dot{U} = 2e^{\int P\,dx}\,\dot{\vartheta}' \text{ and } \dot{V} = -2e^{\int P\,dx}\,\dot{\theta}'.$$

57. Hence

$$\dot{U}U+\dot{V}V = 4e^{\int (p+P)\,dx}\,(\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta') = 4mx^2\,(\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta'),$$

and $$\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta' = (x\dot{u}''-\dot{u}')(xu''-u')+(x\dot{v}''-\dot{v}')(xv''-v')$$

$$= x^2\,(\dot{u}''u''+\dot{v}''v'')-x\,(\dot{u}''u'+\dot{v}''v')$$

$$-x\,(\dot{u}'u''+\dot{v}'v'')+\dot{u}'u'+\dot{v}'v'$$

$$= x^2I-xG-xH+D, \text{ (say } = \Theta_3),$$

when expressed by the notation used for Synthemes; therefore

$$x^2I-xG-xH+D = \Theta_3 = \frac{-1}{8m\dot{\lambda}} = \frac{1}{8m\lambda} = M,$$

where M is an arbitrary constant meaning the same thing as Θ_3.

58. By this mode of proceeding we, in short, obtain the groups of

four relations,

$$\dot{\theta}\theta + \dot{\vartheta}\vartheta = x^2D-2xB-2xC+4A, \quad \text{say} = \Theta_0,$$
$$\dot{\theta}'\theta + \dot{\vartheta}'\vartheta = x^2G-xD-2xE+2B, \quad = \Theta_1,$$
$$\dot{\theta}\theta' + \dot{\vartheta}\vartheta' = x^2H-xD-2xF+2C, \quad = \Theta_2,$$
$$\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta' = x^2I-xG-xH+D, \quad = \Theta_3.$$

59. We also have

$$\dot{\lambda}\dot{U} = x^3u''' = -x^2(1+x^2)u''+x(1+2x^2)u'-2x^2u$$
$$= -x(1+x^2)(xu''-u')+x^2(xu'-2u) = -x(1+x^2)\theta'+x^2\theta,$$

and, in short, the groups of four relations,

$$\dot{\lambda}\dot{U} = -x(1+x^2)\theta'+x^2\theta = 2\dot{\lambda}e^{\int P\,dx}\dot{\vartheta}' \quad \ldots\ldots\ldots\ldots\{1\},$$
$$\dot{\lambda}\dot{V} = -x(1+x^2)\vartheta'+x^2\vartheta = -2\dot{\lambda}e^{\int P\,dx}\dot{\theta}' \quad \ldots\ldots\ldots\{2\},$$
$$\lambda U = -x(1-x^2)\dot{\theta}'-x^2\dot{\theta} = 2\lambda e^{\int p\,dx}\vartheta' \quad \ldots\ldots\ldots\ldots\{3\},$$
$$\lambda V = -x(1-x^2)\dot{\vartheta}'-x^2\dot{\vartheta} = -2\lambda e^{\int p\,dx}\theta' \ldots\ldots\ldots\ldots\{4\}.$$

60. Hence

$$\lambda\dot{\lambda}(U\dot{U}+V\dot{V}) = x^2(1-x^4)(\theta'\dot{\theta}'+\vartheta'\dot{\vartheta}')+x^3(1+x^2)(\theta'\dot{\theta}+\vartheta'\dot{\vartheta})$$
$$-x^3(1-x^2)(\theta\dot{\theta}'+\vartheta\dot{\vartheta}')-x^4(\theta\dot{\theta}+\vartheta\dot{\vartheta})$$
$$= 4\lambda\dot{\lambda}e^{\int(p+P)dx}(\theta'\dot{\theta}'+\vartheta'\dot{\vartheta}') = 4m\lambda\dot{\lambda}x^2(\theta'\dot{\theta}'+\vartheta'\dot{\vartheta}');$$

whence, in the notation of Synthemes,

$$(1-4m\lambda\dot{\lambda}-x^4)\,\Theta_3+x(1+x^2)\,\Theta_2-x(1-x^2)\,\Theta_1-x^2\Theta_0 = 0.$$

61. The value of $U\dot{U}+V\dot{V}$ just indicated ought, and will be shown, to agree, in the results which it gives, with that of Art. 57.

