On a Group of Triangles Inscribed in a Given Triangle ABC, whose sides are Parallel to Connectors of any Point P with A, B, C. *By* R. Tucker, M.A. Received September, 1892. Read November 10th, 1892. Revised February, 1893.

1. Let DEF be one of the in-triangles, having its sides DE, EF FD respectively parallel to BP, CP, AP, and suppose

$$BD = pa, \quad CD = qa, \quad p+q \equiv 1.$$

If the trilinear coordinates of P are λ, μ, ν, the equations to DF, DE are

$$\begin{vmatrix} \alpha & \beta & \gamma \\ 0 & qc & pb \\ \begin{vmatrix} \gamma & -\mu \\ b & c \end{vmatrix} & \begin{vmatrix} -\mu & 0 \\ c & a \end{vmatrix} & \begin{vmatrix} 0 & \nu \\ a & b \end{vmatrix} \end{vmatrix} = 0,$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ 0 & qc & pb \\ \begin{vmatrix} 0 & -\lambda \\ b & c \end{vmatrix} & \begin{vmatrix} -\lambda & \nu \\ c & a \end{vmatrix} & \begin{vmatrix} \nu & 0 \\ a & b \end{vmatrix} \end{vmatrix} = 0;$$

i.e.,
$$a\alpha\,(bp\mu - cq\nu) + pb\beta u - qc\gamma u = 0,$$
$$\alpha\,(c\nu + pa\lambda) + pb\beta\lambda - qc\gamma\lambda = 0.$$

For shortness, write

$$u = b\mu + c\nu, \quad v = c\nu + a\lambda, \quad w = a\lambda + b\mu,$$
$$u+v+w = 2\Sigma_1, \quad a\lambda u + b\mu v + c\nu w = 2\Sigma_2.$$

From the above equations, after reduction, we obtain the equation to EF to be

$$-a\alpha\,(c\nu + pa\lambda)(cq\nu - bp\mu) + pb\beta u\,(c\nu + pa\lambda) + acq\lambda\gamma\,(cq\nu - bp\mu) = 0,$$

and this is parallel to $\quad a\mu - \beta\lambda = 0,$

i.e., CP; therefore

$$\frac{p}{ca\nu\lambda} = \frac{q}{b\mu v} = \frac{1}{\Sigma_2}.$$

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Writing for p, q their values in the above equations, we get the equations to DE, EF, FD to be

$$\left.\begin{array}{r} \nu v w a + a b \lambda^2 \nu \beta - b \lambda \mu v \gamma = 0 \\ -c \mu \nu w a + \lambda w u \beta + b c \mu^2 \lambda \gamma = 0 \\ c a \nu^2 \mu a - a \nu \lambda u \beta + \mu u v \gamma = 0 \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots (\text{i.}).$$

2. In a similar manner we find the equations to $D'E'$, $E'F'$, $F'D'$, the sides of the other in-triangle, to be

$$\left.\begin{array}{r} a b \mu^2 \nu a + \nu w u \beta - a \lambda \mu u \gamma = 0 \\ -b \mu \nu v a + b c \nu^2 \lambda \beta + \lambda u v \gamma = 0 \\ \mu v w a - c \nu \lambda w \beta + c a \lambda^2 \mu \gamma = 0 \end{array}\right\} \dots\dots\dots\dots\dots\dots (\text{ii.}).$$

3. From (i.), (ii.), we can write the coordinates of the angular points

$$\left\|\begin{array}{llll} D, & 0, & \mu v, & a\nu\lambda \\ E, & b\lambda\mu, & 0, & \nu w \\ F, & \lambda u, & c\mu\nu, & 0 \end{array}\right\| ; \quad \left|\begin{array}{llll} D', & 0, & a\lambda\mu, & \nu w \\ E', & \lambda u, & 0, & b\mu\nu \\ F', & c\nu\lambda, & \mu v, & 0 \end{array}\right|, \ (\text{modulus } \Sigma_2) \ \dots (\text{iii.}).$$

4. If L, M, N, are the orthogonal projections of P on BC, CA, AB, we have

$$\left.\begin{array}{llll} L, & 0, & \mu + \lambda \cos C, & \lambda \cos B + \nu \\ M, & \mu \cos C + \lambda, & 0, & \nu + \mu \cos A \\ N, & \lambda + \nu \cos B, & \nu \cos A + \mu, & 0 \end{array}\right|, \ (\text{modulus } \Sigma_1) \ \dots (\text{iv.}).$$

