ON THE EVALUATION OF CERTAIN DEFINITE INTEGRALS BY MEANS OF GAMMA FUNCTIONS

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THE following communication consists of two distinct parts. In Part I., I investigate a formula for the multiple integral

$$\int_{(n-1)} \frac{\xi_1^{\beta_1-1}\xi_2^{\beta_2-1}\dots\xi_n^{\beta_n-1}}{\prod\limits_{t=1}^{t=m}\left(\sum\limits_{s=1}^{s=n} A_{st}\,\xi_s\right)^{\gamma_t}}\,d\xi_1\,d\xi_2\dots d\xi_{n-1},$$

where $\xi_n = 1-\xi_1-\xi_2-\dots-\xi_{n-1}$, and in which β_s, γ_t, A_{st} are quantities which have their real parts positive, and in which, moreover, $\Sigma\beta-\Sigma\gamma$ is either zero or has its real part positive. This formula is modelled on Lejeune Dirichlet's formula

$$\int_{(n-1)} \xi_1^{\beta_1-1}\xi_2^{\beta_2-1}\dots\xi_n^{\beta_n-1}\,d\xi_1\dots d\xi_{n-1} = \frac{\prod\limits_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma(\Sigma\beta_s)},$$

both integrals being taken for all real positive values of $\xi_1, \xi_2, \dots, \xi_{n-1}$ which make ξ_n positive.

In Part II. I wish to find integrals which will take the place of the two given above in the cases when β_s and γ_t are not restricted to have their real parts positive. This second part is founded on Pochhammer's investigation* of a double circuit integral constructed to replace the Eulerian integral of the second kind $\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$, in cases when the real parts of l and m are not positive.

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* *Math. Ann.*, Vol. xxxv., p. 495, "Zur Theorie der Euler'schen Integrale."

I.

1. The multiple integral

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \left(\sum_{s=1}^{s=n} A_{st}\,\xi_s\right)^{-\gamma_t} d\xi_1 d\xi_2 \dots d\xi_{n-1},$$

in which $$\xi_n = 1-\xi_1-\xi_2-\dots-\xi_{n-1},$$

and $$\sum_{s=1}^{s=n} \beta_s = \sum_{t=1}^{t=m} \gamma_t,$$

and in which β_s, γ_t, A_{st} denote constants which have their real parts positive for all values of s and t, taken over a field of integration for which $\xi_1, \xi_2, \dots, \xi_n$ are all real and positive, is equal to

$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\prod_{t=1}^{t=m} \Gamma(\gamma_t)} \int_{(m-1)} \prod_{t=1}^{t=m} \theta_t^{\gamma_t-1} \prod_{s=1}^{s=n} \left(\sum_{t=1}^{t=m} A_{st}\,\theta_t\right)^{-\beta_s} d\theta_1 d\theta_2 \dots d\theta_{m-1},$$

in which $$\theta_m = 1-\theta_1-\theta_2-\dots-\theta_{m-1},$$

taken over a field of integration for which $\theta_1, \theta_2, \dots, \theta_m$ are all real and positive. In particular when $m = 1$,

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \left(\sum_{s=1}^{s=m} A_s \xi_s\right)^{-\Sigma\beta} d\xi_1 d\xi_2 \dots d\xi_{n-1} = \frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma\left(\sum_{s=1}^{s=n} \beta_s\right)} \prod_{s=1}^{n} A_s^{-\beta_s}.$$

2. Consider first the case when $m = 1$, and put $\gamma = \Sigma\beta_s$. In the integral

$$\int_{(n-1)} \prod_1^n \xi_s^{\beta_s-1} \left(\sum_1^n A_s \xi_s\right)^{-\gamma} d\xi_1 d\xi_2 \dots d\xi_{n-1},$$

make in the first place the transformation

$$\xi_s = \xi_n \eta_s \quad (s = 1, 2, \dots, n-1),$$

so that $$\xi_n = 1-\sum_1^{n-1} \xi_s = \left(1+\sum_1^{n-1} \eta_s\right)^{-1},$$

and $$\xi_s = \eta_s/(1+\Sigma\eta_s).$$

Then the limits of η_s are 0 and ∞, and the transformed integral is

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \prod_{s=1}^{s=n-1} \eta_s^{\beta_s-1} \left(A_n + \sum_{s=1}^{s=n} A_s \eta_s\right)^{-\gamma} d\eta_1 d\eta_2 \dots d\eta_{n-1},$$

the factor $(1+\Sigma\eta_s)^{\gamma-\Sigma\beta}$ disappearing, since $\gamma = \Sigma\beta$.

Now substitute for $\left(A_n+\sum_1^{n-1} A_s\eta_s\right)^{-\gamma}$ by the formula

$$\left(A_n+\sum_1^{n-1} A_s\eta_s\right)^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty \zeta^{\gamma-1} e^{-\zeta(A_n+\Sigma A_s\eta_s)}\,d\zeta.$$

Then we shall get

$$\frac{1}{\Gamma(\gamma)}\int_0^\infty \cdots \int_0^\infty \prod_1^{n-1} \eta_s^{\beta_s-1}\zeta^{\gamma-1} e^{-\zeta(A_n+\Sigma A_s\eta_s)}\,d\eta_1 \ldots d\eta_{n-1}\,d\zeta.$$

