while (27), (28) lead to

$$9f'^2 mn\,(u'''m - u''n) + 4\,(c_2 m - b_3 n)^3 = 0,$$

or $4c_2^3 m^3 - 3\,(4b_3 c_2^2 - 3f'^2 u''')\, m^2 n + 3\,(4b_3^2 c_2 - 3f'^2 u'')\, mn^2 - 4b_3^3 n^3 = 0$...(29).

The resultant of (28), (29), after division by f'^3, is

$$\begin{aligned}
&27f'\,(3f'b - 4b_3^2)(3f'c - 4c_2^2) \\
&\qquad \times \{(3f'c - 4c_2^2)\,u''^2 - 2\,(3f'f - 4b_3 c_2)\,u''u''' + (3f'b - 4b_3^2)\,u'''^2\} \\
&+ 6f'\,\{(3f'c - 4c_2^2)(c_2^2 b - b_3^2 c)\,(c_2 b - b_3 f) - 2c_2\,(3f'b - 4b_3^2)(b_3 c - c_2 f)^2\}\,u'' \\
&+ 6f'\,\{(3f'b - 4b_3^2)(b_3^2 c - c_2^2 b)(b_3 c - c_2 f) - 2b_3(3f'c - 4c_2^2)(c_2 b - b_3 f)^2\}\,u''' \\
&+ 64\,(c_2^2 b + b_3^2 c - 2b_2 c_3 f)^3 = 0 \quad\ldots\ldots(30).
\end{aligned}$$

This equation is of the twelfth order in xyz, and is reducible therefore to the sixth, when the differential coefficients are expressed explicitly in terms of the coefficients of the quartic. It is of the ninth order in those coefficients, and in this respect is not reducible, as I have found by examining some particular cases.

In using the term Inflexion-Tangential Equation, of course it is understood that it is so only to *ku près*, where k is a quadratic form; just as two forms of the Bitangential Equation may be arrived at, pursuing different methods of investigation, which shall differ by a sum of terms in which u is a factor.

Hereafter, if I should find leisure to resume the study of this concomitant of the ternary quartic, I hope to be able to lay further results before the Society.

On the Singularities of the Modular Equations and Curves. By HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read February 14th, and April 11th*, 1878.]

Art. 1.—It is proposed, in this paper, to examine the characteristic singularities of the modular equations and curves. The method employed is applicable to all the modular equations hitherto considered by geometers;* but, for brevity, the discussion is confined to the

* [The modular equations considered by Jacobi in the *Fundamenta Nova* are (2) the equation between $u = \phi(\omega)$, and $v = \phi(\Omega)$, (see Art. 4 of this paper, equations 3 and 4,) and (5) the equation between u^8 and v^8, of which the characteristics are discussed here. M. Kronecker, in his researches on the modules which admit of complex multiplication, would seem to have also employed (3) the equation between u^2 and v^2, and (4) the equation between u^4 and v^4. (See the account of these researches in the Reports of the British Association for 1865, pp. 332 and 368; see also Professor Cayley, *Phil. Trans.*, Vol. clxiv. p. 450.) M. Joubert (*Comptes Rendus*, Vol. xlviii., pp. 290—294) was the first to consider (6) the equation between $u^8(1-u^8)$ and $v^8(1-v^8)$. Dr. Felix Müller, in his Inaugural Dissertation (Berlin, 1867), drew attention to the equation (7) between

$$T(\omega) = \frac{(1-u^8+u^{16})^3}{u^{16}(1-u^8)^2} \quad\text{and}\quad T(\Omega) = \frac{(1-v^8+v^{16})^3}{v^{16}(1-v^8)^2};$$

equations containing the squares of the modules, and to the case in which the order of the transformation is uneven.

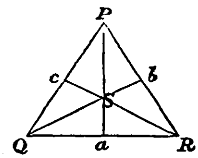
Art. 2.—We represent by q, the square of the modulus of a given elliptic function; by p, the square of the transformed modulus, the transformation being primary,* and of the uneven order N; by

$$(1) \;\ldots\ldots\; F(p,\, q,\, 1) = 0$$

the modular equation subsisting between p and q; in connection with this equation, we consider the modular curve C, of which the trilinear equation

$$(2) \;\ldots\ldots\; F(\alpha,\, \beta,\, \gamma) = 0$$

is obtained by writing $p = \frac{\alpha}{\gamma}$, $q = \frac{\beta}{\gamma}$. We denote by P, Q, R the vertices of the triangle $\alpha\beta\gamma$; and by S the point $\alpha = \beta = \gamma$, or $p = q = 1$; the three intersections (PS, QR), (QS, RP), (RS, PQ) we represent by a, b, c. We regard p and q as the



and the discussion of this equation has recently been resumed by Professor Klein (*Mathematische Annalen*, Vol. xiv., p. 112, May 1878). These geometers have expressed $T(\omega)$ and $T(\Omega)$ rationally in terms of a third indeterminate, in the cases $N = 2, 4$; $3, 5, 7, 13$; the deficiency of the equation (7) being zero in these six cases; but neither of them has given any example of the equation in its explicit form. Writing x for $T(\omega)$ and y for $T(\Omega)$, I find, when $N = 3$,

$$x(x + 2^7.3.5^3)^3 + y(y + 2^7.3.5^3)^3 - 2^{16}x^3y^3 + 2^{11}.3^2.31x^2y^2(x+y)$$
$$- 2^2.3^3.9907xy(x^2 + y^2) + 2.3^4.13.193.6367x^2y^2$$
$$+ 2^9.3^5.5^3.4471xy(x+y) - 2^{15}.5^6.22973xy = 0.$$

The equations (2) ... (7) are symmetrical with regard to the two indeterminates, and, the number N being uneven, they are of the order $A + B$ (Art. 3) in the indeterminates separately, and of the order $2A$ in the indeterminates jointly. I have recently found that the Eulerian functions $\chi(\omega)$ and $\chi(\Omega)$, defined by the equation

$$\chi(\omega) = \sqrt[6]{2}\,\epsilon^{\frac{i\pi\omega}{24}} \times \prod_1^\infty \left(1 - (-1)^s \epsilon^{si\pi\omega}\right),$$

satisfy an equation (1) having the same properties; for the cube $\chi(\omega)$ this had already been shown by M. Kœnigsberger (*Borchardt's Journal*, Vol. lxxii., p. 182 sqq.) Writing $u_1 = \chi(\omega)$, $v_1 = \chi(\Omega)$, we have, in the cases $N = 5, 7, 11$,

$$u_1^6 + 2^{\frac{4}{3}}u_1^5v_1^5 - 2^{\frac{2}{3}}u_1v_1 + v^6 = 0,$$

$$u_1^{12} + 8.2^{\frac{1}{3}}u_1^{11}v_1^{11} - 44u_1^9v_1^9 + 44.2^{\frac{2}{3}}u_1^7v_1^7 - 44.2^{\frac{1}{3}}u_1^5v_1^5 + 22u_1^3v_1^3 - 2.2^{\frac{2}{3}}u_1v_1 + v_1^{12} = 0,$$

$$u_1^8 - 16u_1^7v_1^7 + 7u_1^4v_1^4 - 2u_1v_1 + v_1^8 = 0.$$

The function $\chi(\omega)$ is a twenty-fourth root of $u^8(1-u^8)$; the formulæ relating to its linear transformation have been given by M. Hermite (*Comptes Rendus*, 1858, Vol. xlvi., p. 721). In respect of simplicity of form, the equations (1), (2) ... (7) appear to arrange themselves in this numerical order; but, in respect of simplicity of algebraical theory, the order is reversed, as the deficiency decreases from (1) to (7).—H. J. S. S.]

* A transformation $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ of the uneven order N is primary when it satisfies the congruence $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$, mod. 2. For the theory of the elliptic multiplier it is convenient to fix the signs of $abcd$ by the additional condition $a \equiv 1$, mod. 4; but for our present purpose this restriction is unnecessary.

parameters of two pencils of lines $\alpha - p\gamma$, $\beta - q\gamma$, of which the centres are Q and P, and between the rays of which a correspondence is established by the equation (1); we observe that, to the values 0, 1, ∞ of either parameter, there answer in the two pencils respectively the rays QR, QS, QP; PR, PS, PQ; and that, by a known property of the modular equation, the pairs of rays (QR, PR), (QS, PS), (QP, PQ) are corresponding rays in the two pencils, either ray of any of these pairs being the only ray answering to the other ray of the same pair; the modular curve C is the locus of the intersections of corresponding rays in the two pencils.*

We denote by m and n the order and class of C; by H its deficiency; by K and I its cuspidal and inflexional indices; by D and T its discriminantal order and class; by $E(p) = E(q)$ the highest, by $E'(p) = E'(q)$ the lowest, exponents of p and q which present themselves in the equation (1).

ARTS. 3—8. *N not divisible by a square.*

Art. 3.—We confine ourselves, in the first instance, to the case in which N is not divisible by any square; and we represent, in this case, by A the sum of the divisors of N which surpass $\sqrt{N}$; by B the sum of the divisors of N which are less than $\sqrt{N}$; by ν the number of divisors of either sort, so that 2ν is the whole number of divisors of N. We then have the formulæ

$$\text{(i.)} \ldots\ldots \quad m = 2A,$$
$$\text{(ii.)} \ldots\ldots \quad n = 3A - B,$$
$$\text{(iii.)} \ldots\ldots \quad H = \tfrac{1}{2}(A+B) - 3\nu + 1,$$
$$\text{(iv.)} \ldots\ldots \quad K = 2A + 2B - 6\nu,$$
$$\text{(v.)} \ldots\ldots \quad I - K = 3(A - B),$$
$$\text{(vi.)} \ldots\ldots \quad D = 4A^2 - 5A + B,$$
$$\text{(vii.)} \ldots\ldots \quad T - D = (A - B)(5A - B).$$

To these we may add the equations

$$\text{(viii.)} \ldots\ldots \begin{cases} E(p) = E(q) = A + B, \\ E'(p) = E'(q) = 2B. \end{cases}$$

Of these formulæ (i.) and (viii.) are well known; of the remainder, it will suffice to attend to (iii.) and (iv.), because, when the values of m, H, and K are given, the values of n, I, D, and T are known from the equations of Plücker.

* [In a paper which I hope shortly to lay before the Society, I have discussed, with some fulness of detail, the relation of the algebraical singularities of the parametric equation $F(p, q, 1) = 0$ to the characteristic singularities of the curves of which the equations are included in the formula $F\left(\frac{A}{B}, \frac{C}{D}, 1\right) = 0$, A, B, C, D being the equations of straight lines. The discussion comprises an examination of the effect of any quadric transformation on the singularities of a curve.—H. J. S. S.]

Art. 4.—*The Deficiency.*—The demonstration of the formulæ (iii) and (iv.) depends on the simultaneous expression of the modular parameters p and q as transcendental functions of the quotient of the periods of the given elliptic function. As we have already developed these considerations elsewhere,* we shall in this place assume the results of the discussion as known, and shall confine ourselves to their application to the formulæ (iii.) and (iv.)

