THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS DEFINED BY GENERALISED HYPERGEOMETRIC SERIES

By E. W. BARNES, Sc.D.

[Received December 3rd, 1906.—Read December 13th. 1906.]

1. The present paper is one of a series* in which the author has endeavoured to make a contribution to the theory of integral functions defined by Taylor's series.

Generalised hypergeometric functions form a wide class of integral functions whose asymptotic expansions are closely connected with the theory of linear differential equations. They appear to originate with Clausen†; two important papers are due to Thomae,‡ and the investigations of Goursat§ should also be consulted. But the detailed

* Barnes, (α) "The Asymptotic Expansion of $\sum_{n=0}^{\infty} \frac{x^n}{n!\,(x+\theta)}$ and the Singularities of $g(x, \theta) = \sum_{n=0}^{\infty} \frac{x^n}{n+\theta}$," *Quarterly Journal of Mathematics*, Vol. XXXVII., pp. 289–313.

——— (β) "The Asymptotic Expansion of Integral Functions defined by Taylor's Series," *Philosophical Transactions of the Royal Society* (A), Vol. 206, pp. 249–297.

——— (γ) "On certain Functions defined by Taylor's Series of Finite Radius of Convergence," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 4, pp. 284–316.

——— (δ) "On the Asymptotic Expansion of the Integral Functions

$$\sum_{n=0}^{\infty} \frac{x^n\,\Gamma(1+an)}{\Gamma(1+n)} \quad\text{and}\quad \sum_{n=0}^{\infty} \frac{x^n\,\Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}."$$

Cambridge Philosophical Transactions, Vol. XX., pp. 215–232.

——— (ε) "On the Use of Factorial Series in an Asymptotic Expansion." *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 116–140.

——— (η) "On Functions defined by Simple Types of Hypergeometric Series." *Cambridge Philosophical Transactions*, Vol. XX., pp. 253–279.

† Clausen, (1828) *Crelle*, Bd. III., pp. 89–92.

‡ Thomae, (1869) *Mathematische Annalen*, Bd. II., pp. 427–444.

——— (1879) *Crelle*, Bd. LXXXVII., pp. 222–349.

§ Goursat, (1883) *Annales de l'Ecole Normale Supérieure*, Sér. 2. T. XII., pp. 261–395.

——— (1884) *Acta Mathematica*, Bd. V., pp. 97–120.

development of the theory is largely due to Pochhammer,* whose voluminous writings form an interesting study.

In the present paper I take substantially the notation of Pochhammer. The general type of series considered is

$$1+\frac{a_1 \dots a_p}{1.\rho_1 \dots \rho_q} x+\frac{a_1(a_1+1) \dots a_p(a_p+1)}{1.2.\rho_1(\rho_1+1) \dots \rho_q(\rho_q+1)} x^2+\dots$$
$$=\frac{\Gamma(\rho_1) \dots \Gamma(\rho_q)}{\Gamma(a_1) \dots \Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n) \dots \Gamma(a_p+n)}{n!\,\Gamma(\rho_1+n) \dots \Gamma(\rho_q+n)} x^n$$

wherein $p \leqslant q$. This series we shall denote by

$${}_pF_q\{a_1, \dots, a_p;\ \rho_1, \dots, \rho_q;\ x\}$$

or, briefly, by ${}_pF_q(x)$.

The series satisfies the differential equation

$$\left[(\vartheta+a_1) \dots (\vartheta+a_p)-\frac{d}{dx}(\vartheta+\rho_1-1) \dots (\vartheta+\rho_q-1)\right] y=0 \tag{1}$$

wherein $\vartheta = x(d/dx)$. The equation is of order $(q+1)$, and q other linearly independent solutions are given by

$$x^{1-\rho_m}{}_pF_q\{a_1-\rho_m+1, \dots, a_p-\rho_m+1;\ 2-\rho_m, \rho_1-\rho_m+1, \dots, \rho_q-\rho_m+1;\ x\}$$

where $m = 1, 2, \dots, q$. Among the quantities $\rho_r-\rho_m+1$, that corresponding to $m = r$ is to be omitted.

It subsequently proves convenient to make a change in our notation; so that, for brevity, we write

$$Q_0(x)=\frac{\prod_{r=1}^{q} \Gamma(1-\rho_r)}{\prod_{r=1}^{p} \Gamma(1-a_r)}\, {}_pF_q\{a_1, \dots, a_p;\ \rho_1, \dots, \rho_q;\ (-)^{\mu} x\},$$

* Pochhammer, (1886) "Ueber die Differentialgleichung der allgemeineren hypergeometrische Reihe mit zwei endlichen singulären Punkten," *Crelle*, Bd. CII., pp. 76–159.

——— (1888) "Ueber gewisse partielle Differentialgleichungen denen hypergeometrische Integrale genügen," *Mathematische Annalen*, Bd. XXXIII., pp. 353–371.

——— (1891) "Ueber die Differentialgleichung der allgemeineren F-Reihe," *Mathematische Annalen*, Bd. XXXVIII., pp. 586–597.