In this change the order of integration,* and integrate with respect to the η by the formula

$$\int_0^\infty \eta_s^{\beta_s-1} e^{-\eta_s A_s\zeta}\,d\eta_s = \Gamma(\beta_s)\, A_s^{-\beta_s}\zeta^{-\beta_s},$$

and we shall get
$$\frac{\prod_{s=1}^{s=n-1} \Gamma(\beta_s) A_s^{-\beta_s}}{\Gamma(\gamma)}\int_0^\infty \zeta^{\beta_n-1} e^{-\zeta A_n}\,d\zeta,$$

since
$$\gamma-\sum_{s=1}^{s=n-1}\beta_s = \beta_n.$$

That is, we have, finally,

$$\frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma\left(\sum_{s=1}^{s=n}\beta_s\right)}\prod_{s=1}^{s=n} A_s^{-\beta_s}$$

as the value of the integral

$$\int_{(n-1)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\left(\sum_{s=1}^{s=n} A_s\xi_s\right)^{-\Sigma\beta} d\xi_1\, d\xi_2 \ldots d\xi_{n-1}.$$

3. This result being established, let us apply it to the multiple integral

$$\int_{(n+m-2)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}\left(\sum_{s=1}^{s=n}\sum_{t=1}^{t=m} A_{st}\xi_s\theta_t\right)^{-\sigma} d\xi_1\ldots d\xi_{n-1}\,d\theta_1\ldots d\theta_{m-1}$$

where
$$\sigma = \sum_{s=1}^{s=n}\beta_s = \sum_{t=1}^{t=m}\gamma_t$$

and
$$\sum_{s=1}^{s=n}\xi_s = 1, \qquad \sum_{t=1}^{t=m}\theta_t = 1,$$

the field of integration being for all real positive values of

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_{n-1},\ \xi_n,\ \theta_1,\ \theta_2,\ \ldots,\ \theta_{m-1},\ \theta_m.$$

* For a justification of this see later, § 5.

Since A_{st} has its real part positive for all values of s and t, $\sum\limits_{s=1}^{s=n} A_{st}\xi_s$ and $\sum\limits_{t=1}^{t=m} A_{st}\theta_t$ also have their real parts positive, and so we get, integrating first with regard to the θ, and secondly with regard to the ξ, that

$$\prod_{t=1}^{t=m}\Gamma(\gamma_t)\int_{(n-1)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\prod_{t=1}^{t=m}\left(\sum_{s=1}^{s=n}A_{st}\xi_s\right)^{-\gamma_t}d\xi_1\,d\xi_2\ldots d\xi_{n-1}$$

is equal to

$$\prod_{s=1}^{s=n}\Gamma(\beta_s)\int_{(m-1)}\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}\prod_{s=1}^{s=n}\left(\sum_{t=1}^{t=m}A_{st}\theta_t\right)^{-\beta_s}d\theta_1\,d\theta_2\ldots d\theta_{m-1}.$$

4. Another formula of the same type is obtained by integrating

$$e^{-(\Sigma A_r x_r+\Sigma B_s y_s+\Sigma\Sigma C_{rs}x_r y_s)}\prod_{r=1}^{r=n}x_r^{\beta_r-1}\prod_{s=1}^{s=m}y_s^{\gamma_s-1}$$

first with regard to the x, and then with regard to the y, and equating the two results obtained. In this way we get the equality

$$\prod_{r=1}^{r=n}\Gamma(\beta_r)\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{\prod\limits_{s=1}^{s=m}e^{-B_s y_s}y_s^{\gamma_s-1}}{\prod\limits_{r=1}^{r=n}(A_r+\Sigma_s C_{rs}y_s)^{\beta_r}}\,dy_1\,dy_2\ldots dy_m$$

$$=\prod_{s=1}^{s=m}\Gamma(\gamma_s)\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{\prod\limits_{r=1}^{r=n}e^{-A_r x_r}x_r^{\beta_r-1}}{\prod\limits_{s=1}^{s=m}(B_s+\Sigma_r C_{rs}x_r)^{\gamma_s}}\,dx_1\,dx_2\ldots dx_n,$$

in which β_r, γ_s, A_r, B_s, C_{rs} are constants which have their real parts positive for all values of s and t, and the field of integration in both integrals is for all positive real values of the variables.

5. It is necessary to justify the inversion of the order of integration in the multiple integral with infinite limits.

Taking the integral just considered in § 4, I suppose the limits of x_r to be 0 to h_r and of y_s to be 0 to k_s, where h_r and k_s are real positive quantities; then it is sufficient to show that, as h_r and k_s are indefinitely increased, in any order and in any manner, the integral

$$\int_{h_1}^\infty\int_{h_2}^\infty\cdots\int_{h_n}^\infty\int_{k_1}^\infty\int_{k_2}^\infty\cdots\int_{k_m}^\infty e^{-V}\prod_r x_r^{\beta_r-1}\prod_s y_s^{\gamma_s-1}\,dx_1\ldots dx_n\,dy_1\ldots dy_m,$$

where $$V=\sum_r A_r x_r+\sum_s B_s y_s+\sum_r\sum_s C_{rs}x_r y_s,$$

can be made less than any assignable quantity.