Denoting by x and y real quantities, of which y is essentially posi-

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* In an Introduction (now in the press) to Mr. J. W. L. Glaisher's Tables of the Theta Functions. The method indicated in this article has been already employed by Professor Klein, in the paper to which reference has already been made (*Math. Ann.*, Vol. xiv., p. 111; see especially §§ 6—8, 7—13), and by Professor Dedekind, in a letter addressed to M. Borchardt (June 1877, see *Borchardt's Journal*, Vol. lxxxiii., p. 265, especially §§ 1—4 and 7). In the year 1873, I submitted to the Academy of Sciences in Paris a Mémoire *Sur les Équations Modulaires* (*Comptes Rendus*, August 1873, Vol. lxxvii., p. 472). In this Mémoire (which was ultimately presented, without alteration, to the Accademia dei Lincei, and was printed in their *Atti*, Vol. i., series 3, p. 136, February 1877), I had employed the same method (see Arts. ii., ix., and x. of the Mémoire) to establish the relation which exists between the modular equations of order N and the binary quadratic forms of the positive determinant N. The Mémoire was devoted to that theory alone, as I attached more importance to it than to any other result relating to the modular equations at which I had then arrived. But I had already in the year 1873 obtained:—(i.) a proof of the existence of the modular equations, simpler perhaps than that of M. Dedekind, and based solely on those elementary properties of the function $\phi(\omega)$, which were deduced from the theorem of Fourier by Cauchy and Poisson, without employing any elliptic formulæ; (ii.) a determination in the simpler cases of the Plückerian characteristics of the modular curves; (iii.) a solution of one part at least of the problem relating to complex modules, proposed by Jacobi in Art. 32 of the *Fundamenta Nova*. I communicated to Professor Cayley, in 1873, the formulæ for the deficiency of the equations (2) ... (5) when N is an uneven prime (see his Memoir on the Transformation of Elliptic Functions, presented to the Royal Society in that year); the formulæ for the cuspidal index I obtained by transforming into normal developments the parametric developments which give the deficiency (see Art. 5 of this paper); thus, the order of the curves being known, all their Plückerian characteristics were determined. The case when N is a product of uneven primes presents no greater difficulty than the case when N is a prime; and I had (in fact) obtained the formula for this more general case as early as 1873. The case when N is divisible by a square, and still more the case when N is itself a square, appeared to involve some difficulty; and these I left untouched till the spring of the present year, when I found that the introduction of the arithmetical function f' (see Art. 9 of this paper) caused the supposed difficulty to disappear. To the more exact determination of the indices characteristic of each special singularity of the modular curves, I was guided by the methods employed in a former paper on the Higher Singularities of Plane Curves.

A complete system of formulæ, analogous to that given in the present paper for the modular equation (5), I have already obtained for the equations (2), (3), and (4); with the equation (7), and with the eight equations between corresponding powers of $\chi(\omega)$ and $\chi(\Omega)$, I have not advanced equally far, but I have not found that they offer any peculiar difficulty.

In the Mémoire sur les Équations Modulaires, I have confined myself (as in the present paper) to the equation (5) between the squares of the modules. At the time when the Mémoire was written, I was well acquainted with the characteristic property of the function $T(\omega)$; viz., that it is unchanged by any linear transformation of the elliptic functions; and I even thought of employing it in the Mémoire instead of the function $\phi^8(\omega)$. I had conjectured (erroneously however) that the modular curves T derived from the equation (7) would represent ordinary periodic continued fractions with positive integral quotients, in the same way in which the tive, and by ω the complex variable $x+iy$, which is thus subject to the restriction that the coefficient of i in its imaginary part is positive, we define the function $\phi(\omega)$ by the equation

$$(3) \ldots\ldots \phi(\omega) = \sqrt{2} \, . \, \epsilon^{\frac{i\pi\omega}{8}} \prod_{s=1}^{s=\infty} \frac{1+\epsilon^{2si\pi\omega}}{1+\epsilon^{(2s-1)i\pi\omega}};$$

and we consider the $A+B$ quantities Ω, of which the values are given by the equation

$$(4) \ldots\ldots \Omega = \frac{g'\omega+2k}{g},$$

g, g' being any two conjugate divisors of N, and k being any term of a complete system of residues for the modulus g. We then have the fundamental theorem,

"If $q = \phi^8(\omega)$, the $A+B$ corresponding values of p are included in the formula $$p = \phi^8(\Omega);$$
or, which is the same thing,

$$F(p,\ q,\ 1) = \Pi\,[p-\phi^8(\Omega)],$$

the sign of multiplication extending to the $A+B$ values of Ω."

It results from the discussion to which we have referred, that, if we regard q as an independent complex variable, and represent its values in the usual manner by the points of a plane, p, considered as a function q, has no spiral points other than the three $q=0$, $q=1$, $q=\infty$ (it will be remembered that in the plane of double algebra the infinitely

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modular curves C derived from the equation (5) represent periodic continued fractions with even quotients. But I was deterred from employing the equation (7) by the consideration that there was not a single calculated example of it; indeed, at that time I was not acquainted with the researches of Dr. Felix Müller, and did not know that the equation had attracted any attention. I have since found that the curves T do not precisely represent the reduced forms of Gauss, but instead a system of forms determined by a different regulative principle. I am disposed to think (notwithstanding the considerations mentioned in the note on Art. 1), that there is some advantage in continuing to regard the equation between the squares of the modules, as the principal modular equation, rather than either the equation (1) or (7). At least, as far as concerns the arithmetical theory to which the Mémoire relates, and which I have since extended to the equations (1), (2), and (7), (the theory of the equations (3), (4), and (6) hardly requiring a separate discussion,) the modular curves (5) present phenomena in some respects simpler than those presented by the curves (7), and in all respects simpler than those presented by the curves (1) ... (4).

Both in the Mémoire and in this paper, I have given especial attention to the case in which N is a square, because the solution of the problem of Jacobi for the transformations of order N depends on a consideration of the spaces into which modular curves of order N^2 divide a plane. A note published in the *Transunti* of the Accademia dei Lincei (Vol. i., p. 42, 7 Jan., 1877), contains what is in fact a solution of Jacobi's problem for the case $N=1$; to this particular case of the general problem, the attention of geometers had been called by M. Hermite in the note appended to the second volume of M. Serret's edition (1862) of the Differential Calculus of Lacroix, pp. 421—425.—H. J. S. S.]

distant is a point). Thus, if we cause q to describe any closed contour, not including one of the three points 0, 1, ∞, the values of p will undergo no interchange; but each root of the equation $F(p, q, 1) = 0$ will return, when the contour is closed, to the same value which it had at the beginning; although the contour may include points (other than 0, 1, ∞) at which two of the values of p become equal. But the case is different if we cause q to describe a closed contour round one of the three exceptional points. At each of these points all the values of p become equal to one another, and to the value of q indicated by the point. As the general result is the same for each of the three points, it will suffice to attend to one of them only; for example, to the point $q = 0$. If, then, we cause q to describe a closed contour round 0, the g values of p, or of the expression $\phi^8\left(\frac{g'\omega+2k}{g}\right)$, which contain the divisor g in the denominator, change into one another cyclically, and thus the $\Sigma g = A+B$ roots of the equation arrange themselves in 2ν cycles, corresponding to the different divisors of N; or, which is the same thing, the developments obtained by expanding the different values of p in series proceeding by ascending powers of q are singular; being, in fact, of the type

$$(5) \;\ldots\ldots\ldots\; p = \lambda q^{\frac{g'}{g}} + \ldots .$$

Similarly, at the points $+1$ and ∞, we have singular developments of the types

$$(6) \;\ldots\ldots\; p-1 = \lambda\,(q-1)^{\frac{g'}{g}} + \ldots,$$

$$(7) \;\ldots\ldots\; \frac{1}{p} = \lambda\left(\frac{1}{q}\right)^{\frac{g'}{g}} + \ldots .$$

As these are all the singular developments that can exist, we infer that, if $W(p)$ represent the number of the spiral points of p, each point being reckoned with its proper multiplicity,

$$W(p) = 3\Sigma\,(g-1) = 3\,(A+B)-6\nu.$$

Substituting, in the equation of Riemann,

$$(8) \;\ldots\ldots\; 2H = W(p)-2E\,(p)+2,$$

the value of $E\,(p)$ (equation viii., Art. 3), and the value just obtained for $W(p)$, we find $\qquad H = \tfrac{1}{2}\,(A+B)-3\nu+1,$

which is the equation (iii.)

Art. 5. *The Cuspidal Index.*—To determine the cuspidal index of C, we first consider the developments (5) which appertain to the point R. Since N is not a square, we cannot have $g = g'$; if $g' > g$, we have the normal development*

$$(9) \ldots\ldots \frac{\alpha}{\gamma} = \lambda\left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}} + \ldots;$$

if $g > g'$, we have the normal development

$$(10) \ldots\ldots \frac{\beta}{\gamma} = \mu\left(\frac{\alpha}{\gamma}\right)^{\frac{g}{g'}} + \ldots.$$

The branches corresponding to the developments (9) and (10) have, for their cuspidal indices, $g-1$ and $g'-1$ respectively. Hence each of the lines QR, PR is touched at the point R by a set of branches of which the aggregate cuspidal index is $B-\nu$.

In the same manner, it will be found that at the point S each of the lines QS, PS is touched by a set of branches having the same cuspidal indices as the branches which touch QR or PR at R.

Lastly, from the developments (7), we deduce normal developments of the types

$$(11) \ldots\ldots \frac{\gamma}{\alpha} = \mu\left(\frac{\beta}{\alpha}\right)^{\frac{g'}{g'-g}} + \ldots, \quad g' > g;$$

$$(12) \ldots\ldots \frac{\gamma}{\beta} = \mu\left(\frac{\alpha}{\beta}\right)^{\frac{g'}{g-g'}} + \ldots, \quad g > g';$$

and from these we infer that the line PQ is touched at each of the points P and Q by a set of branches of which the aggregate cuspidal index is $A-B-\nu$.

Since C can have no other cuspidal branches, we find

$$\begin{aligned} K &= 2(B-\nu)+2(B-\nu)+2(A-B-\nu) \\ &= 2(A+B)-6\nu, \end{aligned}$$

which is the formula (iv.).

Art. 6. *The discriminant of* $F(p, q, 1)$.—The values of m, $E(p)$, $E(q)$, $E'(p)$, $E'(q)$ are inferred from the equation (1), by a method, due (as it would seem) to M. Kronecker, of which examples are given in the Report on the Theory of Numbers (*Reports of the British Association* for 1865, p. 349 sqq.) This method is also applicable to the

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* If A, B, C represent straight lines forming a triangle, a development of the type

$$\frac{A}{C} = \lambda_1\left(\frac{B}{C}\right)^{\alpha_1} + \lambda_2\left(\frac{B}{C}\right)^{\alpha_2} + \ldots,$$

in which $\alpha_1, \alpha_2 \ldots$ are positive and increasing, and α_1 is greater than unity, is termed a normal development; A is, of course, the tangent to the branch, B is any line passing through the point to which the development refers.

discriminant of the equation (1)* ; *i.e.*, to the expression

$$(13) \ldots\ldots \nabla(q) = \Pi[\phi^8(\Omega_1)-\phi^8(\Omega_2)]^2,$$

where the sign of multiplication extends to every pair of values of Ω.

Let $$(14) \ldots\ldots (A+B)^2 = A_2+B_2,$$

where A_2 comprises all the terms in $(A+B)^2$ which are greater than N, and B_2 all the terms which are less than N; as for the terms which are equal to N (of which the sum is evidently $2\nu N$), we divide them equally between A_2 and B_2; thus, if $g_1, g_2, \ldots g_t$ are all the divisors of N (unity and N included), we have

$$(15) \ldots\ldots \begin{cases} A_2 = \nu N + \sum\limits_{r=1}^{r=t}\sum\limits_{s=1}^{s=t} g_r g_s, & g_r g_s > N; \\ B_2 = \nu N + \sum\limits_{r=1}^{r=t}\sum\limits_{s=1}^{s=t} g_r g_s, & g_r g_s < N; \\ A_2 + B_2 = \sum\limits_{r=1}^{r=t}\sum\limits_{s=1}^{s=t} g_r g_s = (A+B)^2. \end{cases}$$

Applying the method of M. Kronecker, we find that the highest power of q in $\nabla(q)$ is $2A_2-A-B$, and that the lowest power of q is $2B_2-A-B$. By a known property of the modular equation,

$$F(1-p, 1-q, 1) = F(p, q, 1);$$

hence $\nabla(q)$ must be divisible by $1-q$ as often as by q; we have, therefore,

$$(16) \ldots\ldots \nabla(q) = (q-q^2)^{2B_2-A-B} \times \chi(q);$$

where $\chi(q)$ is a rational and integral function of q, not divisible by q or $1-q$, and of the order $2A_2-4B_2+(A+B)$.