62. Replacing, in Art. 59, $e^{\int p\,dx}$ by $kxe^{\frac{1}{2}x^2}$ and $e^{\int P\,dx}$ by $\dot{k}xe^{-\frac{1}{2}x^2}$, an x will divide out from the second and third columns. Form the four equations,

$$2k\lambda e^{\frac{1}{2}x^2}\{1\}+(1+x^2)\{4\}, \qquad -2k\lambda e^{\frac{1}{2}x^2}\{2\}+(1+x^2)\{3\},$$
$$2\dot{k}\dot{\lambda}e^{-\frac{1}{2}x^2}\{4\}-(1-x^2)\{1\}, \qquad -2\dot{k}\dot{\lambda}e^{-\frac{1}{2}x^2}\{3\}-(1-x^2)\{2\},$$

from which θ', ϑ', $\dot{\vartheta}'$ and $\dot{\theta}'$ are, respectively, absent, and we get a system of four equations,

$$(x^4-1-4\dot{k}\dot{\lambda}k\lambda)\,\dot{\vartheta}'-x\,(1+x^2)\,\dot{\vartheta} = -2k\lambda e^{\frac{1}{2}x^2}x\theta \;\ldots\ldots\ldots\ldots [1],$$

$$(x^4-1-4\dot{k}\dot{\lambda}k\lambda)\,\dot{\theta}'-x\,(1+x^2)\,\dot{\theta} = \;\;2k\lambda e^{\frac{1}{2}x^2}x\vartheta \;\ldots\ldots\ldots\ldots [2],$$

$$(x^4-1-4\dot{k}\dot{\lambda}k\lambda)\,\theta'+x\,(1-x^2)\,\theta = -2\dot{k}\dot{\lambda}e^{-\frac{1}{2}x^2}x\dot{\vartheta} \;\ldots\ldots\ldots [3],$$

$$(x^4-1-4\dot{k}\dot{\lambda}k\lambda)\,\vartheta'+x\,(1-x^2)\,\vartheta = \;\;2\dot{k}\dot{\lambda}e^{-\frac{1}{2}x^2}x\dot{\theta} \;\ldots\ldots\ldots [4].$$

63. Put $x^4-1-4\dot{k}\dot{\lambda}k\lambda = X$, then $\vartheta\,[1]+\theta\,[2]$ gives

$$x\,(\theta\dot{\theta}'+\vartheta\dot{\vartheta}')-x\,(1+x^2)(\theta\dot{\theta}+\vartheta\dot{\vartheta}) = x\Theta_1-x\,(1+x^2)\,\Theta_0 = 0 \;\ldots [5],$$

and in like manner $\dot{\theta}\,[3]+\dot{\vartheta}\,[4]$ gives

$$x\,(\theta'\dot{\theta}+\vartheta'\dot{\vartheta})+x\,(1-x^2)(\theta\dot{\theta}+\vartheta\dot{\vartheta}) = x\Theta_2+x\,(1-x^2)\,\Theta_0 = 0 \;\ldots [6].$$

64. And from $\dot{\theta}'\,\{1\}+\dot{\vartheta}'\,\{2\}$, see Art. 59, we get

$$-(1+x^2)(\theta'\dot{\theta}'+\vartheta'\dot{\vartheta}')+x\,(\theta\dot{\theta}'+\vartheta\dot{\vartheta}') = -(1+x^2)\,\Theta_3+x\Theta_1 = 0 \ldots [7],$$

and from $\theta'\,\{3\}+\vartheta'\,\{4\}$ we find

$$-(1-x^2)(\theta'\dot{\theta}'+\vartheta'\dot{\vartheta}')-x^2\,(\theta'\dot{\theta}+\vartheta'\dot{\vartheta}) = -(1-x^2)\,\Theta_3-x\Theta_2 = 0 \ldots [8].$$