The equation to the circle LMN is

$$2\Delta\, \Sigma\,(a\mu\nu)\, \Sigma\,(a\beta\gamma) = \Sigma\,(a a)\, \Sigma \left\{\lambda b c\,.\,\nu + \mu \cos A\,.\,\mu + \nu \cos A\,.\,a\right\} \dots (\text{v.}).$$

5. From (iii.), we obtain

$$DD' = BD' \sim BD = a\,(c\nu w - c a \nu \lambda)/\Sigma_2 = a b c \mu \nu/\Sigma_2 = FE' \dots\dots (\text{vi.}),$$

$$FF' = D'E = a b c \lambda \mu/\Sigma_2,$$

and $DE' = abc\left\{\Sigma\,(\lambda^2\mu^2) + 2\lambda\mu\nu\,(\lambda \cos A + \mu \cos B - \nu \cos C)\right\}^{\frac{1}{2}}/\Sigma_2$.

Hence the perimeter of the hexagon

$$DD'EE'FF' = 2abc\, \Sigma\,(\lambda\mu)/\Sigma_2.$$

When P is the in-centre, this hexagon is equilateral, each side $= abc/\Sigma\,(ab)$. *Cf. Educational Times*, October, 1892, Quest. 11706.

6. From (i.) and (ii.), we see that DE, $D'F'$ intersect in

$$\frac{\beta}{\mu} = \frac{\gamma}{\nu} = \frac{avw}{bc\lambda\mu\nu}, \quad i.e., \text{ on } AP \quad \ldots\ldots\ldots\ldots\text{(vii.)},$$

and EF, $E'F'$ in a_1 (say), given by

$$\frac{b\beta}{v} = \frac{c\gamma}{w} = \frac{bc\mu\nu a}{\lambda\,(b^2\mu^2 + c^2\nu^2 + \Sigma_2)};$$

hence Aa_1, Bb_1, Cc_1, where b_1, c_1, are analogous points to a_1, cointersect in P', *i.e.*,

$$a\alpha/u = b\beta/v = c\gamma/w \quad \ldots\ldots\ldots\ldots\text{(viii.)}.$$

If P is the orthocentre,	P' is the circumcentre;	
,, circumcentre,	,, nine-point centre;	
,, symmedian point,	,, isotomic of the inverse of the centre of the Brocard ellipse;	

and if P' is the symmedian point, then P is the point

$$a^2\alpha \sec A = b^2\beta \sec B = c^2\gamma \sec C.$$

If P is the centroid, P' evidently coincides with P. In the case of P being one of the cosine-orthocentres (see *Messenger of Mathematics*, No. 199, p. 100), P' is on $\sigma_1 G$, or on $\sigma_2 G$.

7. The equation to PP' is

$$a\alpha\,(b\mu - c\nu) + \ldots + \ldots = 0 \ldots\ldots\ldots\ldots\text{(ix.)},$$

which evidently passes through the centroid (G), and, further, G divides PP' so that

$$PG = 2P'G.$$

8. The equations to BE', CF are

$$\alpha/\lambda u = \gamma/b\mu\nu, \quad \alpha/\lambda u = \beta/c\nu\mu;$$

hence they intersect on the median through A ....(x.).

9. Again, the equations to BE, CF' are

$$\alpha/b\mu\lambda = \gamma/\nu w, \quad \alpha/c\nu\lambda = \beta/\mu v;$$

hence, if these intersect in a_2, CF, AD' in b_2, AD, BE' in c_2, then Aa_2, Bb_2, Cc_2 cointersect in P_1, given by

$$\alpha/a\lambda^2 u = \beta/b\mu^2 v = \gamma/c\nu^2 w \quad \ldots\ldots\ldots\ldots\text{(xi.)},$$

which is a point on PP'.

10. If a', b', c' are the lengths of the sides of LMN, we have

$$\left.\begin{aligned} a'^2 &= \mu^2+\nu^2+2\mu\nu\cos A, \\ b'^2 &= \nu^2+\lambda^2+2\nu\lambda\cos B, \\ c'^2 &= \lambda^2+\mu^2+2\lambda\mu\cos C. \end{aligned}\right\}$$

Now $$BD = ca^2\lambda\nu/\Sigma_2, \quad BF = ca\lambda u/\Sigma_2;$$

hence $$DF^2 = a^2b^2c^2\lambda^2(\mu^2+\nu^2+2\mu\nu\cos A)/\Sigma_2^2,$$

and therefore $$\left.\begin{aligned} DF &= abc\lambda a'/\Sigma_2 = D'E' \\ DE &= abc\mu b'/\Sigma_2 = E'F' \\ EF &= abc\nu c'/\Sigma_2 = D'F' \end{aligned}\right\} \text{......................(xii.).}$$