By another property of the modular equation we have the identity

$$(pq)^{A+B} \times F\left(\frac{1}{p}, \frac{1}{q}, 1\right) = F(p, q, 1).$$

If therefore we write $p \div p'$ for p, and $q \div q'$ for q, the dialytic discriminant of the bipartite binary quantic

$$(p'q')^{A+B} \times F\left(\frac{p}{p'}, \frac{q}{q'}, 1\right)$$

is symmetric with regard to q and q'; *i.e.*, it is of the form

$$(17) \ldots\ldots [qq'(q'-q)]^{2B_2-A-B} \times \chi(q, q');$$

where $\chi(q, q')$ is symmetric and of the order $2A_2-4B_2+A+B$.

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* It has been applied by M. Koenigsberger (*Vorlesungen über die Theorie der Elliptischen Funktionen*, Vol. II., p. 154) to the discriminant of the modular equations between u and v, in the case in which N is not divisible by any square; the result had already been given by M. Hermite in his Mémoire sur la Théorie des Équations Modulaires (*Comptes Rendus*, Vol. xlviii., p. 1079.) The discriminant of the modular equation between $\chi^3(\omega)$ and $\chi^3(\Omega)$ has been similarly treated by M. Krause (*Math. Annalen*, Vol. xii., pp. 1—3).

It will be observed that the order of the discriminant (17) is

$$\begin{aligned} 3\,(2B_2-A-B)+2A_2-4B_2+A+B \\ = 2\,(A_2+B_2)-2\,(A+B) \\ = 2\,(A+B)\,(A+B-1), \end{aligned}$$

as it ought to be; and that the equation $\nabla(q)=0$ is to be regarded as having $2B_2-A-B$ infinite roots, and as having lost the same number of dimensions.

We may add that $\chi(q)$ is a perfect square. For, if $q-q_0$ be any factor of $\chi(q)$, all the developments of p, proceeding by powers of $q-q_0$, have integral exponents; the exponent of $q-q_0$ in $\chi(q)$ is the sum of the discriminantal indices of these developments taken in pairs; and this sum is always even.

Art. 7. *The discriminantal indices of P, Q, R, S.*—We next examine the discriminantal indices in the curve C of the points P, Q, R, S. Representing these indices by $D(P)$, $D(Q)$, $D(R)$, $D(S)$, we have

$$(18) \;\ldots\ldots \begin{cases} D(P) = 2A^2-A_2-A = D(Q), \\ D(R) = 2B_2-2A \quad = D(S). \end{cases}$$

To establish these formulæ, it will suffice to consider the points P and R. At P we have ν superlinear branches of the aggregate order $A-B$ touching PQ; we may symbolize the branch of which (11) is the normal development by (g', g), where $gg'=N$, and $g' > \sqrt{N} > g$. The discriminantal index of the branch (g', g) taken by itself is $g'(g'-g-1)$; the joint discriminantal index of the two branches (g', g) and (g'_1, g_1) is $2g'(g'_1-g_1)$, if $g' > g'_1$, and consequently $g < g_1$. Hence we have

$$(19) \;\ldots\ldots D(P) = \Sigma g'(g'-g-1) + 2\Sigma\Sigma g'(g'_1-g_1),$$

the summations extending to all values of g' and g'_1 which satisfy the inequalities $\quad g' > \sqrt{N}, \;\; g'_1 < g', \;\; g'_1 > \sqrt{N}.$

Attending to these inequalities, we find

$$\begin{aligned} \Sigma g'^2 + 2\Sigma\Sigma g'g'_1 = A^2, \\ \Sigma g' = A, \\ \Sigma g'g + 2\Sigma\Sigma g'g_1 = A_2 - A^2; \end{aligned}$$

and substituting these values in (19), we obtain the value of $D(P)$, given by the formula (18).

Again, at the point R, we first consider the branches touching PR, the normal developments of which are of the type (10). For the discriminantal index of these branches, taken singly and in pairs, we have the expression

$$(20) \;\ldots\ldots \Sigma g\,(g'-1) + 2\Sigma\Sigma g_1 g',$$

the summations extending to all divisors g and g_1 which satisfy the inequalities

$$(21) \;\ldots\ldots\; g > \sqrt{N}, \quad g_1 < g, \quad g_1 > \sqrt{N}.$$

The discriminantal index of the branches touching QR is evidently the same as that of the branches touching PR; and as the aggregate order of each of the two sets of branches is B, they intersect one another in B^2 points, and the part of $D(R)$ which arises from their crossing one another at R is $2B^2$. We have, therefore,

$$D(R) = 2\Sigma g(g'-1) + 4\Sigma\Sigma g_1 g' + 2B^2;$$

but, attending to the inequalities (21), we also have

$$\Sigma g = A, \quad \Sigma g'g + 2\Sigma g_1 g' = B_2 - B^2,$$

whence, in accordance with (18),

$$D(R) = 2B_2 - 2A.$$

Art. 8.—*The intersections with C of the polar curves of P and Q.*—Since the branches which touch PR at R are of the aggregate class $A-B$, the line PR, considered as a tangent drawn from P, counts $A-B$ times as a tangent at R. Similarly PS counts $A-B$ times as a tangent at S. Again, since the branches which touch PQ at P and at Q, are of the aggregate order $A-B$, and of the aggregate class B, PQ, considered as a tangent drawn from P, counts $A-B+B = A$ times as a tangent at P, and B times as a tangent at Q. Thus the three lines PR, PS, PQ count as

$$(A-B)+(A-B)+(A+B) = 3A-B$$

tangents from P; *i.e.*, no other tangents can be drawn to C from P. Again, the polar curve of P intersects C at P, Q, R, S, in

$$\begin{aligned} D(P)+A &= 2A^2-A_2, \\ D(Q)+B &= 2A^2-A_2+B-A, \\ D(R)+A-B &= 2B_2-B-A, \\ D(S)+A-B &= 2B_2-B-A, \end{aligned}$$

points respectively; or, in all, in $4A^2+4B_2-2A_2-3A-B$ points. The whole number of intersections of C by any one of its first polars is $2A(2A-1)$; hence the polar of P intersects C in

$$2A(2A-1)-(4A^2+4B_2-2A_2-3A-B) = 2A_2-4B_2+A+B$$

points, other than P, Q, R, S. These $2A_2-4B_2+A+B$ intersections correspond in the discriminant of $F(p, q, 1)$ to the factors of $\chi(q)$, of which the aggregate order is the same.

As the intersections other than those at P, Q, R, S correspond to the factor $\chi(q)$, so also the intersections at R correspond to the factor q^{2B_1-A-B}, and the intersections at S to the factor $(1-q)^{2B_1-A-B}$. But it is proper to observe that the remaining intersections at P and Q (of which the aggregate number is $4A^2-2A_2+B-A$) surpass the order of the remaining factor q'^{2B_1-A-B} of the complete dialytic discriminant; the difference

$$(4A^2-2A_2+B-A)-(2B_2-A-B)$$
$$=2A(2A-1)-2(A+B)(A+B-1)$$

being, as it ought to be, equal to the difference between the number of intersections of O by any one of its polars, and the order of the dialytic discriminant. We reserve, for a future communication to the Society, a complete discussion of the relations which subsist between the exponents of the factors of the dialytic discriminant of any parametric equation, and the corresponding intersections of the locus curve by the polars of the centres of the generating pencils.

The points, other than P, Q, R, S, in which O is intersected by the polar of P, are all ordinary double (or it may be multiple) points, free from superlinearity, and having tangents which do not pass through P. For O has no superlinear branches beside those at P, Q, R, S, and the only tangents which pass through P are PQ, PR, PS. The same thing is also evident from what has been said in Art. 6 of the exponents of the factors of $\chi(q)$.

Arts. 9—11. *N divisible by a square.*

Art. 9.—*Definition of certain Arithmetical Functions.*—We now pass to the general case in which N is any uneven number whatever. Let

$$N=a_1^{e_1}a_2^{e_2}a_3^{e_3}\ldots,$$

$a_1, a_2 \ldots$ being different uneven primes; let g, g', as before, be two conjugate divisors of N, and let η be the greatest common divisor of g and g'. We resolve g into the product of two factors γ_1 and γ_2, of which γ_1 contains only those prime divisors of g which do not occur in η and g'; and γ_2 contains only those prime divisors of g which do occur in η and g'. Representing by $f(z)$ the number of numbers prime to any given number z and not surpassing it, we write

$$f'(g)=\gamma_1 f(\gamma_2),$$

and we observe that we have the equations

$$\frac{f(\eta)}{\eta}=\frac{f'(g)}{g}=\frac{f'(g')}{g'};$$

each of these quotients being equal to

$$\Pi\left(1-\frac{1}{\epsilon}\right),$$

if ϵ denote any prime divisor of η. We still retain the symbols ν, A, B, A_2, B_2; but with extended significations, which we proceed to explain.

We define ν by the equation

$$(20) \ldots\ldots 2\nu = \Sigma f(\eta),$$

the sign of summation extending to every divisor g of N. We observe that, in general, each term $f(\eta)$ occurs twice in $\Sigma f(\eta)$, because η is the same for each of any two conjugate divisors; but that, if $N = \theta^2$ is a perfect square, the term $f(\theta) = f'(\theta)$ occurs only once in $\Sigma f(\eta)$.

Again, we define A and B by the equations

$$(21) \ldots\ldots A = \Sigma f'(g), \; g \geqq \sqrt{N},$$

$$(22) \ldots\ldots B = \Sigma f'(g), \; g \leqq \sqrt{N},$$

the summations extending to all divisors g of N which satisfy the inequalities (21) and (22) respectively; when $N = \theta^2$ is a perfect square, we divide the term $f'(\theta) = f(\theta)$ equally between A and B. We thus have in every case

$$A + B = \Sigma f'(g),$$

the summation extending to every divisor g of N.

Lastly, we define A_2 and B_2 by the equations

$$(23) \ldots\ldots A_2 = \Sigma\Sigma f'(g_1) f'(g_2), \; g_1 g_2 \geqq N,$$

$$(24) \ldots\ldots B_2 = \Sigma\Sigma f'(g_1) f'(g_2), \; g_1 g_2 \leqq N;$$

in which g_1 and g_2 are any two divisors of N (the same or different) which satisfy the inequalities specified; so that, if g_1 and g_2 are different, the term $f'(g_1) f'(g_2)$ occurs twice in A_2, or in B_2, as the case may be. If $g_1 g_2 = N$, we divide the double term $2f'(g_1) f'(g_2)$, corresponding to these two divisors, equally between A_2 and B_2; if, in particular, $N = \theta^2$ is a perfect square, the single term $[f(\theta)]^2$ is to be divided equally between A_2 and B_2. It is evident that we have, in every case,

$$A_2 + B_2 = (A + B)^2.$$

The sums $2\nu = \Sigma f(\eta)$, and $A + B = \Sigma f'(g)$, may be conveniently expressed in terms of the prime divisors of N. Observing (1) that the terms of the product

$$\Pi\,[1 + a + a^2 + \ldots a^e]$$

represent, after development, all the divisors of N, and (2) that

$$f'(g) = f'(h_1) \times f'(h_2),$$

if g be a product of two relatively prime factors h_1 and h_2, we find

$$\begin{aligned}\Sigma f'(g) &= \Pi\,[f'(1)+f'(a)+\ldots f'(a^s)] \\ &= \Pi\,[1+(a-1)+a(a-1)+\ldots+a^{s-2}(a-1)+a^s] \\ &= \Pi\,[a^{s-1}+a^s];\end{aligned}$$

whence (25) ...... $A+B = N\times\Pi\left(1+\frac{1}{a}\right)$.