In order to include the integral in § 2, zero values of A_r and B_s will not be excluded from consideration. This we may prove as follows:— Throughout the range of integration we have

$$x_r \geqslant h_r \quad \text{and} \quad y_s \geqslant k_s,$$

and therefore $$x_r y_s \geqslant \tfrac{1}{2}(h_r y_s + k_s x_r),$$

and so $$V \geqslant \sum_r x_r(A_r + \tfrac{1}{2}\sum_s C_{rs} k_s) + \sum_s y_s(B_s + \tfrac{1}{2}\sum_r C_{rs} h_r);$$

so that the above integral is less than

$$\prod_{r=1}^{r=n} \int_{h_r}^{\infty} e^{-x_r(A_r+\frac{1}{2}\sum_s C_{rs}k_s)} x_r^{\beta_r-1}\, dx_r \prod_{s=1}^{s=m} \int_{k_s}^{\infty} e^{-y_s(B_s+\frac{1}{2}\sum_r C_{rs}h_r)} y_s^{\gamma_s-1}\, dy_s.$$

This is a product of factors which tend to zero separately and independently as h_r and k_s are indefinitely increased, and so the inversion of the order of integration is justified.

6. If in the formula of § 1 we take $A_{sm} = 1$ for all values of s, we shall have $\sum_s A_{sm}\xi_s = \sum_s \xi_s = 1$, and so, changing $m-1$ into m, we shall get the following result:—

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \prod_{t=1}^{t=m} \left(\sum_{=1}^{=n} A_{st}\xi_s\right)^{-\gamma_t} d\xi_1 d\xi_2 \ldots d\xi_{n-1},$$

in which $$\xi_n = 1-\xi_1-\xi_2-\ldots-\xi_{n-1},$$

and β_s, γ_t, A_{st} denote constants which have their real parts positive for all values of s and t, with the condition that the real part of $\Sigma\beta-\Sigma\gamma$ is positive, taken over a field of integration for which $\xi_1, \xi_2, \ldots, \xi_n$ are all real and positive, is equal to

$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\Sigma\beta-\Sigma\gamma) \prod_{t=1}^{t=m} \Gamma(\gamma_t)} \int_{(m)} \prod_{t=1}^{t=m+1} \theta_t^{\gamma_t-1} \prod_{s=1}^{s=n} \left(\theta_{m+1} + \sum_{t=1}^{t=m} A_{st}\theta_t\right)^{-\beta_s} d\theta_1 d\theta_2 \ldots d\theta_m,$$

where $$\theta_{m+1} = 1 - \sum_{t=1}^{t=m} \theta_t \quad \text{and} \quad \gamma_{m+1} = \sum_{s=1}^{s=n} \beta_s - \sum_{t=1}^{t=m} \gamma_t,$$

the field of integration being for all positive real values of

$$\theta_1, \theta_2, \ldots, \theta_{m+1}.$$

By making the substitution $\theta_t = \theta'\theta_{m+1}$, this last expression may be written in the equivalent form

$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\Sigma\beta-\Sigma\gamma) \prod_{t=1}^{t=m} \Gamma(\gamma_t)} \int_0^\infty \int_0^\infty \dots \int_0^\infty \prod_{t=1}^{t=m} \theta_t'^{\gamma_t-1} \prod_{s=1}^{s=n} \left(1+\sum_{t=1}^{t=m} A_{st}\theta'\right)^{-\beta_s}$$

$$\times d\theta_1' d\theta_2' \dots d\theta_m'.$$

In particular, when $m = 1$ we have that

$$\int_{(n-1)} \prod_{s=1}^{s=n} \xi_s^{\beta_s-1} \{\Sigma A_s \xi_s\}^{-\gamma} d\xi_1 d\xi_2 \dots d\xi_{n-1}$$

is equal to
$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\Sigma\beta-\gamma)\Gamma(\gamma)} \int_0^1 \frac{\theta^{\gamma-1}(1-\theta)^{\Sigma\beta-\gamma-1} d\theta}{\prod_{s=1}^{s=n} (1-\theta+A_s\theta)^{\beta_s}},$$

which may also be written

$$\frac{\prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\Sigma\beta-\gamma)\Gamma(\gamma)} \int_0^\infty \frac{\theta^{\gamma-1} d\theta}{\prod_{s=1}^{s=n} (1+A_s\theta)^{\beta_s}}.$$

7. Generalisations of the known formulæ for definite integrals of Lejeune Dirichlet's type, in which $\Phi(\xi_1+\xi_2+\dots\xi_{n-1})$ takes the place of $\xi_n^{\beta_n-1}$, are easily obtained from the fundamental formulæ of § 3. Thus, for example,

$$\prod_{t=1}^{t=m} \Gamma(\gamma_t) \int_{(n)} \frac{\Phi\left(\sum_{s=1}^{s=n} \xi_s\right) \prod_{s=1}^{s=n} \xi_s^{\beta_s-1}}{\prod_{t=1}^{t=m} \left(A+\sum_{s=1}^{s=n} A_{st}\xi_s\right)^{\gamma_t}} d\xi_1 d\xi_2 \dots d\xi_n$$

is equal to
$$\prod_{s=1}^{s=n} \Gamma(\beta_s) \int_{(m)} \frac{\Phi\left(\sum_{t=1}^{t=m} \theta_t\right) \prod_{t=1}^{t=m} \theta_t^{\gamma_t-1}}{\prod_{s=1}^{s=n} \left(A+\sum_{t=1}^{t=m} A_{st}\theta_t\right)^{\beta_s}} d\theta_1 d\theta_2 \dots d\theta_m,$$

where
$$\Sigma\beta_s = \Sigma\gamma_t;$$

the field of integration for both integrals being over all real positive values of the variables, such that their sum is less than h (*i.e.*, we have $\Sigma\xi_s \ngtr h$ and $\Sigma\theta_t \ngtr h$) and β_s, γ_t, $A+A_{st}h$ being constants which have their real parts positive for all values of s and t.