The $$\text{triangle } DCE = \Delta \,.\, ab^2\lambda\mu^2 v/\Sigma_2^2;$$

hence

$$\text{triangle } DEF = \Delta\left\{1-\frac{a\lambda(b\mu)^2 v+b\mu(c\nu)^2 w+c\nu(a\lambda)^2 u}{\Sigma_2^2}\right\}$$

$$= \Delta abc\lambda\mu\nu\Sigma_1/\Sigma_2^2 = \Delta D'E'F' \text{......................(xiii.).}$$

11. The equations to DE', $D'F$ respectively are

$$\left.\begin{aligned} b\mu^2\nu v\alpha+a\nu\lambda^2 u\beta-\lambda\mu u v\gamma &= 0 \\ c\nu^2\mu w\alpha-\nu\lambda w u\beta+a\lambda^2\mu u\gamma &= 0 \end{aligned}\right\} \text{..................(xiv.);}$$

these intersect in P''; *i.e.*,

$$\alpha/\lambda u = \beta/\mu v = \gamma/\nu w, \quad \text{(modulus } 2\Sigma_2).$$

This point is obviously the centre of perspective of the pair of triangles, and also the centre of the conic through their six vertices.

If P is the circumcentre, P'' is the point $a\cos A\cos(B-C)$, ..., ...;

,,	orthocentre,	,,	symmedian point;
,,	symmedian point,	,,	centre of the Brocard ellipse;
,,	centroid,	,,	centroid;
,,	in-centre,	,,	point $b+c$, $c+a$, $a+b$.

12. The equations to PP'' and $P'P''$ respectively are

$$\left.\begin{aligned} \mu\nu\alpha(b\mu-c\nu)+\ldots+\ldots &= 0 \\ \text{and} \quad a\alpha v w(b\mu-c\nu)+\ldots+\ldots &= 0 \end{aligned}\right\}, \text{ .... (xv.).}$$

13. If g, g' are the centroids of DEF, $D'E'F'$, their coordinates are determined by

$$\left.\begin{array}{ll} & \lambda(b\mu+u),\quad \mu(c\nu+v),\quad \nu(a\lambda+w) \\ \text{and} & \lambda(c\nu+u),\quad \mu(a\lambda+v),\quad \nu(b\mu+w) \end{array}\right|,\ (\text{modulus } 3\Sigma_2)\ldots(\text{xvi.});$$

hence P'' is the mid-point of gg'; it is also the centroid of the hexagon, and is further the centre of the in-ellipse, touching at the points where AP, BP, CP meet the sides.

14. The conic about the hexagon has for its equation

$$\Sigma\left(\frac{a^2}{\lambda^2 u}\right) = \frac{1}{abc\lambda\mu\nu}\,\Sigma\left\{1+\frac{a^2\lambda^2}{vw}\right\} bc\beta\gamma\ldots\ldots\ldots\ldots(\text{xvii.}).$$

This is a circle when P is the orthocentre, and P' therefore the symmedian point; in fact, the in-triangles are then the "cosine"-triangles, and the circle the "cosine"-circle. We hence obtain the equation to this circle under the form

$$\Sigma\,(bc\cos A\,.\,a^2) = \Sigma\,(bc+a^2\cos B\cos C)\,\beta\gamma.$$

15. The D-symmedian line of DEF cuts EF in

$$\lambda\mu^2 ub'^2+\lambda^3\mu a'^2,\quad c\mu^3\nu b'^2,\quad \nu\lambda^2 wa'^2;$$

hence its equation is

$$\nu a\left[ca\mu^2\nu b'^2-\lambda vwa'^2\right]-a\nu\lambda\beta\,(\mu ub'^2+b\lambda^2 a'^2)+\mu\gamma v\,(\mu ub'^2+b\lambda^2 a'^2)=0;$$

and hence to that through E is

$$\nu aw\,(\nu vc'^2+c\mu^2 b'^2)+\lambda\beta\,(ab\nu^2\lambda c'^2-\mu wub'^2)-b\lambda\mu\gamma\,(\nu vc'^2+c\mu^2 b'^2)=0.$$

From these two equations we get the symmedian point of DEF to be given by

$$\left.\begin{array}{l} \dfrac{\nu a}{\mu ub'^2+b\lambda^2 a'^2}=\dfrac{\lambda\beta}{\nu vc'^2+c\mu^2 b'^2}=\dfrac{\mu\gamma}{\lambda wa'^2+a\nu^2 c'^2} \\ \text{and of } D'E'F' \text{ by} \\ \dfrac{\mu a}{\nu uc'^2+c\lambda^2 a'^2}=\ldots=\ldots \end{array}\right\}\ldots\ldots\ldots(\text{xviii.}).$$