Again, if we write $f''(g)$ for $f(\eta)$, and give to h_1, h_2 the same signification as before, $f''(g)$ satisfies the equation

$$f''(g) = f''(h_1)\times f''(h_2);$$

whence we infer that

$$\Sigma f(\eta) = \Pi\,[f''(1)+f''(a)+\ldots+f''(a^s)].$$

First, let $a = 2\mu+1$; then

$$\begin{aligned}&f''(1)+f''(a)+\ldots f''(a^{2\mu+1}) \\ &\quad= 2\,[1+(a-1)+a(a-1)+\ldots+a^{\mu-1}(a-1)] = 2a^{\mu}.\end{aligned}$$

Secondly, let $a = 2\mu$; then

$$\begin{aligned}&f''(1)+f''(a)+\ldots+f''(a^{2\mu}) \\ &\quad= 2\,[1+(a-1)+a(a-1)+\ldots+a^{\mu-2}(a-1)]+a^{\mu-1}(a-1) \\ &\quad= a^{\mu}+a^{\mu-1}.\end{aligned}$$

If therefore $N = \Pi b^{2\beta+1}\times\Pi c^{2\gamma}$, where b, ..., c, ..., are different prime numbers, we have

$$(26)\ \ldots\ldots\ \Sigma f(\eta) = 2\nu = \Pi\, 2b^{\beta}\times\Pi c^{\gamma}\left(1+\frac{1}{c}\right).$$

It will be observed that the definitions which we have now given of the symbols ν, A, B, A_2, B_2, coincide, in the case in which N is not divisible by any square, with the definitions of Arts. 3 and 6.

Art. 10. *Case when N is not a square.*—Excluding, for the present, the case in which N is a perfect square, we have to show that, in all other cases, the formulæ of Arts. 3—8 hold without further modification. For brevity, we shall establish only a few of the assertions included in this general statement, as the method to be pursued with regard to all of them is the same.

(i.) If $\Phi(N)$ is the sum of the divisors of N, and e_1, e_2, ... are the primes of which the squares divide N, the order of the irreducible modular equation of order N is

$$\Phi(N)-\Sigma\Phi\left(\frac{N}{e^2}\right)+\Sigma\Phi\left(\frac{N}{e_1^2\,e_2^2}\right)-\ldots$$

(See M. Joubert, *Comptes Rendus*, Vol. l., p, 1041; Report on the

Theory of Numbers, *loc. cit.*, p. 332). But this expression has for its value $N\Pi\left(1+\frac{1}{a}\right)$; *i.e.*, the order of the modular equation in p or in q is $A+B$ (equation 25).

(ii.) If we write $q=\phi^8(\omega)$, the $A+B$ corresponding values of p are given by the equation

$$p=\phi^8\left(\frac{g'\omega+2k}{g}\right),$$

in which k is any one of the $\frac{f(\eta)}{\eta}g=f'(g)$ residues of g which are prime to η, the greatest common divisor of g and g'. If, as in Art. 4, we cause q to describe a closed contour round 0, the $f'(g)$ values of p, which answer to any given divisor g, arrange themselves in $f(\eta)$ cycles each containing $\frac{g}{\eta}$ roots; and thus the developments (5), which appertain to the simultaneous values $p=0$, $q=0$, assume, in the general case which we are now considering, the form

$$(27) \ldots\ldots\ p=\lambda\,(q)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots;$$

the least common denominator of the exponents being $\frac{g}{\eta}$. It will be observed that there are $f(\eta)$ developments, in which g and g' have the same values; the coefficients λ having different values in these $f(\eta)$ developments. Similarly, there are $\Sigma f(\eta)$ developments of each of the types

$$(28) \ldots\ldots\ (p-1)=\lambda\,(q-1)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots,$$

$$(29) \ldots\ldots\ \frac{1}{p}=\lambda\left(\frac{1}{q}\right)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots.$$

Hence $\quad W(p)=3\Sigma f(\eta)\left[\frac{g}{\eta}-1\right]$

$$=3\Sigma f'(g)-3\Sigma f(\eta)=3\,(A+B)-6\nu;$$

and, consequently, as in Art. 4,

$$H=\tfrac{1}{2}(A+B)-3\nu+1.$$

(iii.) From the developments (27), (28), (29), we can deduce the normal developments of the six sets of branches which touch PR and QR at R, PS and QS at S, PQ at P and Q. Each set comprises ν branches; if h^2 is the greatest square dividing N, h of these are linear in each of the first four sets; all the rest are superlinear. It will suffice here to determine the cuspidal and discriminantal indices of the branches touching PQ at P. The normal developments of these branches are of the type

$$(30) \ldots\ldots \left\{ \begin{array}{l} \dfrac{\gamma}{\alpha} = \mu \left(\dfrac{\beta}{\alpha} \right)^{\frac{g'}{\eta} \div \frac{g'-g}{\eta}} + \ldots \\ \qquad\qquad g' > g. \end{array} \right.$$

Hence their aggregate cuspidal index is

$$\Sigma f(\eta) \left(\frac{g'}{\eta} - \frac{g}{\eta} - 1 \right), \quad g < \sqrt{N};$$

or, which is the same thing,

$$\overset{g > \sqrt{N},}{\Sigma f'(g)} - \overset{g < \sqrt{N},}{\Sigma f'(g)} - \overset{g < \sqrt{N},}{\Sigma f(\eta)} = A - B - \nu.$$

To obtain the discriminantal index, we first consider a single group of $f(\eta)$ branches corresponding to given values of g, g'. The discriminantal index of one of these branches, taken by itself, is

$$\frac{g'}{\eta} \left(\frac{g'}{\eta} - \frac{g}{\eta} - 1 \right);$$

the joint discriminantal index of two different branches of the group is

$$2 \frac{g'}{\eta} \left(\frac{g'}{\eta} - \frac{g}{\eta} \right);$$

so that the aggregate discriminantal index of the group (g', g) is

$$f(\eta) \times \frac{g'}{\eta} \left(\frac{g'}{\eta} - \frac{g}{\eta} - 1 \right) + f(\eta) \left[f(\eta) - 1 \right] \times \frac{g'}{\eta} \left(\frac{g'}{\eta} - \frac{g}{\eta} \right)$$
$$= f'(g') \left[f'(g') - f'(g) - 1 \right].$$

We next consider the two groups (g', g) and (g_1', g_1) consisting respectively of $f(\eta)$ and $f(\eta_1)$ branches. If $g' > g_1'$, or, which is the same thing, if $g' g_1 > N$, the joint discriminantal index of the two groups

is $$2 f(\eta) f(\eta_1) \times \frac{g'}{\eta} \left(\frac{g_1'}{\eta_1} - \frac{g_1}{\eta_1} \right) = 2 f'(g') \left[f'(g_1') - f'(g_1) \right].$$

Thus the aggregate discriminantal index of the branches touching PQ at P is given by the equation

$$D(P) = \Sigma f'(g') \left[f'(g_1') - f'(g) - 1 \right]$$
$$+ 2 \Sigma\Sigma f'(g') \left[f'(g_1') - f'(g_1) \right],$$

the summations extending to all values of g' and g_1' which satisfy the inequalities $\quad g' > \sqrt{N}, \; g_1' < g', \; g_1' > \sqrt{N}.$

But, as in Art. 7,

$$\Sigma\,[f'(g')]^2+2\Sigma\Sigma f'(g')\,f'(g'_1)=A^2,\quad \Sigma f'(g')=A,$$
$$\Sigma f'(g')\,f'(g)+2\Sigma\Sigma f'(g')\,f'(g_1)=A_2-A^2;$$

whence, as before, $\qquad D(P)=2A^2-A_2-A.$

Art. 11.—*Case when N is a square.*—The case in which $N=\theta^2$ is a perfect square requires separate consideration, because the modular curve of order θ^2 meets the line PQ in $f(\theta)$ points distinct from one another and from P and Q; and again, at each of the points R and S, it has $f(\theta)$ linear branches, of which the tangents are different from one another, and from the lines RP, RQ, SP, SQ. Thus some of the characteristics of the singularities at $PQRS$ are changed; and with them some of the characteristic indices of the curve.

We write θ' for $f(\theta)=f'(\theta)$. It will be found, on referring to Art. 10, i. and ii., that

$$E(p)=E(q)=A+B,\quad m=2A,$$
$$H=\tfrac{1}{2}(A+B)-3\nu+1,$$

as in the case when N is not a square. Again, the cuspidal index of each of the four sets of branches which touch PR and QR at R, PS and QS at S, is, as before, $B-\nu$; but the cuspidal index of the branches at P and Q is $A-B-\nu+\frac{1}{2}\theta'$ instead of $A-B-\nu$. For this index is

$$\underset{g>\sqrt{N},}{\Sigma f'(g')}-\underset{g<\sqrt{N},}{\Sigma f'(g)}-\underset{g<\sqrt{N},}{\Sigma f(\eta)},$$

(see Art. 10, iii.); and

$$\overset{g>\sqrt{N}}{\Sigma f'(g)}=A-\tfrac{1}{2}\theta',$$
$$\overset{g<\sqrt{N}}{\Sigma f'(g)}=B-\tfrac{1}{2}\theta',$$
$$\overset{g<\sqrt{N}}{\Sigma f(\eta)}=\nu-\tfrac{1}{2}\theta'.$$

To find the discriminantal indices of $PQRS$, we denote by $\overline{A}$, $\overline{B}$, $\overline{A}_2$, $\overline{B}_2$ the numbers obtained by omitting in ABA_2B_2 the terms depending on θ; we thus have

$$\overline{A}=A-\tfrac{1}{2}\theta',\quad \overline{B}=B-\tfrac{1}{2}\theta',$$
$$\overline{A}_2=A_2-2\theta'A+\tfrac{1}{2}\theta'^2,$$
$$\overline{B}_2=B_2-2\theta'B+\tfrac{1}{2}\theta'^2.$$

Using these expressions, we find, as in Art. 10, iii.,

$$D(P)=D(Q)=2\overline{A}^2-\overline{A}_2-\overline{A};$$

or, substituting for $\overline{A}$ and $\overline{A}_2$ their values,

$$D(P)=D(Q)=2A^2-A_2-A+\tfrac{1}{2}\theta'.$$

To determine $D(R)$, we have:—(i.) For the discriminantal index of the set of branches touching either PR or QR, $\overline{B}_2 - \overline{A} - \overline{B}^2$; (ii.) for the joint discriminantal index of these two sets of branches, $2\overline{B}^2$; (iii.) for the joint discriminantal index of the θ' linear branches, $\theta'(\theta'-1)$; (iv.) for the joint discriminantal index of the linear branches taken with the branches touching either PR or QR, $2\theta'\overline{B}$.

$$\text{Hence}\quad D(R) = D(S) = 2[\overline{B}_2 - \overline{A} - \overline{B}^2] + 2\overline{B}^2 + \theta'(\theta'-1) + 4\theta'\overline{B}$$
$$= 2(B_2 - A),$$

as in the case when N is not a square.

There is no change in the expressions for the order of $\nabla(q)$, and for the exponents of the factors q and $1-q$ in $\nabla(q)$ (see Art. 6); and these expressions agree with the values which we have obtained for $D(R) = D(S)$, and for $D(P) = D(Q)$. For PR, touching the curve at R, counts as $A-B$ tangents drawn from P; and hence the order of q in the discriminant ought to be, what in fact it is,

$$2(B_2 - A) + (A - B) = 2B_2 - A - B.$$

And again, PQ, considered as drawn from P, counts as $A - \frac{1}{2}\theta'$ tangents at P, and as $B - \frac{1}{2}\theta'$ tangents at Q. Thus, the number of intersections of C by the polar of P, which lie on the line PQ, is

$$2(2A^2 - A_2 - A + \tfrac{1}{2}\theta') + (A - \tfrac{1}{2}\theta') + (B - \tfrac{1}{2}\theta') = 4A^2 - 2A_2 + B - A;$$

and this number is, as it ought to be, the excess of $2A(2A-1)$ above the order of $\nabla(q)$; *i.e.*, the excess of the whole number of intersections above the intersections lying on PQ.