——— (1891) "Ueber die Differentialgleichungen der Reihen $F(\rho, \sigma; x)$ und $F(\rho, \sigma, \tau; x)$," *Mathematische Annalen*, Bd. XLI., pp. 197–218.

——— (1893) "Ueber die Reduction der Differentialgleichung der allgemeineren F-Reihe," *Crelle*, Bd. CXII., pp. 58–86.

——— (1895) "Ueber die Differentialgleichungen der F-Reihen 3-ter Ordnung," *Mathematische Annalen*, Bd. XLVI., pp. 584–605.

——— (1898) "Ueber die Differentialgleichungen der F-Reihen 4-ter Ordnung," *Mathematische Annalen*, Bd. L., pp. 285–302.

The reader of Pochhammer's papers will note that an earlier paper (1870, *Crelle*, Bd. LXXIII., pp. 135–157) dealt with series of a different type, to which the name "hypergeometric" was also given.

$$Q_m(x) = x^{1-\rho_m} \frac{\Gamma(\rho_m-1)\prod_{t=1}^{q}{}' \Gamma(\rho_m-\rho_t)}{\prod_{t=1}^{p} \Gamma(\rho_m-a_t)}$$

$$\times {}_pF_q\{1+a_1-\rho_m, \ldots, 1+a_p-\rho_m;\ 2-\rho_m, \ldots, \rho_q-\rho_m+1;\ (-)^\mu x\}$$

$$(m = 1, 2, \ldots, q).$$

Then, evidently, the $(q+1)$ linearly independent solutions of the differential equation (1) are

$$Q_m\{(-)^\mu x\} \quad (m = 0, 1, 2, \ldots q).$$

We shall put $\mu = q+1-p$, so that μ is an integer $\geqslant 1$.

2. Various types of integral functions were considered by the author in the memoir "On the Asymptotic Expansion of Integral Functions defined by Taylor's Series." Parts IX. and X. of this paper contained a *résumé* of results relating to the most simple hypergeometric integral functions ${}_1F_1\{a; \rho; x\}$ and ${}_0F_1\{\rho; x\}$. The proofs of such results were given in detail in a subsequent paper.* By means of contour integrals involving gamma functions of the variable in the subject of integration, it was shewn to be possible to develop the theory of ${}_1F_1\{a; \rho; x\}$ with considerable simplicity; and from it the theory of ${}_0F_1\{\rho; x\}$ was deduced by Kummer's formula

$${}_0F_1\{\rho; x\} = e^{2x^{\frac{1}{2}}}\,{}_1F_1\{\rho-\tfrac{1}{2}; 2\rho-1; -4x^{\frac{1}{2}}\}.$$

These two functions are the most simple examples of the two classes into which higher hypergeometric integral transcendents can be divided, the division corresponding to $\mu = 1$ or $\mu > 1$. References to the history and literature of the asymptotic theory of these two elementary transcendents will be found in the paper just cited.

The corresponding theory of the higher transcendents which forms the subject of the present investigation is by no means an obvious extension of more elementary results. The asymptotic expansions which arise for the two simple functions ${}_1F_1\{a; \rho; x\}$ and ${}_0F_1\{\rho; x\}$ have themselves the form of hypergeometric series. In the more complex cases this not true. Moreover, there does not appear to be any analogue to Kummer's formula which we can use to deduce the cases when $\mu > 1$ from cases when $\mu = 1$. Nor has any analogue to Gauss's expression of ${}_2F_1\{\alpha, \beta; \gamma; 1\}$ in terms of gamma functions been discovered, and Mellin denies the possibility of its existence.

* *Loc. cit.*, § 1, Paper (η).

3. The results of the present theory were adumbrated by Stokes,* who shewed that, *if x be real and positive,*

$$\frac{\prod_{r=1}^{p} \Gamma(a_r)}{\prod_{r=1}^{q} \Gamma(\rho_q)} \, {}_pF_q\{x\} = (2\pi)^{\frac{1}{2}(1-\mu)} \mu^{-\frac{1}{2}} x^{[\Sigma a - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu} \exp\{\mu x^{1/\mu}\} J$$

where J tends to unity as x tends to infinity.

The general theory when x can take any complex value is due to Orr.† In his first paper Orr shewed that, if

$$|\arg x| < \pi\{1+\tfrac{1}{2}\mu\} \quad \text{and} \quad m = 1, 2, \ldots, p,$$

$$\frac{1}{\sin \pi a_m} Q_0(-x) + \sum_{r=1}^{q} \frac{Q_r(-x)}{\sin \pi(\rho_r - a_m)}$$

$$= \frac{\Gamma(a_m) \prod_{r=1}^{q} \Gamma(a_m - \rho_r + 1)}{\pi \prod_{r=1}^{p}{}' \, \Gamma(1 + a_m - a_r)} (x)^{-a_m} {}_{q+1}F_{p-1}\left\{ \begin{array}{l} a_m, a_m+1-\rho_1, \ldots, a_m+1-\rho_q; \\ a_m - a_1 + 1, \ldots, a_m - a_p + 1; -1/x \end{array} \right\}. \tag{A}$$

This result, which is obtained almost intuitively in Part I. of the present paper, was, with Orr, the main outcome of some thirty pages of laborious analysis in which it was derived by an elaborate process of induction: the notable simplification obtained by the present methods will be at once apparent to the reader.