16. To find the orthocentre of DEF, we note that the perpendicular from D on EF is also perpendicular to CP; therefore its equation is

$$(\nu\lambda w-\mu^2\cos B v+\lambda^2\mu a\cos A)\,a+(\mu+\lambda\cos C)\,a\nu\lambda\beta$$
$$-(\mu+\lambda\cos C)\,\mu v\gamma=0;$$

and to the perpendicular from E on FD is

$$-(\nu+\mu\cos A)\,\nu w\alpha+(\lambda\mu u-\nu^2\cos Cw+\nu b\mu^2\cos B)\,\beta$$
$$+(\nu+\mu\cos A)\,b\lambda\mu\gamma=0,$$

whence, after reductions, the coordinates of the orthocentre (H_1) are

$$\left.\begin{array}{l}\alpha\equiv(\mu+\lambda\cos C)\left[\lambda\mu+\nu\lambda\cos A+\mu\nu\cos B-\nu^2\cos C\right]\\ \beta\equiv(\nu+\mu\cos A)\left[\mu\nu-\lambda^2\cos A+\lambda\mu\cos B+\nu\lambda\cos C\right]\\ \gamma\equiv(\lambda+\nu\cos B)\left[\nu\lambda+\lambda\mu\cos A-\mu^2\cos B+\mu\nu\cos C\right]\end{array}\right|,$$

$$(\text{modulus } 4R^2/uvw)\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(xix.)}.$$

In like manner, the coordinates of orthocentre (H_2) of $D'E'F'$ are

$$\left.\begin{array}{l}\alpha\equiv(\nu+\lambda\cos B)\left[\nu\lambda+\lambda\mu\cos A+\mu\nu\cos C-\mu^2\cos B\right]\\ \beta\equiv\ldots\\ \gamma\equiv\ldots\end{array}\right\}\ldots\text{(xx.)},$$

with the same modulus as before. From these values we find P'' to be the mid-point of H_1H_2.

17. The perpendicular from P'' on EF (multiplied by EF)

$$=abc\lambda\mu\nu\Delta w/\Sigma_2^2,$$

whence we obtain (xiii.) in another way.

18. If p_1, p_2, p_3; p_1', p_2', p_3' are the perpendiculars from P on EF, FD, DE, and $E'F'$, $F'D'$, $D'E'$ respectively, we have

$$p_1p_2p_3=\frac{b\lambda\mu^2\Sigma_1}{c'\Sigma_2}\times\frac{c\mu\nu^2\Sigma_1}{a'\Sigma_2}\times\frac{a\nu\lambda^2\Sigma_1}{b'\Sigma_2}=p_1'p_2'p_3'.$$

19. The equation to the circle DEF is

$$\Sigma_1.\Sigma_2^2.\Sigma(\alpha\beta\gamma)=\Sigma(a\alpha)\,\Sigma\left\{\mu\nu cw\,(a\lambda b^2\nu-a^2\nu v+cuv)\,\alpha\right\}$$

and to $D'E'F'$ is

$$\Sigma_1.\Sigma_2^2.\Sigma(\alpha\beta\gamma)=\Sigma(a\alpha)\,\Sigma\left\{\mu\nu bv\,(a\lambda c^2\mu-a^2\mu w+buw)\,\alpha\right\}\quad\ldots\text{(xxi.)}.$$

20. If e', f, are the mid-points of $D'F'$, DE respectively, they are given by

$$c\nu\lambda,\ \mu(a\lambda+v),\ \nu w;\quad b\mu\lambda,\ \mu\nu,\ \nu(a\lambda+w);$$

whence Be', Cf intersect in

$$\frac{\alpha}{bc\lambda} = \frac{\beta}{cv} = \frac{\gamma}{bw}, \text{ i.e., on } AP' \text{ ....(xxii.).}$$

21. The circles CDE, AEF, BDF have for their equations

$$\left.\begin{aligned} \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(b\nu w\alpha + a^2\lambda\nu\beta)/\Sigma_2, & (a) \\ \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(c\lambda u\beta + b^2\mu\lambda\gamma)/\Sigma_2, & (b) \\ \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(a\mu v\gamma + c^2\nu\mu\alpha)/\Sigma_2, & (c) \end{aligned}\right\} \text{.........(xxiii.).}$$

The radical axes of (a) and (b), and of (b) and (c) are

$$b\nu w\alpha + \lambda(a^2\nu - cu)\beta - b^2\lambda\mu\gamma = 0,$$

$$-c^2\mu\nu\alpha + c\lambda u\beta + \mu(b^2\lambda - av)\gamma = 0\,;$$

hence the radical centre of the three circles is

$$b\nu\alpha / \left[2ab\lambda\nu\cos A + b\mu v + \nu^2(c^2 - a^2)\right] = \ldots = \ldots.$$

Whence, if P is the orthocentre, the radical centre is the negative Brocard point, and if P is this Brocard point, then the circles pass through the circumcentre.