ARTS. 12—14. *Formulæ applicable to all values of N.*

Art. 12.—If, in the formulæ relating to the case when N is a square, we omit the terms containing the symbol θ' defined by the equation

$$\theta' = f(\theta) = f'(\theta') = f(\sqrt{N}),$$

we obtain the corresponding formulæ for the case when N is not a square. We shall henceforward denote by θ' a number which is equal to zero when N is not a square, and which is equal to $f(\sqrt{N})$ when N is a square; and we shall treat the two cases simultaneously, except when it is necessary to call attention to the difference between them.

Art. 13.—*The discriminantal class of the superlinear branches.*—In the paper on the Higher Singularities of Plane Curves* (Arts. 12 and

* Proceedings of the Society, Vol. vi., p. 153.

13), it has been shown that, if d and t are the order and class of a superlinear branch, D and T its discriminantal order and class, we have the equation $$T-D = t^2-d^2.$$
And again, that if there be a second superlinear branch of the order d' and class t' touching the first, and if we represent by $\overline{T}$ and $\overline{D}$ the joint discriminantal indices (of order and class) appertaining to the two branches, we have the equation

$$\overline{T}-\overline{D} = 2\,(tt'-dd').$$

Combining these two results, we obtain the theorem—

"If any number of branches touch one another at the same point, the difference between the discriminantal order and class of the singularity is equal to the difference between the squares of its order and of its class."

Employing a notation explained in Art. 14, we apply this theorem to determine the discriminantal class of the branches (PPQ), (PQQ), (PRR), (QRR), (PSS), (QSS). We thus find

$$(31) \;\ldots\ldots\; T\,(PPQ)-D\,(PPQ) = T\,(PQQ)-D\,(PQQ) = (B-\tfrac{1}{2}\theta')^2-(A-B)^2.$$

$$(32) \;\ldots\ldots\; T\,(PRR)-D\,(PRR) = T\,(QRR)-D\,(QRR) = T\,(PSS)-D\,(PSS) = T\,(QSS)-D\,(QSS) = (A-B)^2-(B-\tfrac{1}{2}\theta')^2;$$

so that
$$T\,(PPQ) = T\,(PQQ) = B_2-B^2-A-\theta'B+\tfrac{1}{2}\theta'+\tfrac{1}{4}\theta'^2 = \overline{B}_2-\overline{B}^2-\overline{A} = D\,(PRR) = D\,(QRR) = D\,(PSS) = D\,(QSS),$$
and
$$T\,(PRR) = T\,(QRR) = T\,(PSS) = T\,(QSS) = 2A^2-A_2-A+\tfrac{1}{2}\theta' = 2\overline{A}^2-\overline{A}_2-\overline{A} = D\,(PQQ) = D\,(PPQ).$$

Art. 14.—*Summary of the results.*—For convenience of reference, we exhibit the preceding results in a tabular form.

Characteristics and Singularities of the Modular Curve C.

I. *Explanation of the symbols.*—

(1) The order of the transformation is the uneven number N.

(2) g and g' are conjugate divisors of N; h^2 is the greatest square dividing N.

(3) η is the greatest common divisor of g and g'.

(4) $f\,(\eta)$ is the number of numbers not surpassing η and prime to it.

(5) $f'(g)$ and $f'(g')$ are defined by the equation

$$\frac{f'(g)}{g} = \frac{f(\eta)}{\eta} = \frac{f'(g')}{g'}.$$

(6) $2\nu = \Sigma f(\eta)$, $A+B = \Sigma f'(g)$, $A_2+B_2 = (A+B)^2$.

In these equations the summations Σ extend to all divisors g of N; A comprehends all the terms $f'(g)$ in which $g > \sqrt{N}$, and also, if $N=\theta^2$, the term $\frac{1}{2}\theta' = \frac{1}{2}f(\theta)$; A_2 comprehends all the terms of $\Sigma f'(g_1) \times \Sigma f'(g_2)$, in which $g_1 g_2 > N$, and one-half of every term in which $g_1 g_2 = N$, g_1 and g_2 denoting any two divisors of N, the same or different. The definitions of B and B_2 follow from those of A and A_2.

(7) m, n, K, I, D, T, H denote respectively the order, the class, the cuspidal index, the inflexional index, the discriminantal order, the discriminantal class, and the deficiency of the curve.

(8) The symbol (XXY) or (YXX) denotes a branch, or an aggregate of branches, touching the line XY at the point X.

(9) The symbols $O(XXY)$, $C(XXY)$, $K(XXY)$, $I(XXY)$, $D(XXY)$, $T(XXY)$ denote the order, class, cuspidal index, inflexional index, discriminantal order, discriminantal class of the branches (XXY). The symbols $O(X)$, $K(X)$, $D(X)$, $C(XY)$, $I(XY)$, $T(XY)$ are to be similarly interpreted with regard to the branches which pass through a given point X or touch a given line XY. Lastly, the symbols $D(XXY, XXZ)$ and $T(XXY, XYY)$ denote respectively twice the number of points common to the branches (XXY), (XXZ), and twice the number of tangents common to the branches (XXY), (XYY).

II. *Characteristics of the Curve.**

$$m = 2A, \quad n = 3A-B-\theta',$$
$$H = \tfrac{1}{2}(A+B)-3\nu+1,$$
$$K = 2(A+B)-6\nu+\theta',$$
$$I = 5A-B-6\nu-2\theta',$$
$$I-K = 3A-3B-3\theta',$$
$$D = 4A^2-5A+B+\theta',$$
$$T = (3A-B-\theta')^2-5A+B+\theta',$$
$$T-D = (3A-B-\theta')^2-4A^2.$$

III. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPQ) *and* (PQQ).

$$O(PPQ) = A-B; \quad C(PPQ) = B-\tfrac{1}{2}\theta',$$
$$K(PPQ) = A-B-\nu+\tfrac{1}{2}\theta', \quad I(PPQ) = B-\nu,$$
$$D(PPQ) = 2A^2-A_2-A+\tfrac{1}{2}\theta',$$
$$T(PPQ) = B_2-B^2-A-\theta'B+\tfrac{3}{4}\theta'+\tfrac{1}{4}\theta'^2.$$

* Several of the formulæ which follow may be more simply expressed by using the symbols $\overline{A}$, $\overline{B}$, $\overline{A}_2$, $\overline{B}_2$ of Art. 11, and by writing $\overline{\nu} = \nu-\frac{1}{2}\theta'$.

The number of distinct branches is $\nu-\frac{1}{2}\theta'$. They are all superlinear; viz., corresponding to every divisor g of N, which is less than $\sqrt{N}$, there are in (PPQ) $f(\eta)$ superlinear branches, each of the order $\frac{g'-g}{\eta}$, and of the class $\frac{g}{\eta}$; (PQQ) is of the same type as (PPQ).

(ii.) *Characteristics of* (PRR), (QRR), (PSS), (QSS).

All these singularities are of the same type.

$$
\begin{aligned}
O(PRR) &= B-\tfrac{1}{2}\theta'; \quad C(PRR) = A-B,\\
K(PRR) &= B-\nu,\\
I(PRR) &= A-B-\nu+\tfrac{1}{2}\theta',\\
D(PRR) &= B_2-B^2-A-\theta' B+\tfrac{1}{2}\theta'+\tfrac{1}{4}\theta'^2,\\
T(PRR) &= 2A^2-A_2-A+\tfrac{1}{2}\theta'.
\end{aligned}
$$

The number of distinct branches in (PRR) is $\nu-\frac{1}{2}\theta'$; of these, $h-\theta'$ are linear (Art. 10, iii.); the characteristics of (PRR) and (PPQ) are reciprocal; viz., corresponding to any divisor g of N, which is less than $\sqrt{N}$, there are in (PRR) $f(\eta)$ branches, each of the order $\frac{g}{\eta}$ and of the class $\frac{g'-g}{\eta}$.

(iii.) *Characteristics of* (PQ).

$$
\begin{aligned}
C(PQ) &= 2C(PPQ) = 2B-\theta';\\
I(PQ) &= 2I(PPQ) = 2B-2\nu,\\
T(PQ) &= T(PPQ)+T(PQQ)+T(PPQ, PQQ)\\
&= 2T(PPQ)+2(B-\tfrac{1}{2}\theta')^2\\
&= 2B_2-2A-4\theta' B+\theta'(\theta'+1).
\end{aligned}
$$

(iv.) *Characteristics of* (R) *and* (S).

These are the same for the two points.

$$
\begin{aligned}
(1) \ldots\ldots\ O(R) &= O(PRR)+O(QRR)+O(\theta)\\
&= 2B.\\
(2) \ldots\ldots\ K(R) &= K(PRR)+K(QRR)\\
&= 2B-2\nu.\\
(3) \ldots\ldots\ D(R) &= D(PRR)+D(QRR)+D(PRR, QRR)\\
&\qquad +D(\theta)+D(\theta, PRR)+D(\theta, QRR)\\
&= 2[B_2-B^2-A-\theta' B+\tfrac{1}{2}\theta'+\tfrac{1}{4}\theta'^2]+2(B-\tfrac{1}{2}\theta')^2\\
&\qquad +\theta'(\theta'-1)+4\theta'(B-\tfrac{1}{2}\theta')\\
&= 2(B_2-A).
\end{aligned}
$$

The symbol (θ) is used to represent the θ' linear branches which pass through R, having tangents distinct from one another and from PR, QR.

(v.) *Tangents to the Curve from PQRS.*

(1) PQ, considered as drawn from P, counts as $A-\frac{1}{2}\theta'$ tangents at P, and as $B-\frac{1}{2}\theta'$ tangents at Q; PR counts as $A-B$ tangents at R; thus PQ. PR, PS count as $3A-B-\theta'=n$ tangents drawn from P.

(2) The tangents to the branches (PRR), (QRR), and (θ) count as $2(A-\frac{1}{2}\theta')+2\theta'=2A+\theta'$ tangents drawn from R. Thus, there are $A-B-2\theta'$ other tangents which can be drawn to the curve from R.*

(vi.) *Intersections with the Curve of the sides of the quadrangle PQRS.*

(1) PQ meets the curve in $A-\frac{1}{2}\theta'$ points at P, and in as many at Q; and in θ' non-singular points distinct from either P or Q.

(2) PR meets the curve in $A-B$ points at P; at R it meets the branches (PRR) in $A-\frac{1}{2}\theta'$ points; the branches (QRR) in $B-\frac{1}{2}\theta'$ points; the branches (θ) in θ' points; in all in $2A$ points. The same statements hold, *mutatis mutandis*, for the lines QR, PS, QS.

(3) RS meets the curve $2(B-\frac{1}{2}\theta')+\theta'=2B$ points at R, and in as many at S; and also in $2(A-2B)$ other points.†

IV. *Residual singularities of the Curve.*

Designating by $K_1I_1D_1T_1$ the parts of the indices $KIDT$ which arise from the singularities connected with the points and lines of the quadrangle $PQRS$, we find, from the preceding formulæ,

$$\begin{cases} K_1 = 2(A+B)-6\nu+\theta', \\ I_1 = 4A-2B-6\nu+2\theta', \\ D_1 = 4A^2+4B_2-2A_2-6A+\theta', \\ T_1 = 8A^2+2B_2-4A_2-6A-4\theta'B+3\theta'+\theta'^2; \end{cases}$$

and for the residual singularities we have

$$\begin{cases} K_2 = 0, \\ I_2 = A+B-4\theta', \\ D_2 = 2A_2-4B_2+A+B, \\ T_2 = 4A^2+4B^2+A_2-5B_2+6\theta'(B-A)+A+B-2\theta'. \end{cases}$$

---

* If $\chi\left(\frac{1}{M}, q\right)=0$ represent the equation of the multiplier, which is of the order $A+B$ in $\frac{1}{M}$, and of the order $\frac{1}{2}(A-B)$ in q, the values of q appertaining to the points of contact of these tangents are determined by the equations

$$\chi\left(\sqrt{n}, \frac{q}{q-1}\right)=0, \quad \chi\left(-\sqrt{n}, \frac{q}{q-1}\right)=0;$$

when $N=\theta'^2$, the first of these equations has θ' roots equal to zero, and θ' infinite roots; both these sets of roots are to be rejected.