To prove this it is only necessary to make the substitutions $\Sigma\xi_s = t$, $\xi_s = tx_s$, and $\Sigma\theta_t = u$, $\theta_t = uy_t$, so that $\Sigma x_s = 1$ and $\Sigma y_t = 1$.

If in this formula we take $m = 2$, write γ for γ_1, and put $A_{s2} = 0$ for all values of s, we shall get that

$$\int_{(n)} \frac{\Phi\left(\sum_{s=1}^{s=n} \xi_s\right) \prod_{s=1}^{s=n} \xi_s^{\beta_s - 1}}{\left(A + \sum_{s=1}^{s=n} A_s \xi_s\right)^{\gamma}} d\xi_1 d\xi_2 \dots d\xi_n$$

is equal to

$$\frac{A^{\Sigma\beta-\gamma} \prod_{s=1}^{s=n} \Gamma(\beta_s)}{\Gamma(\gamma)\Gamma(\Sigma\beta-\gamma)} \int_0^h d\theta_1 \frac{\theta_1^{\gamma-1}}{\prod_{s=1}^{s=n} (A+A_s\theta_1)^{\beta_s}} \int_0^{h-\theta_1} d\theta_2 \, \Phi(\theta_1+\theta_2)\, \theta_2^{\Sigma\beta-\gamma-1}$$

when the real part of $\Sigma\beta-\gamma$ is positive.

II.

8. In a paper in the *Mathematische Annalen*, Vol. xxxv., p. 495, Pochhammer has investigated a double circuit integral, constructed to take the place of the Eulerian integral of the first kind,

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx,$$

when the real parts of a and b are not restricted to be positive.

The double circuit integral in question is

$$\int^{(1+,\,0+,\,1-,\,0-)} x^{a-1}(1-x)^{b-1}\, dx,$$

by which is denoted an integral taken along a path which starts from some point c, lying on the real axis between 0 and 1, makes a circuit round 1 in the positive direction, and returns to c, then a circuit round 0 in the positive direction in the same way, and then circuits round 1 and 0, respectively, in the negative direction so that the integrand returns to c with its initial value.

The value of this double circuit integral is independent of the position of c, and we may suppose the path of the variable to be the real axis except at points in the immediate neighbourhood of the points 0 and 1, where we may suppose it to describe small circles about 0 and 1.

Now when the real parts of a and b are positive the values of the integral round these small circles become infinitely small at the same time as the radii of the circles become infinitely small, and the circles may be reduced to their centres so that the integral is reduced to one taken along the real axis between the points 0 and 1. Moreover the integrand is multiplied by $e^{2\pi\iota a}$ when a circuit is made round 0, and by $e^{2\pi\iota b}$ when a circuit is made round 1.

Denoting this integral by $e^{\pi\iota(a+b)}\mathfrak{E}(a, b)$, Pochhammer shows that

$$\mathfrak{E}(a+1, b) = -\frac{a}{a+b}\mathfrak{E}(a, b),$$

$$\mathfrak{E}(a, b+1) = -\frac{b}{a+b}\mathfrak{E}(a, b);$$

and further that when the real parts of a and b are positive

$$e^{\pi\iota(a+b)}\mathfrak{E}(a, b) = (e^{2\pi\iota a}-1)(e^{2\pi\iota b}-1)\int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

Leaving out of consideration the cases in which a or b is an integer, for which some modification is necessary, it is easy to see that these equations establish the formula

$$\int^{(1+,\,0+,\,1-,\,0-)} x^{a-1}(1-x)^{b-1}\,dx = (e^{2\pi\iota a}-1)(e^{2\pi\iota b}-1)\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

for all values of a and b.

This particular double circuit integral is, of course, not the only one which can be so used; any circuit which encloses each of the points 0 and 1, the same number of times positively and negatively, and for which, when the real parts of a and b are positive, the integral reduces to

$$f(e^{2\pi\iota a}, e^{2\pi\iota b})\int_0^1 x^{a-1}(1-x)^{b-1}\,dx,$$

where f is a rational integral function, will do as well. Thus, for instance, if the circuit $(1+,\,0+,\,1-,\,0-)$ be denoted by C, the circuit $1+,\,C,\,1-,\,C$ will make $f(e^{2\pi\iota a}, e^{2\pi\iota b})$ equal to $(e^{2\pi\iota b}-1)^2(e^{2\pi\iota a}-1)$; the circuit $1+,\,C,\,1-$ will make f equal to $e^{2\pi\iota b}(e^{2\pi\iota b}-1)(e^{2\pi\iota a}-1)$; and the circuit $1+,\,0+,\,C,\,0-,\,1-,\,C-$ will make f equal to

$$(e^{2\pi\iota(a+b)}-1)(e^{2\pi\iota b}-1)(e^{2\pi\iota a}-1).$$

9. Consider the integral

$$\int_{(n-1)} x_1^{\beta_1-1}x_2^{\beta_2-1}\dots x_{n-1}^{\beta_{n-1}-1}x_n^{\beta_n-1}\,dx_1\,dx_2\dots dx_{n-1},$$

where
$$x_n \equiv 1-x_1-x_2-\dots-x_{n-1},$$

and the field of integration is such that each of the boundaries $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$ is enclosed by the same number of positive and negative circuits, these circuits being drawn so as not to meet any of the boundaries, so that after the field has been covered each factor of the integrand returns to its original value, and, moreover, is such that, when $\beta_1, \beta_2, \dots$ have their real parts positive, the integral can be reduced to one taken throughout the content bounded by $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$, where $x_1, x_2, \dots$, are real and positive. Since the integrand is multiplied by a factor $e^{2\pi i\beta_r}$ when a circuit is made round $x_r = 0$, the value of the integral, when $\beta_1, \beta_2, \dots$ have their real parts positive, will be