In like manner, the circle $CD'E'$ has its equation

$$\Sigma(a\beta\gamma) = \Sigma(a\alpha)(b^2\mu\nu\alpha + a\nu w\beta)/\Sigma_2,$$

and the radical centre of the three analogous circles is

$$c\mu\alpha / \left[2ca\lambda\mu\cos A + c\nu w - \mu^2(a^2 - b^2)\right] = \ldots = \ldots.$$

Whence, if P is the orthocentre, then the radical centre is the positive Brocard point, and if P is this point, then the three circles pass through the circumcentre.

22. The radical axis of CDE, $BD'F'$ is

$$\lambda\alpha = \mu\beta\,;$$

therefore the radical axes of these and the analogous circles meet in

$$\alpha\lambda = \beta\mu = \gamma\nu,$$

i.e., in the inverse of P.

23. Since the sides of DEF are parallel to AP, BP, CP, we have

$$\left.\begin{aligned} \cot D &= (\mu\nu - \lambda^2\cos A + \lambda\mu\cos B + \nu\lambda\cos C)/\lambda D \\ \cot E &= (\nu\lambda + \lambda\mu\cos A - \mu^2\cos B + \mu\nu\cos C)/\mu D \\ \cot F &= (\lambda\mu + \nu\lambda\cos A + \mu\nu\cos B - \nu^2\cos C)/\nu D \end{aligned}\right\} \text{...(xxiv.),}$$

where $$D \equiv \lambda \sin A + \mu \sin B + \nu \sin C;$$

hence $\cot$ (Brocard angle) $= \Sigma(\lambda^2\mu^2) + \lambda\mu\nu\Sigma(\lambda\cos A)/\lambda\mu\nu D$.

The same result, of course, holds for $D'E'F'$. Compare (xxiv.) with (xix.).

Again, $$DE.DF : AP.BP = 2\Delta ab\lambda\mu\Sigma_1 : \Sigma_2^2;$$

therefore $DE.EF.FD : AP.BP.CP = 2\Delta abc\lambda\mu\nu\Sigma_1\sqrt{2\Delta\Sigma_1} : \Sigma_2^3$.

If ρ is the circum-radius of $DEF\ (\Delta_1)$, then, since

$$DE.EF.FD = 4\rho\Delta_1,$$

we have $$AP.BP.CP = \rho\Sigma_2/\Delta, \quad \text{by (xiii.)}.$$

[24. The parallels to AB, BC, CA through D, E, F respectively, are

$$\left.\begin{aligned} a^2\nu\lambda\alpha + ab\nu\lambda\beta - b\mu\nu\gamma &= 0 \\ -c\nu w\alpha + b^2\lambda\mu\beta + bc\lambda\mu\gamma &= 0 \\ ca\mu\nu\alpha - a\lambda u\beta + c^2\mu\nu\gamma &= 0 \end{aligned}\right\} \text{..................(xxv.)}.$$

These cointersect in Q, $(b\lambda\mu,\ c\mu\nu,\ a\nu\lambda)$ ....(xxvi.).

In like manner, the parallel through D' to CA is

$$a^2\lambda\mu\alpha - c\nu w\beta + ca\lambda\mu\gamma = 0;$$

and this and the analogous lines for E', F' intersect in Q',

$(c\nu\lambda,\ a\lambda\mu,\ b\mu\nu)$ ....(xxvii.).

QQ' passes through P'', and is bisected by it.

25. The equations to DF', ED' are

$$\left.\begin{aligned} a\mu v\alpha - ca\nu\lambda\beta + c\mu v\gamma &= 0 \\ a\nu w\alpha + b\nu w\beta - ab\lambda\mu\gamma &= 0 \end{aligned}\right\} \text{.............. (xxviii.)},$$

which intersect in $\beta/\mu v = \gamma/\nu w$, *i.e.*, on AP''.

26. We collect a few results of interest.

DE, $D'E'$ intersect on CP'; DF, $E'F'$ on CP;

CF, BE' „ AG; CF', BE on AP_1 (§ 9);

DE, $D'F'$ „ AP.