65. Now, recurring to Art. 60, and therein replacing m by $\dot{k}k$, we have

$$4\dot{k}\dot{\lambda}k\lambda\Theta_3 = (1-x^4)\,\Theta_3+x\,(1+x^2)\,\Theta_2-x\,(1-x^2)\,\Theta_1-x^2\Theta_0$$
$$= x\,(1+x^2)\,\Theta_2-x^2\Theta_0 = -(1-x^4)\,\Theta_3-x^2\Theta_0,$$

by [7] and [8],

therefore $\qquad (x^4-1-4\dot{k}\dot{\lambda}k\lambda)\,\Theta_3-x^2\Theta_0 = X\Theta_3-x^2\Theta_0 = 0,$

and $$\Theta_0 = \frac{X}{x^2}\,\Theta_3, \quad \Theta_1 = \left(x+\frac{1}{x}\right)\Theta_3,$$

and $$\Theta_2 = \left(x-\frac{1}{x}\right)\Theta_3.$$

66. Differentiate the first of the equations of Art. 58, and eliminate A', B', C', and D' by means of (*a*), (*b*), (*c*), and (*d*) of Art. 9. Then we see that, identically,

$$\Theta_0' = \Theta_1+\Theta_2;$$

moreover, by Art. 65, we have

$$\Theta_1+\Theta_2=2x\Theta_3;$$

hence $$\Theta_0'=2x\Theta_3,$$

and (see Art. 65) $$\frac{\Theta_0'}{\Theta_0}=\frac{2x^3}{X}=\frac{2x^3}{x^4-1-4\dot{k}\dot{\lambda}k\lambda};$$

and $$\Theta_0=C\sqrt{X}.$$

67. Equating the values of Θ_0, in Arts. 65 and 66,

$$\frac{X}{x^2}\Theta_3=C\sqrt{X},$$

$$\Theta_3^2X=\Theta_3^2(x^4-1-4\dot{k}\dot{\lambda}k\lambda)=C^2x^4;$$

and $$C^2=\Theta_3^2,$$

and $$1+4\dot{k}\dot{\lambda}k\lambda=0=1+4m\dot{\lambda}\lambda.$$

68. Hence the system of equations of Art. 58 may now be written

$$x^2D-2x(B+C)+4A=\Theta_0=Mx^2,$$

$$x^2G-xD-2xE+2B=\Theta_1=M\left(x+\frac{1}{x}\right),$$

$$x^2H-xD-2xF+2C=\Theta_2=M\left(x-\frac{1}{x}\right),$$

$$x^2I-xG-xH+D=\Theta_3=M=\tfrac{1}{2}\lambda=-\tfrac{1}{2}\dot{\lambda};$$

where M and Θ_3 mean the same expression, but M is used to remind us that the Θ's have a double character, one as compounds of $A, B, \ldots I$, and the other as functions of M (an arbitrary constant) and x, and that in each they must be shown to fulfil the requisite conditions.

69. It is now evident that the two different forms of $\dot{U}U+\dot{V}V$ give coincident results; for, on making

$$4m\lambda\dot{\lambda}\,(=4\dot{k}\dot{\lambda}k\lambda)=-1,$$

the last equation of Art. 60 becomes, on replacing the Θ's by their respective values,

$$M\left\{(2-x^4)+x^4-1-(1-x^4)-x^4\right\}=0, \quad \text{an identity.}$$

As another verification, write the first equation of Art. 53 in the form

$$U' = \left(\frac{1}{x}+x\right) U + 2k\dot{\lambda}\,\frac{e^{\frac{1}{2}x^2}}{x}\,V;$$

then, differentiating, and eliminating U'' by means of Art. 51, U' also disappears, and we get

$$U = 2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{x\dot{V}'+(x^2-1)\,\dot{V}\right\};$$

differentiate again, and also, by Art. 51, eliminate $\dot{V}''$, we get

$$U' = 2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{(x^2+1)\,\dot{V}'+x^3\dot{V}\right\};$$

and, equating these values of U',

$$\left(\frac{1}{x}+x\right) U+2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{-(x^2+1)\,\dot{V}'+\left(\frac{1}{x}-x^3\right)\dot{V}\right\};$$

whence, multiplying into x and dividing out $1+x^2$, we get the former value of U.