† At each of these points we have $p=q$. The equation $F(p, p, 1)=0$ is divisible by $[p(p-1)]^{2B}$; the remaining roots, which are $2A-4B$ in number, give the intersections of the curve by RS at points other than R and S. These roots may be determined by the method (due to M. Kronecker) described in the Report on the Theory of Numbers, Arts. 131—133.

ARTS. 15, 16.—*Case when N is a square. The Linear Branches* (θ).

Art. 15. The developments appertaining to the θ' linear branches U, which intersect PQ at points other than P and Q, are

$$(33) \ldots\ldots -\frac{1}{p} = \frac{e^{iu}}{q} + \frac{e^{iu}(1+e^{iu})}{2q^2} + \frac{e^{iu}(1+e^{iu})(21+11e^{iu})}{64q^3} + \ldots\ldots,$$

where $u = \frac{2l+1}{\theta}\pi$, $2l+1$ being any term of a system of residues prime to 2θ. We hence obtain the normal developments

$$(34) \ldots\ldots \frac{\beta-\frac{1}{2}\gamma+e^{iu}(\alpha-\frac{1}{2}\gamma)}{\beta} = 5(1-e^{2iu})\frac{\gamma^2}{\beta^2}+\ldots\ldots,$$

so that the tangents of the θ' branches are the lines

$$\beta-\tfrac{1}{2}\gamma+e^{iu}(\alpha-\tfrac{1}{2}\gamma) = 0,$$

which meet one another in the point $\alpha=\beta=\frac{1}{2}\gamma$; *i.e.*, in the point O in which RS intersects ab.

The developments appertaining to the θ' linear branches at R are

$$(35) \ldots\ldots p = e^{iv}q+\tfrac{1}{2}e^{iv}(1-e^{v})q^2+\ldots\ldots,$$

where $v = \frac{2h}{\theta}\pi$, h being any term of a system of residues prime to θ;

so that the tangents are $\qquad \alpha - e^{v}\beta = 0,$

none of the branches being inflected at R.

Similarly the developments appertaining to the θ' linear branches at S are

$$(36) \ldots\ldots p-1 = e^{iv}(q-1)+\tfrac{1}{2}e^{iv}(1-e^{iv})(q-1)^2+\ldots\ldots,$$

and the tangents are $\qquad \alpha-\gamma = e^{iv}(\beta-\gamma),$

there being no inflexion.

The two sets of tangents at R and S meet PQ in the same points in which it is intersected by the linear branches U; for, if

$$2l+1+2h = (2k+1)\theta,$$

we have $u+v = (2k+1)\pi$, whence $e^{iv} = -e^{-iu}$.

Art. 16. The developments (33—36) may be obtained as follows, with the help of formulæ established in the Report on the Theory of Numbers already cited.

If $\omega = 1+\frac{i}{\sigma}$, where σ is positive and increases without limit, $q=\phi^8(\omega) = 1-\phi^{-8}(i\sigma)$ increases without limit; and the limit of $q \div 16e^{\pi\sigma}$ is unity. The corresponding values of p are comprised in the formula

$$\phi^8\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right),$$

where g and g' are conjugate divisors of N, and g, g', k have no common divisor. But this expression may be exhibited in a form from which the dimensions of $\phi(\Omega)$, as compared with $\phi(\omega)$, may be inferred; viz., we have (Report, *loc. cit.*, p. 350.)

$$\phi^8\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right)=\phi^{-8}\left(\frac{d'i\sigma+2l+1}{d}\right),$$

where d' is the greatest common divisor of $g'+2k$ and g, $d=\frac{N}{d'}$, and $2l+1$ is determined by a certain congruence for the modulus d. In order that the development of $\frac{1}{p}$, in a series proceeding by powers of $\frac{1}{q}$, should correspond to a branch intersecting PQ elsewhere than at P or Q, $\frac{1}{p}$ and $\frac{1}{q}$ must be of the same dimensions. But

$$\text{Lim.}\, p \div e^{\frac{d'}{d}\pi\sigma} = 1, \quad \text{Lim.}\, q \div e^{\pi\sigma} = 1;$$

hence $d'=d$, or N is necessarily a square, and $d=d'=\theta$. Since $\theta=d'$ is the greatest common divisor of $g'+2k$ and g, let $g=\lambda\theta$, $g'=\frac{\theta}{\lambda}$; then θ divides $\frac{\theta}{\lambda}+2k$; *i.e.*, $\frac{\theta}{\lambda}$ divides g, g', and $2k$, which are relatively prime. Hence $\lambda=\theta$, $g=\theta^2=N$, $g'=1$. Now there are just θ' values of $2k$ for which θ is the greatest common divisor of θ^2 and $1+2k$; viz., if $2\mu+1$ be any number less than 2θ and prime to θ, the θ' values of $2k$ are included in the formula $2k=(2\mu+1)\,\theta-1$; and it will be found that the congruence determining $2l+1$ is $(2\mu+1)(2l+1)\equiv -1$, mod. θ. Hence we have, for the θ' values of $2k$ which we are considering,

$$\phi^8\left(\frac{\frac{i}{\sigma}+1+2k}{N}\right)=\phi^{-8}\left(i\sigma+\frac{2l+1}{\theta}\right)=\phi^{-8}\left(i\sigma+\frac{u}{\pi}\right).$$

Expanding the values of

$$\frac{1}{q}=-\frac{\phi^8(i\sigma)}{1-\phi^8(i\sigma)} \quad \text{and of } \frac{1}{p}=\phi^8\left(i\sigma+\frac{u}{\pi}\right),$$

by means of the formula

$$(37) \;\ldots\ldots\; \phi^8(\omega)=16e^{i\pi\omega}(1-8e^{i\pi\omega}+44e^{2i\pi\omega}-\ldots\ldots),$$

which arises from the expansion of (3); and equating the coefficients of like powers of $e^{-\pi\sigma}$ in the series

$$\frac{1}{p}=\frac{A}{q}+\frac{B}{q^2}+\frac{C}{q^3}+\ldots\ldots$$

we obtain the developments (33).

Similarly the developments (35), which appertain to the linear branches at R, may be obtained by substituting the expansions of $p = \phi^8\left(i\sigma + \frac{v}{\pi}\right)$ and $q = \phi^8(i\sigma)$ in the assumed series

$$p = Aq + Bq^2 + \ldots\ldots$$

ARTS. 17—19. *The Six Modulur Curves.*

Art. 17. If we represent by $\epsilon(x)$ any one of the six anharmonic functions

$$(38) \ldots\ldots x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x},$$

the modular equation (1) is unchanged by the simultaneous substitution of $\epsilon(p)$ for p and $\epsilon(q)$ for q. Hence, if $\epsilon_1(x)$, $\epsilon_2(x)$ denote any two, the same or different, of the functions (38), the thirty-six substitutions

$$(39) \ldots\ldots F[\epsilon_1(p), \; \epsilon_2(q), \; 1]$$

give only six different equations. As representatives of these, we take the following

$$\text{(i.)} \ldots\ldots F(1-p, \; q, \; 1) = 0,$$

$$\text{(ii.)} \ldots\ldots F\left(p, \; \frac{1}{q}, \; 1\right) = 0,$$

$$\text{(iii.)} \ldots\ldots F\left(1-p, \; \frac{1}{1-q}, \; 1\right) = 0,$$

$$\text{(iv.)} \ldots\ldots F(p, \; q, \; 1) = 0,$$

$$\text{(v.)} \ldots\ldots F\left(1-p, \; \frac{1}{q}, \; 1\right) = 0,$$

$$\text{(vi.)} \ldots\ldots F\left(p, \; \frac{1}{1-q}, \; 1\right) = 0.$$

The equation $F(p, q, 1) = 0$ is symmetric with regard to p and q; and it will be found that the equations (i.), (ii.), (iii.) possess the same property; thus, for example, the equations

$$F\left(p, \; \frac{1}{q}, \; 1\right) = 0, \quad \text{and} \quad F\left(q, \; \frac{1}{p}, \; 1\right) = 0,$$

are the same, because

$$F(x, y, 1) = F(y, x, 1) = (xy)^{A+B} F\left(\frac{1}{x}, \; \frac{1}{y}, \; 1\right).$$

The fifth and sixth equations, on the other hand, are changed, each into the other, by the interchange of p and q.

Art. 18. Denoting by X and Y rectangular Cartesian coordinates, and writing in the equations (i.) ... (vi.),

$$(40) \ldots\ldots \begin{cases} p = \tfrac{1}{2} + X - iY \\ q = \tfrac{1}{2} + X + iY, \end{cases}$$

we obtain the equations of six curves, which, in the Mémoire *Sur les Équations Modulaires*, we have called the first, second, third, fourth, fifth, and sixth modular curves. The equations of the first four of these curves are real, as appears from the symmetry of the equations (i.)—(iv.) with regard to p and q; the equations of the fifth and sixth curves are imaginary and conjugate to one another.

The first and fourth curves are each of them symmetric with regard to both axes; the fourth curve is its own inverse (anallagmatic) with regard to each of the two real circles

$$(X \pm \tfrac{1}{2})^2 + Y^2 = 1;$$

and the first curve with regard to each of the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2}i)^2 = -1.$$

The second and third curves are symmetric with regard to the axis of X, and symmetric to one another with regard to the axis of Y; the fifth and sixth (imaginary) curves are symmetric with regard to the axis of Y, and symmetric to one another with regard to the axis of X. The second and third curves are the inverses of the first, with regard to the circles $(X-\tfrac{1}{2})^2+Y^2=1$, $(X+\tfrac{1}{2})^2+Y^2=1$, respectively; similarly the two imaginary curves are the inverses of the fourth curve with regard to the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2}i)^2 = -1.$$

Lastly, the substitution $X = iY'$, $Y = -iX'$ changes the first curve into the fourth, the second into the fifth, the third into the sixth, and *vice versâ*.

These assertions are the geometrical equivalents of the properties of the modular equation stated in Art. 17; it will suffice to verify one of them. The equation of the first modular curve is

$$F(\tfrac{1}{2}-X+iY,\ \tfrac{1}{2}+X+iY,\ 1) = 0;$$

its inverse with regard to the circle $(X+\tfrac{1}{2})^2+Y^2=1$ is obtained by writing

$$\tfrac{1}{2}-X+iY = \frac{1}{\tfrac{1}{2}-X'-iY'},$$

$$\tfrac{1}{2}-X-iY = \frac{1}{\tfrac{1}{2}-X'+iY'},$$

so that

$$\tfrac{1}{2}+X+iY = 1 - \frac{1}{\tfrac{1}{2}-X'+iY'}.$$

The equation of the inverse curve is therefore

$$F\left(\frac{1}{\tfrac{1}{2}-X'-iY'},\ 1-\frac{1}{\tfrac{1}{2}-X'+iY'},\ 1\right)=0;$$

and this is identical with the equation

$$F\left(\tfrac{1}{2}+X'-iY',\ \frac{1}{\tfrac{1}{2}+X'+iY'},\ 1\right)=0;$$

i.e., with the equation of the second modular curve, because

$$F\left(\frac{1}{1-y},\ 1-\frac{1}{1-x},\ 1\right)=0$$

is identical with $$F\left(x,\ \frac{1}{y},\ 1\right)=0.$$

Art. 19. The equations of the first and fourth modular curves are included in the general equation

$$F(\alpha, \beta, \gamma) = 0;$$

viz., to obtain the first curve, we write

$$\alpha=\tfrac{1}{2}-X+iY,\quad \beta=\tfrac{1}{2}+X+iY,\quad \gamma=1;$$

and, to obtain the fourth curve, we write

$$\alpha=\tfrac{1}{2}+X-iY,\quad \beta=\tfrac{1}{2}+X+iY,\quad \gamma=1.$$

Thus the theory of the singularities of these two curves is implicitly contained in the preceding discussion of the singularities of C.