CF', FP intersect DE in W so that

$$DW : WE = EE' : EC,$$

and CP' meets AB in L', so that

$$AL' : BL' = u : v.$$

Again, FP'' cuts DE in R, $DR : RE = u : w$;

CP cuts FD in R', $DR' : R'F = v : b\mu$;

BP' cuts FD in R'', $DR'' : R''F = c\nu : w$;

BP cuts DF in R_1, $DR_1 : R_1F = a\lambda : u$;

and BE cuts DF in R_2, $DR_2 : R_2F = a\lambda b\mu : wu$.

27. If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ are (for the moment) the coordinates of P, P', P'', G, then

$$\frac{\alpha_1\alpha_2}{\alpha_3\alpha_4} = \frac{\beta_1\beta_2}{\beta_3\beta_4} = \frac{\gamma_1\gamma_2}{\gamma_3\gamma_4};$$

hence, if (1, 2) are inverse points, so also are (3, 4).

28. If P_2 is the isotomic conjugate of P, *i.e.* $(\mu\nu/a^2, \nu\lambda/b^2, \lambda\mu/c^2)$, then P_2P is parallel to $P'P''$, and $P''G = \frac{1}{3}P''P_2$.

29. $$EQ : FQ : DQ = \mu\nu : \nu\lambda : \lambda\mu = F'Q' : D'Q' : E'Q';$$

hence, if P is the in-centre of ABC, Q, Q' are the circumcentres of DEF, $D'E'F'$ respectively.

30. Lines through A, B, C parallel to BP, CP, AP are given by

$$\left.\begin{aligned} b\nu\beta + v\gamma &= 0 \\ c\lambda\gamma + w\alpha &= 0 \\ a\mu\alpha + u\beta &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxix.)};$$

these intersect in a', b', c', given by

$$(bc\lambda\nu,\ vw,\ -b\nu w),\quad (-c\lambda u,\ ca\mu\lambda,\ wu),\quad (uv,\ -a\mu v,\ ab\mu\nu),$$

and P' is the centre of perspective of ABC, $a'b'c'$. Similarly the parallel through A to CP is given by

$$w\beta + c\mu\gamma = 0,$$

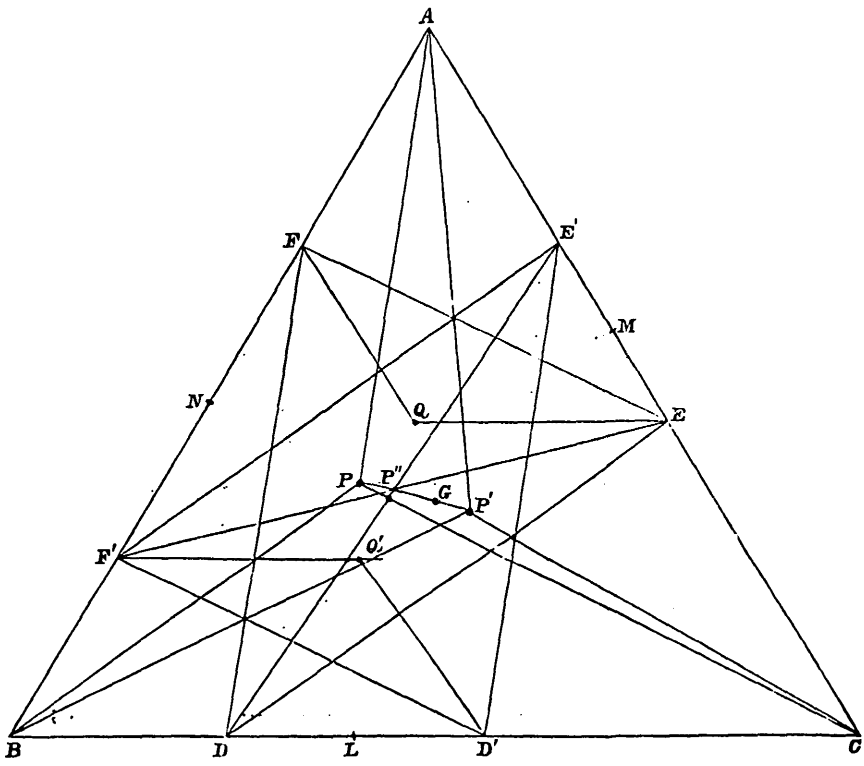
and the three corresponding lines intersect in a'', b'', c'', *i.e.*

$$(wu,\ ca\mu\nu,\ -a\nu w),\quad (-b\lambda u,\ uv,\ ab\nu\lambda),\quad (bc\lambda\mu,\ -c\mu v,\ vw).$$

31. The circle ADD' has for its equation

$$\Sigma(\alpha\beta\gamma) = \Sigma(a\alpha)\, a^2\lambda\mu\nu\left(\frac{cvw}{\mu}\beta + \frac{b\mu v}{\nu}\gamma\right)\Big/\Sigma_2^2\,]$$

32. We now proceed to examine a few cases of envelopes of some of the prominent lines in the figure for different loci of P.