$$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)\int_{(n-1)} x_1^{\beta_1-1}\dots dx_1\,dx_2\dots dx_{n-1},$$

taken throughout the content bounded by $x_1 = 0$, $x_2 = 0$, ..., that is, it will be

$$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)\frac{\prod_{r=1}^{r=n}\Gamma(\beta_r)}{\Gamma\left(\sum_{r=1}^{r=n}\beta_r\right)},$$

f being the appropriate rational integral function which belongs to the particular field chosen, and I wis˙ι to show that the value of the integral will be given by the same formul., for all values of $\beta_1, \beta_2, \dots$.

The form of the field of integration will in general be independent of the values of $\beta_1, \beta_2, \dots$, but modifications are necessary if any of them are positive integers, in order to avoid the occurrence of zero factors in f. If β_r is a positive integer, the field of integration will start from and end in $x_r = 0$, just as in the case of Pochhammer's integrals $\int x^{a-1}(1-x)^{b-1}\,dx$, in which, if a is a positive integer, the double circuit is replaced by a single contour which starts from 0, makes a circuit round 1, and returns to 0 again. Negative integral values of β_r are excluded from consideration.

10. One such field of integration is obtained, and, moreover, the formula is proved for the particular case by applying to the integral

$$\int_{(n-1)}\prod_{s=1}^{s=n} x_s^{\beta_s-1}\,dx_1\,dx_2\dots dx_{n-1}$$

the transformation which is commonly used to obtain its value when β_s is restricted to be positive for all values of s.

If we put

$$\begin{aligned} x_1 &= 1-v_1, \\ x_2 &= v_1(1-v_2), \\ x_3 &= v_1 v_2(1-v_3), \\ &\dots\ \dots\ \dots\ \dots \\ x_{n-1} &= v_1 v_2 \dots v_{n-2}(1-v_{n-1}), \\ x_n &= v_1 v_2 \dots v_{n-2} v_{n-1}, \end{aligned}$$

$$\int_{(n-1)} \prod_{s=1}^{s=n} x_s^{\beta_s-1}\, dx_1\, dx_2 \dots dx_{n-1}$$

is transformed into

$$\int (1-v_1)^{\beta_1-1} v_1^{\beta_2+\beta_3+\dots+\beta_n-1}\, dv_1 \int (1-v_2)^{\beta_2-1} v_2^{\beta_3+\beta_4+\dots+\beta_n-1}\, dv_2 \dots$$
$$\times \int (1-v_{n-1})^{\beta_{n-1}-1} v_{n-1}^{\beta_n-1}\, dv_{n-1},$$

and now, if we suppose all the variables v to make double circuits about the points 0 and 1, so that each factor of the above expression is a Pochhammer integral, we get as the value of the expression

$$(e^{2\pi i\beta_1}-1)(e^{2\pi i(\beta_2+\beta_3+\dots+\beta_n)}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i(\beta_3+\dots+\beta_n)}-1)\dots \frac{\prod_{s=1}^{s=n}\Gamma(\beta_s)}{\Gamma\left(\sum_{s=1}^{s=n}\beta_s\right)}.$$

The original variables x will in this case describe a closed continuum of $(n-1)$ dimensions in a space of $2(n-1)$ dimensions, such that each of the boundaries $x_r = 0$ is enclosed by the same number of positive and negative circuits, because this is true of the variables v, and, moreover, this continuum will not meet any of the boundaries. Also any factor $x_s^{\beta_s-1}$ of the integrand will return to its original value after the continuum has been covered, since this is true of any of its factors $v_1^{\beta_s-1} v_2^{\beta_s-1} \dots (1-v_s)^{\beta_s-1}$. Further, when the real part of β_s is positive for all values of s, each of the double circuits can be reduced to the part of the real axis between the points 0 and 1, described four times with different values of the integrand, as explained by Pochhammer; so that the continuum described by the variables x can be reduced to that bounded by $x_1 = 0$, $x_2 = 0$, ..., where all the variables may be taken to be real and positive. This field of integration, therefore, is such a one as has been described, and the value of the function $f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)$ for this particular field is

$$(e^{2\pi i\beta_1}-1)(e^{2\pi i(\beta_2+\beta_3+\dots+\beta_n)}-1)(e^{2\pi i\beta_2}-1)(e^{2\pi i(\beta_3+\dots+\beta_n)}-1)\dots$$
$$\times (e^{2\pi i\beta_{n-1}}-1)(e^{2\pi i\beta_n}-1).$$

If any of the β are positive integers, it is necessary to modify the field of integration. As the order of the letters is immaterial, we may suppose all the positive integral indices to come at the end; thus, if β_n is a positive integer, the path of the variable v_{n-1} is a contour which starts from 0 and makes a circuit round 1 and returns to 0*; if β_{n-1} and β_n are both positive integers, v_{n-2} makes the circuit just mentioned, while the path of v_{n-1} is the real axis from 0 to 1.