In both curves the points P, Q are the cyclic points, and (ab, RS) or O is the origin: in the first curve ab and RS are the axes of X and Y; a, b being the points $(\pm\tfrac{1}{2}, 0)$, and R, S the points $(0, \pm\tfrac{1}{2}i)$: c is the point at an infinite distance on the axis of Y; in the fourth curve R, S are the points $(\pm\tfrac{1}{2}, 0)$, and a, b the points $(0, \pm\tfrac{1}{2}i)$, c being the point at an infinite distance on the axis of X. Both equations (as has been already said) are real; and it follows, from the theory explained in the Mémoire cited, that both of them represent real curves, except when $N\equiv 3$, mod. 4; in which case the fourth curve reduces itself to the pair of conjugate points $(\pm\tfrac{1}{2}, 0)$.

When N is not a square, both curves are completely and parabolically cyclic, having at each cyclic point ν branches, of the aggregate order $A-B$ and class B, touching the line at an infinite distance.

When N is a square, each of the two curves has θ' real infinite branches. The fourth curve has also θ' real branches passing through each of the points $(\pm\tfrac{1}{2}, 0)$; (these two points always belong to the curve, though, when N is not a square, only as isolated points:) the tangents to the θ' branches are parallel to the asymptotes of the curve. Similarly the first modular curve acquires θ' linear, but imaginary, branches at each of the points $(0, \pm\frac{1}{2}i)$; the tangents to these branches being imaginary lines parallel to the real asymptotes.

The equations of the asymptotes of the first and fourth curves are respectively

$$(41) \ldots\ldots \begin{cases} Y\cos\frac{u}{2} - X\sin\frac{u}{2} = 0, \\ Y\sin\frac{u}{2} + X\cos\frac{u}{2} = 0; \end{cases}$$

u denoting $\frac{2l+1}{\theta}\pi$, as in Art. 15. And it may be inferred from the developments (33) and (34), given in that Article, that the rectangular hyperbola

$$\left(Y\cos\frac{u}{2} - X\sin\frac{u}{2}\right)\left(Y\sin\frac{u}{2} + X\cos\frac{u}{2}\right) = 5\sin u$$

osculates at an infinite distance the branches asymptotic to the two lines (41).

Lastly, if $v = \frac{2h}{\theta}\pi$, as in Art. 15, the tangents of the fourth curve at the points $(\pm\frac{1}{2}, 0)$ are

$$Y\cos\frac{v}{2} + (X\pm\tfrac{1}{2})\sin\frac{v}{2} = 0;$$

the tangents of the first curve at the imaginary points $(0, \pm\frac{1}{2}i)$ are

$$(Y\pm\tfrac{1}{2}i)\sin\frac{v}{2} - X\cos\frac{v}{2} = 0:$$

and these tangents are parallel to the asymptotes of the curves to which they respectively appertain; because, if

$$2l+1+2h = (2k+1)\theta,$$

$$\tan\frac{u}{2} = \cot\frac{v}{2}.$$

The points $(\pm\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{2}i)$ are foci, and indeed the only foci, of both curves: of these, the points $(0, \pm\frac{1}{2}i)$ lie on the first curve, and the two real points $(\pm\frac{1}{2}, 0)$ are its two foci (properly so called); the axis of Y being the only corresponding cyclic axis, or directrix. The points $(\pm\frac{1}{2}, 0)$ belong to the fourth curve (only as isolated points, when N is not a square), and this curve has, properly speaking, only the pair of imaginary foci $(0, \pm\frac{1}{2}i)$.

Art. 19. The second and third modular curves may be regarded as derived from the equation (1), by the substitution

$$(42) \ldots\ldots \begin{cases} p = \dfrac{\alpha}{\gamma}, \quad q = \dfrac{\gamma}{\beta}, \\ \alpha = \frac{1}{2} \pm (X - iY), \\ \beta = \frac{1}{2} \pm (X + iY), \\ \gamma = 1. \end{cases}$$

X and Y being rectangular Cartesian coordinates, and the upper signs relating to the second curve, the lower to the third.

Thus the theory of each of these curves is comprehended in that of the curve C', of which the trilinear equation is

$$(43) \ldots\ldots \begin{cases} (\beta\gamma)^{B-A} \times F(\alpha\beta, \gamma^2, \beta\gamma) = 0, \\ \text{or} \\ (\alpha\gamma)^{B-A} \times F(\alpha\beta, \gamma^2, \alpha\gamma) = 0. \end{cases}$$

The singularities of C' may be examined by the method already employed in the case of C. Attending, for brevity, only to the case in which N is not divisible by any square, we write $p = \dfrac{\alpha}{\gamma}$, $q = \dfrac{\gamma}{\beta}$, in the parametric developments of Art. 4, and we deduce, as follows, the normal developments of the singular branches of C'.

(i.) From (6) we obtain

$$\frac{\alpha - \gamma}{\gamma} = \lambda \left(\frac{\gamma - \beta}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots;$$

or, multiplying by $\dfrac{\gamma}{\beta} = 1 + \dfrac{\gamma - \beta}{\beta}$,

$$(44) \ldots\ldots \frac{\alpha - \gamma}{\beta} = \lambda \left(\frac{\gamma - \beta}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots,$$

which is itself a normal development, if $g' > g$, and gives rise, by reversion, to such a development, if $g' < g$. Hence C' has a singularity at S, having the same characteristics as the corresponding singularity of C.

(ii.) From (5) we infer

$$\frac{\alpha}{\gamma} = \lambda \left(\frac{\gamma}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots.$$

Here, when p and q are small, α must be small compared with γ, and γ compared with β; *i.e.*, the coordinates of the point ($p=0$, $q=0$) are $\alpha = 0$, $\gamma = 0$; and the normal development is

$$(45) \ldots\ldots \frac{\alpha}{\beta} = \lambda \left(\frac{\gamma}{\beta}\right)^{\frac{g+g'}{g}} + \ldots\ldots.$$

(iii.) Similarly from the development (7) we deduce

$$\frac{\gamma}{a} = \lambda\left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}} + \ldots\ldots$$

Thus the coordinates of the point ($p = \infty$, $q = \infty$) are $\beta = 0$, $\gamma = 0$; and we find, after reversion and multiplication by $\frac{\gamma}{a}$, the normal development

$$(46) \ldots\ldots \frac{\beta}{a} = \pm\mu\left(\frac{\gamma}{a}\right)^{\frac{g'+g}{g'}} + \ldots\ldots,$$

which is of the same type as (45).

Thus the curve has at P and Q singularities of one and the same self-reciprocal type, not resembling the singularities which C has at the same points. The point R does not lie on C'.

Art. 20.

Characteristics and Singularities of the Modular Curve C'.

I. *Characteristics of the Curve.*

$$\begin{aligned}
m &= 2A+2B, \quad n = 3A+B,\\
H &= \tfrac{1}{2}(A+B)-3\nu+1,\\
K &= 2A+4B-6\nu,\\
I &= 5A+B-6\nu,\\
I-K &= 3(A-B),\\
D &= 4(A+B)^2-5A-3B,\\
T &= (3A+B)^2-5A-3B,\\
T-D &= (3A+B)^2-4(A+B)^2\\
&= (A-B)(5A+3B).
\end{aligned}$$

It will be noticed (1) that these formulæ do not contain θ', although the case when N is a square is included in them; (2) that, when N is not a square, $I-K$ has the same value for C' as for C.

II. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPR) *and* (QQR).

$$\begin{aligned}
O(PPR) &= A+B = C(PPR),\\
K(PPR) &= A+B-2\nu = I(PPR),\\
D(PPR) &= A_2+3B_2-2A-2B = T(PPR).
\end{aligned}$$

The number of distinct branches at each of the points P and Q is 2ν; viz., corresponding to every divisor g of N, there are in (PPR), $f(\eta)$ branches of the order $\frac{g}{\eta}$ and of the class $\frac{g'}{\eta}$; of the 2ν branches, h are linear, and, in particular, when N is a square, θ' of these are also non-inflexional.

(ii.) *Characteristics of* (PSS), (QSS), *and* (S).

These are the same for C' as for C (see Art. 14, III., ii. and iv.). When N is a square the equations of the tangents to the linear branches are

$$\alpha-\gamma=\epsilon^{i\phi}(\gamma-\beta)$$

(see Art. 15, equation 36).

(iii.) *Tangents to the Curve from* $PQRS$.

(1) PR, considered as a tangent drawn from P, counts as $2A+2B$ tangents at P; and PS counts as $A-B$ tangents; hence PR and PS are the only tangents from P.

(2) RP, considered as a tangent drawn from R, counts as $A+B$ tangents at P; and so does RQ at Q; thus there are $A-B$ other tangents which can be drawn to C' from R.

(3) Besides the tangents at S, there are $A-B-2\theta'$ other tangents which can be drawn to the curve from S (see Art. 14, III., v., 2).

(iv.) *Intersections with the Curve of the sides of the quadrangle* $PQRS$.

(1) PQ meets the curve in $A+B$ points at P, and in $A+B$ points at Q. Thus it never meets the curve again, and touches it nowhere.

(2) PR and QR each meet the curve in $2A+2B$ points, touching it at P and Q respectively, and meeting it nowhere else.

(3) PS and QS meet the curve in $A+B$ points at S, and in $A+B$ points at P and Q respectively; thus they never meet the curve again.

(4) RS meets the curve in $2B$ points at S, and in $2A$ other points.*

III. *Residual Singularities of the Curve.*

Employing the notation of Art. 14, IV., we have

$$\begin{cases} K_1=2A+4B-6\nu, \\ I_1\ =4A-6\nu+\theta', \\ D_1=2A_2+8B_2-6A-4B, \\ T_1\ =4A^2+6B_2-6A-4B+\theta'. \end{cases}$$

$$\begin{cases} K_2=0, \\ I_2\ =A+B-\theta', \\ D_2=2A_2-4B_2+A+B, \\ T_2\ =2A^2-2B^2+3A_2-3B_2+A+B-\theta'. \end{cases}$$

The indices K_2, D_2 have the same values for C and C', because these

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* The points of contact of the $A-B$ tangents, iii. 2, and of the $A-B-2\theta'$ tangents, iii. 3, and the $2A$ points of intersection, iv. 4, can be determined by methods similar to those indicated in the Notes on Art. 14.

indices refer to singularities which do not lie, in either figure, upon the fundamental triangle of the quadric transformation by which the curves are changed into one another. The equality of the indices I_2, when N is not a square, implies the theorem:

"Each of the first three modular curves has as many non-singular inflexional tangents as it has osculating circles, which pass through the point $(\frac{1}{2}, 0)$; or again, through the point $(-\frac{1}{2}, 0)$."

Of the formulæ contained in the preceding enumeration, we shall demonstrate only one; viz., the expression for $D(PPR)$ or $D(P)$.