70. That the conditions imposed on the Θ's are fulfilled when they are expressed as functions of M and x, it is not difficult to see. But they must be fulfilled when the Θ's are expressed in terms of $A, B, \ldots I$, or of some of those symbols. Eliminating accented letters from $(x^2D)'-\{2x\,(B+C)\}'+4A'$, the identity $\Theta_1+\Theta_2=\Theta_0'$ is obtained immediately. But that Θ_3' is a constant, is shown in a different manner. Thus,

$$\Theta_3' = (x^2I)'-\{x\,(G+H)\}'+D',$$

and

$$(x^2I)' = x^2I'+2xI$$

$$= x^2\left\{-\frac{2}{x}I+\left(\frac{1}{x^2}-2\right)H+\frac{2}{x}F+\left(\frac{1}{x^2}+2\right)G-\frac{2}{x}E\right\}+2xI$$

$$-(xG)' = -xG'-G$$

$$= -x\left\{I-\left(\frac{1}{x}-x\right)G+\left(\frac{1}{x^2}-2\right)D+\frac{2}{x}C\right\}-G$$

$$-(xH)' = -xH'-H$$

$$= -x\left\{I-\left(\frac{1}{x}+x\right)H+\left(\frac{1}{x^2}+2\right)D-\frac{2}{x}B\right\}-H,$$

$$D' = G+H,$$

and therefore

$$\Theta_3' = -2xI + (2+x^2)\,G + (2-x^2)\,H - 2xE + 2xF - \frac{2}{x}D - 2C + 2B$$

$$= -\frac{2}{x}\Theta_3 + \Theta_1 - \Theta_2 = M\left(-\frac{2}{x} + x + \frac{1}{x} + \frac{1}{x} - x\right) = 0,$$

and Θ_3 is constant.

71. That $\Theta_1' + \Theta_2' = \Theta_0'' = 2M$ will now be shown to hold. We have

$$\Theta_2' = (x^2H)' - (xD)' - (2xF)' + (2C)',$$

and $$(x^2H)' = \{x\,(xH)\}' = x\,(xH)' + xH.$$

But $(xH)'$ is known from Art. 70, and

$$(x^2H)' = x^2\left\{I - \left(\frac{1}{x} + x\right)H + \left(\frac{1}{x^2} + 2\right)D - \frac{2}{x}B\right\} + 2xH$$

$$-(xD)' = -x\,(G+H) - D,$$

$$-(2xF)' = -2x\left\{H - \left(\frac{1}{x} + x\right)F + \left(\frac{1}{x^2} + 2\right)C - \frac{2}{x}A\right\} - 2F,$$

$$2C' = 2D + 2F,$$

therefore

$$\Theta_2' = x^2I - x\,(2+x^2)\,H - xG + 2\,(1+x^2)\,D + 2\,(1+x^2)\,F - \frac{2}{x}(1+2x^2)\,C - 2xB + 4A$$

$$= \Theta_3 + (1+x^2)\left(-xH + D + 2F - \frac{2}{x}C\right) + x^2D - 2xB - 2xC + 4A$$

$$= \Theta_3 - \left(\frac{1}{x} + x\right)\Theta_2 + \Theta_0;$$

also, by calculation,

$$\Theta_1' = \Theta_3 + \left(x - \frac{1}{x}\right)\Theta_1 - \Theta_0;$$

hence $$\Theta_0'' = \Theta_1' + \Theta_2' = 2\Theta_3 - \left(\frac{1}{x} + x\right)\Theta_2 - \left(\frac{1}{x} - x\right)\Theta_1$$

$$= M\left\{2 - \left(x^2 - \frac{1}{x^2}\right) - \left(\frac{1}{x^2} - x^2\right)\right\} = 2M,$$

as it ought.

72. It follows that the four equations of Art. 68 are not independent, but are equivalent to two independent conditions only. Hence, eliminating M, we find that

$$x^2\Theta_3 - x(\Theta_1+\Theta_2)+\Theta_0 = 0,$$

or $$x^4I-2x^3(G+H)+4x^2D+2x^2(E+F)-4x(B+C)+4A = 0\ldots(\text{A}).$$

VII. *Conclusion.*

73. The seven equations (a, b, c, d, 1, 2, 3) of Art. 9, and the two (i., ii.) of Arts. 11, 41, 43, together with (A) just arrived at, constitute a system of ten independent equations, homogeneous in eleven quantities and sufficient to determine all that is sought or possible, viz., ratios. In this paper I shall not attempt the determination. But I shall make some remarks intended to facilitate it.