11. It is also possible to construct such surfaces geometrically. Consider the double integral

$$\iint x_1^{\beta_1-1} x_2^{\beta_2-1} x_3^{\beta_3-1}\, dx_1\, dx_2,$$

where $$x_3 \equiv 1 - x_1 - x_2.$$

If we put $x_1 = y_1 + z_1 \iota$, $x_2 = y_2 + z_2 \iota$, then y_1, z_1, y_2, z_2 are the coordinates of a point in four-dimensional space. The boundaries are the planes $y_1 = 0$, $z_1 = 0$; $y_2 = 0$, $z_2 = 0$; $y_1 + y_2 = 1$, $z_1 + z_2 = 0$. When the real parts of β_1, β_2, and β_3 are positive, the integration is over the area of the triangle in the plane $z_1 = 0$, $z_2 = 0$ which is bounded by $y_1 = 0$, $y_2 = 0$, $y_1 + y_2 = 1$.

Let this triangle be OAB in the figure. Take a point P_1 in the

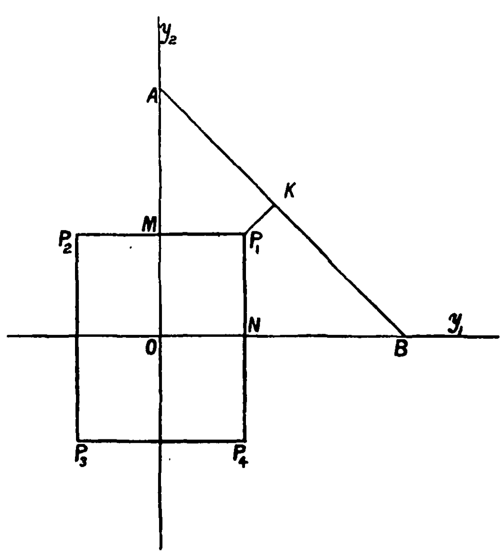

FIG. 1.

* *Cf.* Pochhammer, *loc. cit.*, p. 510.

triangle whose co-ordinates are $(a, b, 0, 0)$; and its images P_2, P_3, P_4 in the other three quadrants.

Consider the surface whose equations are

$$y_1^2+z_1^2 = a^2, \qquad y_2^2+z_2^2 = b^2.$$

This can be represented by two circles, a point on the surface being represented by two points taken together, one on each circle. Thus the

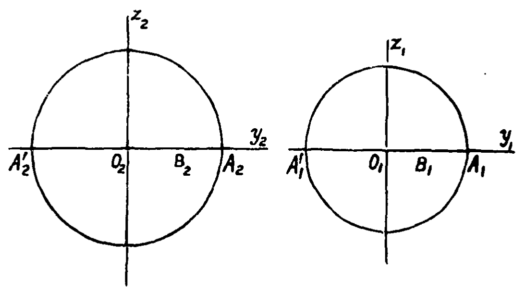

FIG. 2.

points P_1, P_2, P_3, P_4 in Fig. 1 are represented respectively by the pairs (A_1A_2), $(A_1'A_2)$, $(A_1'A_2')$, (A_1A_2').

This surface can be described in four distinct ways, since the circles O_1 and O_2 may each be described either in a positive or negative direction. If, starting from (A_1, A_2), we describe each circle in the positive direction, x_1 and x_2 will each have taken the factor $e^{2\pi\iota}$, and $x_1^{\beta_1-1}x_2^{\beta_2-1}$, therefore, the factor $e^{2\pi\iota(\beta_1+\beta_2)}$, and the whole surface will have been described in a positive sense. If, now, we describe each circle in the negative sense, the whole surface will have been described positively again, but x_1 and x_2 will each have taken the factor $e^{-2\pi\iota}$, and $x_1^{\beta_1-1}x_2^{\beta_2-1}$ will have returned to its original value.*

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* This may be illustrated by putting $\beta_3 = 1$, $y_1 = a\cos\theta$, $z_1 = a\sin\theta$, $y_2 = b\cos\phi$, $z_2 = b\sin\phi$. The integral $\iint x_1^{\beta_1-1}x_2^{\beta_2-1}\,dx_1\,dx_2$ becomes in this case $-a^{\beta_1}b^{\beta_2}\iint e^{\beta_1\theta\iota}e^{\beta_2\phi\iota}\,d\theta\,d\phi$, and its four values will be $-a^{\beta_1}b^{\beta_2}(e^{\pm 2\pi\iota\beta_1}-1)(e^{\pm 2\pi\iota\beta_2}-1)$ according to the directions in which the circles round O_1 and O_2 are described, and the integrand will return to its starting point (A_1, A_2) with the corresponding values $a^{\beta_1-1}b^{\beta_2-1}e^{2\pi\iota(\pm\beta_1\pm\beta_2)}$. If, starting with these values as the initial value of the integrand, the surface were described in the negative sense, that is if the integral were taken over the opposite side of the surface—or over a *variété opposée*—the value of the integral would be changed in sign.

In the case of an anchor ring in ordinary space of three dimensions, we have similarly a surface which has an inside and an outside, and in which two sets of circles drawn on it, viz., those about the axis of revolution and those about the circular axis, can be described independently in either direction.

Let such a double description of the surface be denoted by C_{12}, the surface itself being called S_{12}.

We may deform this surface S_{12} into the surface represented by two contours, one of which starting from A_1 describes the line A_1O_1 to some point near to O_1, then makes a small circle about O_1, and returns to A_1 on the other side of the axis O_1A_1; the other being a similar contour starting from A_2, and making a small circle about O_2. Then corresponding to any point Q in the area OMP_1N we shall have four points on this deformed surface, viz., if the point Q is represented by the two points B_1 and B_2, and if the points infinitesimally near to B_1 and B_2 on opposite sides of O_1A_1 and O_2A_2 respectively are denoted by B_1+, B_1- and B_2+, B_2-, we have corresponding to Q the four points represented by $(B_1\pm, B_2\pm)$.