We have, as in Art. 10, iii.,

$$D(P) = \Sigma f(\eta)\frac{g+g'}{\eta}\left(\frac{g}{\eta}-1\right) + \Sigma f(\eta)\left[f(\eta)-1\right]\frac{g(g+g')}{\eta^2}$$
$$+ 2\Sigma\Sigma_1 f(\eta) f(\eta_1)\frac{g_1(g+g')}{\eta\eta_1};$$

where g is any divisor of N, and g' is the conjugate divisor of g; g_1 is any divisor less than g, so that $g_1' > g'$; the summations Σ_1 and Σ extend to every value of g_1 and g respectively. Hence we find

$$(47) \;\ldots\ldots\; D(P) = \Sigma f'(g)\left[f'(g)+f'(g')\right] - \Sigma\left[f'(g)+f'(g')\right]$$
$$+ 2\Sigma\Sigma_1 f'(g_1)\left[f'(g)+f'(g')\right].$$

But we have, evidently,

$$\Sigma f'(g) f'(g) + 2\Sigma\Sigma_1 f'(g_1) f'(g) = (A+B)^2 = A_2+B_2;$$
$$\Sigma\left[f'(g)+f'(g')\right] = 2A+2B;$$

and, observing that $g'g_1 < N$, we also find

$$\tfrac{1}{2}\Sigma f'(g) f'(g') + \Sigma\Sigma_1 f'(g_1) f'(g') = B_2.$$

Introducing these values into the equation (47), we obtain

$$D(P) = A_2+3B_2-2A-2B,$$

in accordance with the formula II. i. *suprà.*

The following presents were made to the Library in the Vacation :—

"Educational Times," August—November, 1878.

"Reprint of Mathematics from the Educational Times," Vol. xxix.

"Proceedings of the Musical Association for the Investigation and Discussion of Subjects connected with the Art and Science of Music," 4th Session, 1877–8.

"Monatsbericht," Juni, Juli, August, 1878.

"American Journal of Mathematics, Pure and Applied," Vol. i., No. 3; Baltimore, 1878.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^me^ Série, tome 1^er^; "Table des Matières et Noms d'Auteurs." Also Nos. for Fev., Mars, Avril, Juin, 1878.

"Journal of Institute of Actuaries," No. cxi., April 1878; No. cxii., July 1878; and No. cxiii., October 1878.

"Proceedings of Royal Society," Vol. xxvii., Nos. 188, 189.

"Crelle," 85 Band, 3^es^ Heft, 4^es^ Heft; 86 Band, 1^es^ Heft, 2^es^ Heft; Berlin, 1878.

"Atti della R. Accademia dei Lincei," anno cclxxv., 1877–8, Serie terza; "Transunti," vol. ii.; Roma, 1878.

"Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades," von F. Klein, in München (Mai 1878); from "Math. Annalen," Band xiv., pp. 111–172.

"Théorème d'Arithmétique sur la somme des inverses des puissances semblables des nombres premiers," by J. W. L. Glaisher. (Association Française pour l'Avancement des Sciences, 1877.)

"Expressions for Laplace's Coefficients, Bernoullian and Eulerian Numbers, &c., as Determinants." ("Messenger of Mathematics," No. 64, 1876.)

"On a Numerical Continued Product." (Ditto, No. 65, 1876.)

"On Long Successions of Composite Numbers." (Ditto, No. 79, 1877; No. 83, 1878.)

"Algebraical Theorems and Expressions derived from Lagrange's Series." (Ditto, No. 80, 1877.)

"On a Class of Determinants." (Ditto, No. 83, 1878.)

"On the Product $1^1 . 2^2 . 3^3 \dots n^n$." (Ditto, No. 75, 1877.)

"Series and Products for π and Powers of π." (Ditto, No. 77, 1877.)

"On Some Continued Fractions." (Ditto, ditto.)

"Further Note on Certain Numerical Continued Products." (Ditto, No. 72, 1877).

"An Approximate Numerical Theorem involving e and π, and a Theorem in Trigonometry." ("Quarterly Journal of Mathematics," No. 58, 1877.)

"Proof of Stirling's theorem $1.2.3 \dots n = \sqrt{(2n\pi)}\, n^n e^{-n}$" (Ditto, No. 57, 1877).

"On Expressions for the Theta Functions as Definite Integrals" ("Proceedings of Cambridge Philosophical Society," Vol. iii., Pt. iii., May 1877).

"On a Formula of Cauchy's for the Evaluation of a class of Definite Integrals" (Ditto, Vol. iii., Pt. i., Nov. 1876).

"Preliminary Account of an enumeration of the Primes in Burckhardt's Tables" (1 to 3000000), and Dase's Tables (6000000 to 9000000), (Ditto, Vol. iii., pp. 17—23, 47—56).

"On certain Determinants," and "On a Series Summation, leading to an expression for the Theta Function as a Definite Integral" ("Report of British Association," 1876).

"On certain identical Differential Relations" ("Proceedings of London Mathematical Society," Vol. viii., Nos. 106, 107).

"Numerical Values of the first twelve Powers of π, of their Reciprocals, and of certain other related Quantities" ("Proceedings of London Mathematical Society," Vol. viii., Nos. 112, 113).

"On an Elliptic Function Solution of Kepler's Problem" ("Monthly Notices of Royal Astronomical Society," May 1877).

"On the Solution of Kepler's Problem" ("Supplementary Number of the Monthly Notices of R. A. S." for 1877).

"On Factor Tables," with an account of the mode of formation of the Factor Table for the 4th million.

The above 21 pamphlets from the author, Mr. J. W. L. Glaisher, F.R.S.

"Bulletin de la Société Mathématique de France," Tome vi., No. 5; Paris.

"Theorie der algebraischen Gleichungen," von Dr. Julius Petersen (Kopenhagen, 1878): from the Author.

"Jahrbuch über die Fortschritte der Mathematik," achtes Band, Jahrgang 1876 (in 3 Heften), Heft 2, Heft 3; Berlin, 1878.

"Betrachtungen über die Kummer'sche Fläche und ihren Zusammenhang mit den hyperelliptischen Functionen ($p=2$)," von Karl Rohn; München, 1878.

"Boletin de la Institucion libre de Enseñanza," 30 de Junio de 1878, No. 33; and "Suplemento al Número 37" (Año tercero curso de 1878–9).

"Institucion libre de Enseñanza, 1ª Conferencia" (25 Nov. 1877), "Las Elecciones Pontificias por el Exc^mo^ S^r^ D. E. Montero Rios"; Madrid, 1877.

"Ricerche sulle equazioni algebrico-differenziali," (Memoria di F. Casorati a Pavia,) estratto dagli Annali di Matematica (26 pp.)

"M. S. Verification of Pervouchine's first result (Divisibility of $2^{2^{12}}+1$ by $7.2^{14}+1$)"; by T. Muir, M.A.

"Ditto, and also of second result (Divisibility of $2^{2^{23}}+1$ by $5\times 2^{25}+1$)"; by J. Bridge, M.A.

"Reale Istituto Lombardo di Scienze e Lettre,"—"Rendiconti," Serie ii., Volumes ix., x., 1876–7; "Memorie," Classe di Scienze Matematiche e Naturali, Vol. xiii. (iv. della Serie iii., Fasc. iii. e ultimo), 1877; Vol. xiv. (v. della Serie iii., Fasc. i.), 1878; Milano.

"The analytical Theory of Heat," by J. Fourier; translated, with Notes, by A. Freeman, M.A., Cambridge, 1878: from the Translator.

"Publications of the Cincinnati Observatory" (No. 4, "Micrometrical Measurements of Double Stars," 1877).

"Proceedings of Royal Irish Academy" (Vol. ii., Series ii., Jan. 1877, No. 7; Vol. i., Series ii., March 1877, No. 12; Vol. iii., Series ii., Aug. 1877, No. 1).

"Transactions of the Royal Irish Academy" [Vol. xxv., xx. (Nov. 1875). Vol. xxvi., vi. (Nov. 1876); vii, viii., ix. (Aug. 1877); x. (March 1878); xi., xii. (April 1878); xiii., xiv. (July 1878); xv., xvi. (Aug. 1878). Vol. xxvii., Pt. 1 (March 1877)].

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," redigirt von Dr. Rudolf Wolf; 21er Jahrgang, 1es, 2es, 3es, 4es Heft; 22er Jahrgang, 1es, 2es, 3es, 4es Heft.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2e Serie, Tome ii., 3e cahier; Paris, 1878.

APPENDIX.

The result of Professor Cayley's researches on the Double Theta Functions (p. 29) is given in his Memoir "On the Double Theta Functions" ("Crelle," 85 Band, 3es Heft, pp. 214—245, in continuation of 83 Band, pp. 210—233).

Herr Weichold's solution of the Irreducible Case, of which an abstract was given in Vol. viii. (pp. 312—316), is printed *in extenso* in No. 1 of the "American Journal of Mathematics, Pure and Applied" (pp. 32—49).

We may refer to the same Journal, No. 3 (pp. 261—276), No. 4 (pp. 384, 385), for a very full and nearly complete Bibliography of Hyper-space and non-Euclidean Geometry, by G. B. Halsted (*cf.* Note to Vol. viii., London Mathematical Society's "Proceedings," p. 310).

"On the Theory of Groups" (pp. 126—133), see remarks by Prof. Cayley in the American Journal above cited, No. 1 (pp. 50—52), No. 2 (pp. 174—176); see also "Mathematische Annalen," Band xiii., 4es Heft (pp. 561—565).

The same Journal contains a paper by M. Edouard Lucas, "Théorie des Fonctions Numériques simplement Périodiques" (pp. 184—240, 289—321), bearing upon a subject treated of in the "Proceedings" by Prof. H. J. S. Smith and Mr. Samuel Roberts.

In No. 4 of the same Journal (pp. 350—358), in a paper entitled "Applications of Grassmann's Extensive Algebra," Prof. Clifford gives, in a brief form, the purport of his once contemplated paper, "On the Classification of Geometric Algebras" (Vol. vii., p. 135).

Prof. Clifford's "Classification of Loci" (Vol. viii., p. 184) was presented as a paper and received by the Royal Society, April 8, 1878. An abstract of its contents is given in Vol. xxvii., No. 187, of the "Proceedings of Royal Society," pp. 370, 371.

Mr. J. C. Malet's paper, "Proof that every Algebraic Equation has a Root" (Vol. viii., p. 289), is printed in the "Transactions of the Royal Irish Academy" (Vol. xxvi., July 1878, No. xiv.)

In connection with Dr. Hirst's "Note on the Correlation of two Planes" (Vol. viii., pp. 262—273), see "Annali di Matematica" (Dec. 1877, pp. 287—300).

Prof. H. J. S. Smith's Presidential Address (Vol. viii., pp. 6—29) is translated by Dr. H. G. Zeuthen, and appears in the "Tidsskrift for Mathematik" (May, 1877).

There is a passage in Chasles' "Aperçu Historique" (p. 125),—"Si à une épicycloïde, engendrée par un point d'une circonférence de cercle qui roule sur un autre cercle fixe, on circonscrit des angles tous égaux entre eux, leurs sommets seront situés sur une épicycloïde allongée ou raccourcée,"—wherein Prof. Wolstenholme's like discovery (Vol. iv., p. 330) is anticipated.

A paper by Sir W. Thomson, "On a Machine for the Solution of Simultaneous Linear Equations," ("Proceedings of Royal Society," Vol. xxviii., No. 191, pp. 111—113), is, if we mistake not, connected with his communication, "The Integration of the Equations for the Motion of a System acted on by Forces expressed by Linear Functions of the Displacements and Velocities" (Vol. vi., p. 114). An account of the recent communication will be found in "Nature" (Vol. xix., pp. 161, 162).

An account of Robert Flower, the logarithmist (p. 75), is given in a letter from Mr. A. J. Ellis to the Editor of the "Academy" ("Academy," April 20, 1878, pp. 347, 348).

In connection with Dr. Klein's paper (p. 123), see his fuller paper in the "Mathematische Annalen," xiv. Band (Heft i., p. 111, to Heft ii., p. 172), entitled, "Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades."

On p. 148 the reference, in the text, to "Nature" is incorrectly printed: it should be Vol. xviii., and the form should be $2^{2^m}+1$. Verifications of both M. Pervouchine's results have been presented to the Society by Mr. J. Bridge, M.A. (*cf.* "Nature," Vol. xix., pp. 17, and 73, 74); and a verification of the earlier result has also been made and presented to the Society by the author, Mr. T. Muir, M.A. (*cf.* "Nature," Vol. xviii., p. 652).