74. Differentiate (4) of Art. 10, eliminating $\dot{u}'''$ and u''', which are particular values of $\dot{y}'''$ and y'''. After considerable reductions, I get

$$(B-\dot{p}A)\,G+(C-pA)\,H+(pB-D)\,F+(\dot{p}C-D)\,E$$

$$=(B-\dot{p}A)\,\dot{u}''u'+(C-pA)\,\dot{u}'u''+(\dot{p}C-D)\,\dot{u}''u+(pB-D)\,\dot{u}u''$$

$$-(E+F)\,\dot{u}'u'+(G+pF)\,\dot{u}'u+(H+\dot{p}E)\,\dot{u}u'-(pH+\dot{p}G)\ldots(5).$$

75. Such relations as (5), if troublesome to attain, are, when attained, easily verified. They are all of the form

$$\dot{Æ} = \Sigma\,.\,J_{r,s}\,\dot{u}^{(r)}u^{(s)},$$

and since, in Art. 9, it is indifferent whether we eliminate $\dot{u}$ and u, or $\dot{v}$ and v, we have

$$Æ = \Sigma\,.\,J_{r,s}\,\dot{v}^{(r)}v^{(s)}.$$

Hence, adding and adopting the notation of Synthemes, we get

$$2Æ = \Sigma\,.\,J_{r,s}\,(r,\,s),$$

which is an identity. The equations (1, 2, 3, 4, 5), although constructed without reference to this identity, all satisfy it.

76. The mixed integral (which, forgetting that it was the same for the dotted as for the undotted system, I calculated independently for each) must, for our present purpose, be expressed in terms of $\dot{y}$ and y. This is most simply done by first eliminating $\dot{Y}''$ (or Y''), and then substituting for $\dot{Y}'$ and $\dot{Y}$ (or Y' and Y). It may be written in the

form

$$x^6\dot{y}''y''+(x^3-2x^4)\,\dot{y}''y'-(x^2+2x^4)\,\dot{y}'y''+2\,(x^3-x)\,\dot{y}''y+2\,(x^3+x)\,\dot{y}y''$$
$$+4x^3\dot{y}'y'+2\,(1-2x^2)\,\dot{y}'y-2\,(1+2x^2)\,\dot{y}y'+4x\dot{y}y=\frac{c}{x}=-\frac{\dot{c}}{x};$$

from which differentiation will elicit no additional result, because its sinister multiplied into x and then differentiated vanishes identically, as it ought.

77. This mixed integral also vanishes identically when $\dot{y}=x^2$ and when $y=x^2$. And if we deal with it by the process of Art. 75, replacing $\dot{y}''y''$ and $\dot{y}''y'$, &c., by the Synthemes (2, 2) and (2, 1), &c., and reducing, we get

$$x^3\Theta_3-x^2\Theta_3-x^2\Theta_1+x\Theta_0+\dot{\Theta}_2-\Theta_1=\frac{2c}{x}=-\frac{2\dot{c}}{x},$$

which, by (A) and Art. 68, is

$$-\frac{2M}{x}=\frac{2c}{x}=-\frac{2\dot{c}}{x};$$

and $$c=-\dot{c}=M=\Theta_3.$$

I shall write the mixed integral in the forms

$$M=x^6\dot{y}''y''+x^3\,(1-2x^2)\,\dot{y}''y'+\ldots+4x\dot{y}y=\Sigma\,.\,K_{r,s}\,\dot{u}^{(r)}u^{(s)}\ldots,$$

interchanging y and u without further remark.

78. And now arises a process by which any attempt to form directly an algebraical equation in any of the ratios, say $\frac{u'}{u}$, will perhaps be superseded, and which gives a new theorem in the inverse method of differential resolvents.