If $x_1^{\beta_1-1}x_2^{\beta_2-1}$ start from P, with its natural value, viz.,

$$\exp\left[(\beta_1-1)\log a+(\beta_2-1)\log b\right],$$

where $\log a$ and $\log b$ are real, and if its value at (B_1+, B_2+) be $x_1^{\beta_1-1}x_2^{\beta_2-1}$, then its value at (B_1-, B_2+) will be $e^{2\pi\iota\beta_1}x_1^{\beta_1-1}x_2^{\beta_2-1}$, at (B_1+, B_2-) will be $e^{2\pi\iota\beta_2}x_1^{\beta_1-1}x_2^{\beta_2-1}$, and at (B_1-, B_2-) will be $e^{2\pi\iota(\beta_1+\beta_2)}x_1^{\beta_1-1}x_2^{\beta_2-1}$, and, if we consider the area OMP_1N to be described positively as the describing point is passing through (B_1+, B_2+), it will be described negatively at (B_1+, B_2-) and (B_1-, B_2+), but positively at (B_1-, B_2-).

So that, when the real parts of β_1 and β_2 are positive and the circles round O_1 and O_2 may be reduced to their centres, we have that the integral over the surface S_{12} is equivalent to $(e^{2\pi\iota\beta_1}-1)(e^{2\pi\iota\beta_2}-1)$ times the integral over the area OMP_1N. Thus the integral over C_{12} is equivalent to $2(e^{2\pi\iota\beta_1}-1)(e^{2\pi\iota\beta_2}-1)$ times the integral over OMP_1N.

An exactly similar field of integration enclosing the corner A, viz., that over the surface S_{23} whose equations are

$$y_2^2+z_2^2=b^2, \qquad y_3^2+z_3^2=c^2,$$

where $$y_3+\iota z_3 \equiv 1-y_1-y_2-\iota(z_1+z_2),$$

will be denoted by C_{23}, and over a similar surface enclosing the corner B by C_{31}.

In order that when z_1 and z_2 are taken infinitesimally small, the whole area of the triangle OAB, in Fig. 1, may be covered, these surfaces must be so deformed that each meets the others in a single line. That this can be done may be seen by moving B out to a very great distance along Oy_1; then the two surfaces S_{12} and S_{23} become

$$y_1^2+z_1^2 = a^2, \qquad y_2^2+z_2^2 = b^2,$$

and
$$(1-y_1)^2+z_1^2 = (1-a)^2, \qquad y_2^2+z_2^2 = b^2,$$

which meet one another along the circle

$$y_1 = a, \qquad z_1 = 0, \qquad y_2^2+z_2^2 = b^2,$$

and by continuous deformation we may bring B back again, and still have the two surfaces meeting one another along a single line. We may suppose the portions of the triangle OAB which correspond to the three fields of integration C_{12}, C_{23}, C_{31} to be OMP_1N, $MAKP_1$, $KBNP_1$ respectively, and now, if we assume the element of area which describes the field to start from P_1, then move along the line P_1K, make a circuit round AB at K, and return to P_1—so that the integrand has taken the factor $e^{2\pi\iota\beta_3}$—then to describe C_{12} in a positive sense, then the circuit round K in a negative sense, and finally C_{12} in a negative sense, then we shall have as the equivalent of the integral over this field, when the real parts of β_1 and β_2 are positive,

$$2(e^{2\pi\iota\beta_3}-1)(e^{2\pi\iota\beta_2}-1)(e^{2\pi\iota\beta_1}-1)$$

times the integral over the area OMP_1N in the plane $z_1 = 0$, $z_2 = 0$. The integrals along the positive and negative circuits around K destroy one another, since the integrand returns to its original value after C_{12} has been described.

If we construct similar fields corresponding to the other two regions AMP_1K and BNP_1K, these three taken together will give a field of integration such as is required, and the factor $f(e^{2\pi\iota\beta_1}, e^{2\pi\iota\beta_2}, e^{2\pi\iota\beta_3})$ corresponding to it will be

$$2(e^{2\pi\iota\beta_1}-1)(e^{2\pi\iota\beta_2}-1)(e^{2\pi\iota\beta_3}-1).$$

If β_3 is a positive integer, it is necessary to avoid the factor $(e^{2\pi\iota\beta_3}-1)$. The field of integration may be taken to be over the surface S_{12}, so deformed that P_1M and P_1N lie along KA and KB respectively, or rather over the part of such a surface cut off by a boundary curve, which lies in the plane $y_1+y_2 = 1$, $z_1+z_2 = 0$, and encloses the line AB of Fig. 1.

12. I now propose to show that for any field of integration of this sort the value of the integral

$$\int_{(n-1)} x_1^{\beta_1-1} x_2^{\beta_2-1} \ldots x_n^{\beta_n-1}\, dx_1\, dx_2 \ldots dx_{n-1},$$

where $$x_n = 1 - x_1 - x_2 - \dots - x_{n-1},$$

is $$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots) \frac{\prod_{r=1}^{r=n} \Gamma(\beta_r)}{\Gamma(\Sigma\beta_r)}$$

for all values of β_1, β_2, ... (negative integral values excluded), $f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \dots)$ being the appropriate function for the field chosen.

This I shall do by the method of induction, so that the formula, being true for any values of β_1, β_2, ..., which have their real parts all positive, is also true for any values of β_1, β_2, ... whatever.