Let P lie on the line $\quad p\lambda + q\mu + r\nu = 0$ ....(A).

From (i.), the equation to DE is

$$\nu v w \alpha + ab\lambda^2\nu\beta - b\lambda\mu v\gamma = 0.$$

Eliminating μ between this equation and (A), we have

$$\nu^3\left[-bcr\alpha\right] + \nu^2\lambda\left[(caq - bcp - abr)\,\alpha + bcr\gamma\right]$$
$$+\nu\lambda^2\left[(a^2q - abp)\,\alpha + qab\beta + (bcp + abr)\,\gamma\right] + \lambda^3(abp\gamma) = 0.$$

Comparing this with the cubic in Salmon's *Higher Plane Curves* (1873, p. 66), and accentuating the letters in the text-book, we have:

$$a' = -bcr\alpha,$$
$$3b' = (caq - bcp - abr)\,\alpha + bcr\gamma,$$
$$3c' = (a^2q - abp)\,\alpha + qab\beta + (bcp + abr)\,\gamma,$$
$$d' = abp\gamma.$$

Then the equation of the envelope of DE is found by putting the above values in

$$a'^2d'^2 + 4a'c'^3 + 4b'^3d' - 6a'b'c'd' - 3b'^2c'^2 = 0.$$

This shows that the envelope is in general a quartic curve.

33. If the locus of P is a straight line passing through an angular point, we have three cases to consider.

(1) $r = 0$: the envelope reduces to

$$4b'd' = 3c'^2,$$

i.e. $$4abcp\,(aq - bp)\,\alpha\gamma = \left[a\,(aq - bp)\,\alpha + qab\beta + bcp\gamma\right]^2.$$

This is a parabola touching AB and BC, and having

$$a\,(aq - bp)\,\alpha + abq\beta + bcp\gamma = 0$$

for the chord of contact.

(2) $p = 0$: the envelope is $4a'c' = 3b'^2$,

i.e. $$4\,(-abcr\alpha)(aq\alpha + bq\beta + br\gamma) = \left[(caq - abr)\,\alpha + bcr\gamma\right]^2,$$

i.e. the parabola

$$\left[a\alpha\,(cq + br) + bcr\gamma\right]^2 + 4ab^2cqr\alpha\beta = 0.$$

(3) $q = 0$: in this case

$$a' = -bcr\alpha,$$
$$3b' = -b\,(cp + ar)\,\alpha + bcr\gamma,$$
$$3c' = -abp\alpha + b\,(cp + ar)\,\gamma,$$
$$d' = abp\gamma.$$

34. Let P lie on the conic

$$p\mu\nu + q\nu\lambda + r\lambda\mu = 0 \quad \text{.........................} \quad (B).$$

Taking the same line (DE) as before, we have to eliminate μ (say) between (i.) and (B). The result is

$$\nu^2\left[c\,(ap-bq)\,a\right]+\nu\lambda\left[(car+a^2p-abq)\,a+abp\beta+bcq\gamma\right]$$
$$+\lambda^2\left[a^2\nu a+abr\beta+abq\gamma\right]=0.$$

The envelope therefore is

$$\left[aa\,(ap-bq+cr)+abp\beta+bcq\gamma\right]^2=4caa\,(ap-bq)(ara+br\beta+bq\gamma),$$

i.e., if $\lambda'=ap-bq-cr,$

$$(aa\lambda'-abp\beta-bcq\gamma)^2+4aba\beta\,(ap-bq\,.\,ap-cr)=0.$$

Hence $a=0,\ \beta=0$ are tangents to the conic, and

$$ca\lambda'-abp\beta-bcq\gamma=0$$

the chord of contact.

35. If $ap=bq=cr$, and the conic (B) therefore the minimum circum-ellipse, the lines DE, &c. become the line at infinity.

36. If (B) is the circumcircle, then the envelope is the parabola

$$4c^2a^2\cos^2 Aa^2+a^4\beta^2+b^2c^2\gamma^2+4abc^2\cos A\gamma a+2a^2bc\beta\gamma$$
$$+4ca\,(bc-a^2\cos A)\,a\beta=0.$$

This equation can be written under the form

$$(2ac\cos Aa-a^2\beta+bc\gamma)^2+4abc\beta\,(a\gamma+ca)=0;$$

hence $\beta=0,\quad a\gamma+ca=0$

are tangents, and $2ac\cos Aa-a^2\beta+bc\gamma=0$

represents the chord of contact.