79. The six equations (1, 2, ... 6) will give five distinct equations of the form

$$M\Sigma\,.\,J_{r,s}\,\dot{u}^{(r)}u^{(s)}=Æ\Sigma\,.\,K_{r,s}\,\dot{u}^{(r)}u^{(s)},$$

which, divided by $\dot{u}u$, will give five non-homogeneous equations in

$$A,\ B,\ \ldots\ I,\ \dot{u}_1,\ \dot{u}_2,\ u_1,\ u_2,$$

where $$\dot{u}_1,\ \dot{u}_2,\ u_1,\ u_2=\frac{\dot{u}'}{\dot{u}},\ \frac{\dot{u}''}{\dot{u}},\ \frac{u'}{u},\ \frac{u''}{u}\ \text{respectively.}$$

Eliminate $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2, and we get a result of the form

$$f_0(A, B, \ldots I, M, x) = 0.$$

80. Next eliminate I, G, H and D by means of the formulæ in Art. 68. The result will be, say,

$$f_1(A, B, C, E, F, M, x) = 0,$$

which will be changed into, say,

$$f_2(\xi_1, \psi_1, \xi_2, \psi_2, A, M, x) = 0$$

by the assumptions

$$\xi_1 = \frac{1}{x} B - \frac{2}{x^2} A, \quad \psi_1 = x^4 E - x^3 B,$$

$$\xi_2 = \frac{1}{x} C - \frac{2}{x^2} A, \quad \psi_2 = x^4 F - x^3 C.$$

81. But, by differentiating, and substituting for $A', B', \ldots F'$, we get

$$\xi_1' = \frac{1}{x^6}(\psi_1 + Mx^4), \quad \psi_1' = \left(\frac{4}{x} + x\right)\psi_1 - x^6\xi_1 + M(x^3 + x),$$

$$\xi_2' = \frac{1}{x^6}(\psi_2 + Mx^4), \quad \psi_2' = \left(\frac{4}{x} - x\right)\psi_2 + x^6\xi_2 + M(x^3 - x).$$

82. These results enable us to calculate $f_2', f_2'', \ldots$ with comparative ease; but they also lead to a new conclusion. For, by one more differentiation of the last group, we get

$$\xi_1'' + \left(\frac{1}{x} - x\right)\xi_1' + \xi_1 = M\left(\frac{1}{x^4} + \frac{1}{x^2} - 1\right),$$

$$\xi_2'' + \left(\frac{1}{x} + x\right)\xi_2' - \xi_2 = M\left(-\frac{1}{x^4} + \frac{1}{x^2} + 1\right),$$

$$\psi_1'' - \left(\frac{9}{x} + x\right)\psi_1' + \left(\frac{24}{x^2} + 5\right)\psi_1 = -M(4 + 2x^2 + x^4),$$

$$\psi_2'' - \left(\frac{9}{x} - x\right)\psi_2' + \left(\frac{24}{x^2} - 5\right)\psi_2 = \quad M(4 - 2x^2 + x^4),$$

the one system being the correlate of the other; and the four biordinals are, respectively, the Differential Resolvents of four algebraical equations of the respective forms

$$\phi_1(\xi_1, M, x) = 0, \quad \phi_2(\psi_1, M, x) = 0, \quad \phi_3(\xi_2, M, x) = 0,$$

and

$$\phi_4(\psi_2, M, x) = 0.$$

83. The connection of the results of this paper with the Riccatian

$$z'' + x^{1/i-2} z = 0,$$

where i is any integer, is made in § V. The theory of Differential Resolvents only makes its appearance collaterally, otherwise many remarks would arise upon the above forms, which indeed determine those of the roots of the algebraical equations.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics. By R. A. Roberts.

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1. In a paper recently published in the *Proceedings of the London Mathematical Society* (Vol. xvi., p. 242), I noticed the possibility of the inscription of a doubly infinite number of polygons in a quadric such that the sides touch two confocal quadrics, provided these three quadrics are connected by a certain pair of relations depending on the number of sides of the polygon. This result was arrived at by making use of Liouville's form of the differential equations of the system of lines touching two confocal quadrics (*Journal de Mathématiques*, t. xii., p. 418), and was seen to depend upon the finding of the $2n^{\text{th}}$ parts of the complete values of the hyperelliptic integrals $L(x)$, $M(x)$.

2. The object of the present paper is to show how to obtain the actual relations connecting the quadrics corresponding to a given number of sides of the polygon. For this purpose I employ, for the sake of symmetry, Professor Cayley's form of the equation of a system of confocal quadrics, namely,

$$\frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$