For shortness I write F for the integral

$$\int_{(n-1)} x_1^{\beta_1-1} x_2^{\beta_2-1} \dots x_n^{\beta_n-1} \, dx_1 \, dx_2 \dots dx_{n-1},$$

taken over a field of integration such that each factor of the integrand $x_r^{\beta_r-1}$ returns to its initial value after the field has been covered.

Further, I write F_r for the integral obtained by replacing β_r in F by β_r+1. Then, since

$$\frac{d}{dx_r}(x_r^{\beta_r} x_n^{\beta_n}) = \beta_r x_r^{\beta_r-1} x_n^{\beta_n} - \beta_n x_r^{\beta_r} x_n^{\beta_n-1}$$
$$= x_r^{\beta_r-1} x_n^{\beta_n-1} \{\beta_r(1-x_1-x_2-\dots-x_{n-1}) - \beta_n x_r\},$$

I get, on multiplying by $x_1^{\beta_1-1} x_2^{\beta_2-1} \dots x_{r-1}^{\beta_{r-1}-1} x_{r+1}^{\beta_{r+1}-1} \dots x_{n-1}^{\beta_{n-1}-1}$ and integrating throughout the field, the two relations

$$0 = \beta_r F_n - \beta_n F_r, \tag{1}$$

$$0 = \beta_r F - \beta_r F_1 - \beta_r F_2 - \dots - (\beta_r + \beta_n) F_r - \dots - \beta_r F_{n-1}, \tag{2}$$

the left-hand side vanishing since $x_r^{\beta_r} x_n^{\beta_n}$ returns to its initial value.*

In these equations r may have any one of the values 1, 2, ..., $(n-1)$, and, if we write down the series of $n-1$ equations so obtained from (2), and solve them, we get at once

$$F_r = (\beta_r / \Sigma\beta) F \quad (r = 1, 2, 3, \dots, n).$$

13. By the help of this relation the evaluation of the integral

$$\int_{(n-1)} \prod_{s=1}^{s=n} x_s^{\beta_s-1} \, dx_1 \dots dx_{n-1}$$

* Or, if β_r is a positive integer, has zero for its initial and final values.

is reduced to that of another integral of the same form in which β_r can be replaced by β_r+n, where n is a positive integer, that is, finally, to the evaluation of the integral when all the β have their real parts positive, and so the result is established that, if the value of

$$\int_{(n-1)} \prod_{s=1}^{s=n} x_s^{\beta_s-1}\, dx_1 \ldots dx_{n-1}$$

taken over a suitable field is

$$f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \ldots)\frac{\prod_{r=1}^{r=n}\Gamma(\beta_r)}{\Gamma(\Sigma\beta_r)},$$

when the real parts of $\beta_1, \beta_2, \ldots$ are positive, it is given by the same formula for all values of $\beta_1, \beta_2, \ldots$.

14. The same reasoning may be applied to extend to all values of β_s and γ_t—with the exception of negative integral values—the formula proved in Part I., § 3, of this paper.

For, if we have

$$\prod_{t=1}^{t=m}\Gamma(\gamma_t)\,\phi(e^{2\pi i\gamma_1}, e^{2\pi i\gamma_2}, \ldots)\int_{(n-1)}\prod_{s=1}^{s=n}\xi_s^{\beta_s-1}\prod_{t=1}^{t=m}\left(\sum_{s=1}^{s=n}A_{st}\xi_s\right)^{-\gamma_t}d\xi_1 d\xi_2\ldots d\xi_{n-1}$$

$$=\prod_{s=1}^{s=n}\Gamma(\beta_s)\,f(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, \ldots)\int_{(m-1)}\prod_{t=1}^{t=m}\theta_t^{\gamma_t-1}\prod_{s=1}^{s=n}\left(\sum_{t=1}^{t=m}A_{st}\theta_t\right)^{-\beta_s}$$

$$\times\, d\theta_1 d\theta_2\ldots d\theta_{m-1},$$

the integrals being taken over suitable fields, and f and ϕ being the rational integral functions corresponding to the fields of integration which have been chosen, then we shall get on integrating each side with respect to some one coefficient, say $A_{s't'}$, the same formula with $\beta_{s'}$ and $\gamma_{t'}$ changed into $\beta_{s'}-1$ and $\gamma_{t'}-1$ respectively. That the constant of integration introduced in this way is zero may be seen as follows. It is not a function of $A_{s't'}$, and therefore it cannot be a function of A_{st} for any values of s or t, since the integrals are unaltered, if, throughout them both, any two suffixes s, or any two suffixes t, are interchanged.

As it is therefore independent of the coefficients A_{st}, we may put A_{st} equal to unity for all values of s and t, and we get integrals of the form considered in the last paragraph, and the relation is at once verified.

The formula, then, having been proved for values of β_s and γ_t which have their real parts positive, is also true when β_s and γ_t are replaced by

$\beta_s - n$ and $\gamma_t - n$ where n is any integer, and so is true, in general, for all values of β_s and γ_t.

15. A similar extension may be made for the formula of § 4, the field of integration beginning and ending at infinity ($x_r = +\infty$, for all values of r), and making circuits about $x_r = 0$. In fact the formula is at once proved for one such field, by making use of the expression for $\Gamma(p)$ as a contour integral, viz., $\frac{1}{e^{2\pi i p}-1}\int x^{p-1}e^{-x}\,dx$, taken round a contour which starts from infinity on the positive side of the axis of real quantity, makes a circuit about the origin, and returns to infinity on the negative side of the real axis.