[37. The equation to DE' is

$$b\mu^2\nu va+a\lambda^2\nu u\beta-\lambda\mu uv\gamma=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

If the locus of P is $p\lambda+q\mu+r\nu=0,$

eliminating ν between these two equations, we get

$$\mu^4\left[bcq^2a\right]+\mu^3\lambda\left[2bcpqa-c^2q^2\gamma-abqra+bcqr\gamma\right]$$
$$+\mu^2\lambda^2\left[bcp^2a+caq^2\beta-2pqc^2\gamma-r\,\{paba+qab\beta-(caq+bcp)\,\gamma\}-r^2ab\gamma\right]$$
$$+\mu\lambda^3\left[2capq\beta-c^2p^2\gamma-abrp\beta-capr\gamma\right]+\lambda^4\left[cap^2\beta\right]=0.$$

If we take $p=0$, or $q=0$, this equation reduces to a quadratic, and the envelope can be readily found.

38. The parabola through $DF'AC$ is

$$ca\nu\lambda\beta^2 = c\mu v\beta\gamma + a\mu v a\beta + 4ca\Sigma_2\gamma a/b^2,$$

the axis being parallel to the median through B.

39. If P moves in the straight line

$$pa + q\beta + r\gamma = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (L),$$

then P' moves in the parallel straight line

$$aa\,(-pbc + qca + rab) + \dots + \dots = 0.$$

Hence, if P moves along a side of the triangle ABC, P' moves along a parallel which bisects the perpendicular from the opposite angle on to that side. And, if P moves on the conic

$$p\beta\gamma + q\gamma a + ra\beta = 0,$$

then P' moves on the conic

$$\Sigma a^2 a^2\,(-pa + qb + rc) = 2abc\Sigma\,(p\beta\gamma) \quad \dots\dots\dots\dots (D).$$

If the primitive locus is the circumcircle, then (D) becomes

$$\Sigma\,(aa^2 \cos A) = \Sigma\,(a\beta\gamma),$$

i.e. the nine-point circle.

If it is the minimum circum-ellipse, then the locus of P' is the maximum in-ellipse.

40. If the locus of P is (L), then the loci of Q, Q' are respectively

$$br\beta\gamma + cp\gamma a + aqa\beta = 0$$

and

$$cq\beta\gamma + ar\gamma a + bpa\beta = 0.$$

The discussion of these and similar results we leave to the reader.]

Thursday, February 9th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

The following papers were read:—

The Harmonics of a Ring: Mr. W. D. Niven.

The Group of Thirty Cubes composed by Six differently coloured Squares: Major MacMahon.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 12; Band XVII., Stück 1.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VI., 4th Series.—"James Prescott Joule," by Osborne Reynolds.

"Proceedings of the Royal Society," Vol. LII., No. 317.

"Proceedings of the Royal Irish Academy," Vol. II., No. 3, 3rd Series; December, 1892.

Carruthers, G. T.—"The Cause of Gravity," 8vo; Inverness, 1892.

"Memoirs of the Mathematical Section of the Russian Society of Naturalists," Vol. XIV.; Odessa, 1892.

"The Nautical Almanac for 1896."

"Nyt Tidsskrift for Mathematik," A. 3^je Aargang, Nos. 7, 8; Copenhagen.

"Nyt Tidsskrift for Mathematik," B. 3^je Aargang, No. 4; Copenhagen.

"Bulletin of the New York Mathematical Society," Vol. II., No. 4.

"Bulletin des Sciences Mathématiques," 2^me Série, Tome XVI.; December, 1892.

"Transactions of the Canadian Institute," Vol. III., Part I., No. 5; Toronto, December, 1892.

"Rendiconti del Circolo Matematico di Palermo," Tomo VI., Fasc. 6; November-December, 1892.

"Atti della Reale Accademia dei Lincei — Rendiconti," Vol. I., Fasc. 12, 2° Semestre e Indice; Roma, 1892.

"Atti della Reale Accademia dei Lincei—Memorie," Vol. VI.; Roma, 1890.

"Educational Times," February, 1893.

"Annali di Matematica," Tomo XX., Fasc. 4; Milano, 1893.

"Journal für die reine und angewandte Mathematik," Band CXI., Heft 1; Berlin.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VI., Fasc. 7-12; Napoli, 1892.

"Indian Engineering," Vol. XII., Nos. 26, 27; Vol. XIII., Nos. 1, 2.

"Washington Naval Observations for 1888," Washington, 1892.

"On Coaxal Systems of Circles," by R. Lachlan, M.A. (extracted from *Quarterly Journal of Pure and Applied Mathematics*, No. 102, 1892). From the author.