In his second paper Orr obtained the dominant term of the asymptotic equality which is given in Parts II. and III. of the present memoir. In Part III. I shew that, if $|\arg x| < \pi$ and $m = 1, 2, \ldots, \mu$,

$$Q_0(x) + \sum_{r=1}^{q} e^{(\mu-2m)(\rho_r-1)\pi\iota} Q_r(x)$$

$$= \exp\{-\mu e^{(\mu-2m)\pi\iota/\mu} x^{1/\mu}\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} x^{\theta/\mu} e^{(\mu-2m)\theta\pi\iota} \sum_{n=0}^{\infty} \frac{\lambda_n e^{n(2m-\mu)\pi\iota/\mu}}{x^{n/\mu}} \tag{B}$$

where $$\theta = [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu - 1)]/\mu.$$

Orr's second paper bears traces of great compression: the argument is most difficult to follow, and at times the methods seem open to serious objection. Moreover, the method only gives the dominant term of the asymptotic expansion in question, and it is obviously desirable to shew how subsequent terms arise in the general case, even though their complexity is such that they are not actually obtained.

* Stokes, *Proceedings of the Cambridge Philosophical Society*, Vol. VI., pp. 362–366 (1889).

† Orr, *Cambridge Philosophical Transactions*, Vol. XVII., pp. 171–199 (1898); pp. 283–290 (1899).

Further, Orr only adumbrates the converse theory of the expression of the hypergeometric integral functions separately in terms of groups of asymptotic expansions. He suggests that such should be obtained by elimination between the results of the equations (A) and (B). In this paper I shew that the contour integrals which I employ give such expressions even more easily than they give the converse equalities. And the nature of certain apparent redundancies which the latter equations seem to contain is at once apparent from the present theory, which appears to solve every problem which the asymptotic theory of the series presents. The importance of the theory and its generality must be my excuse for the length of this paper.

4. The simple method used in Part I. to obtain the formula (A) of § 3 is an obvious application of ideas which I have employed for many classes of functions. The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle,* whose suggestive paper was the starting point of the investigations of Mellin,† though the type of contour and its use can be traced back to Riemann [*Œuvres Mathématiques* (1898), pp. 166–167]. The formula (B) is, however, only obtained by the use of factorial series of a somewhat complicated type, and the reduction of these series to more simple forms and the development of their properties are the most novel features of the present investigation. This use of factorial series to obtain the asymptotic expansion of integral functions for the region at infinity for which they are exponentially infinite is due to the author, and was employed in a recent paper ‡ in which his asymptotic expansion of $G_\beta(x, \theta)$ when $R(x) > 0$ § was obtained anew. Each type of integral function discussed gives rise to a new type of factorial series; but wide classes of such series possess similar properties, and the investigation in Part III. of the present paper is, to some extent, based on the properties of the more elementary series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{n!\,(n+\theta)^\beta}$$

previously discussed.

5. When $\mu = 1$ the results for generalised hypergeometric integral functions differ in character from, and can be obtained by more simple analysis than, those which hold good for other integral values of μ.

* Pincherle, "Sulle funzioni ipergeometriche generalizzate" (1888), *Atti d. R. Accademia dei Lincei*, Ser. 4, *Rendiconti*, Vol. IV., pp. 694–700, and pp. 792–799.

† Mellin, *Acta Societatis Scientiarum Fennicæ* (1895), T. XX., No. 12.

‡ *Loc. cit.*, § 1, Paper (ε). § *Loc. cit.*, § 1, Paper (β), Part III.

Consequently in Part II. of the present paper this special case is considered separately.

I construct the function ${}_pS_p(s)$ defined when $R(s) > R(\Sigma\alpha - \Sigma\rho)$ by the factorial series

$$\sum_{t=0}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+1)} \prod_{r=1}^{p} \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-\alpha_r+t-s)}$$

and shew by transformations depending on contour integrals

(1) that ${}_pS_p(s)$ admits of analytic continuation over the whole of the finite portion of the s-plane;

(2) that it has simple poles at the points

$$s = \Sigma\alpha - \Sigma\rho - r \quad (r = 0, 1, 2, \ldots, \infty)$$

at which the residues can be calculated with sufficient labour, and no other finite singularities, except the obvious ones which belong to $\Gamma(-s)$ and $\Gamma(1-\rho_r-s)$:

(3) that, if $s = u+\iota v$, when u and v are real, and if u be finite and $\epsilon > 0$, $|{}_pS_p(s)| \exp\{(\frac{3}{2}\pi - \epsilon)|v|\}$ can, by taking $|v| >$ a sufficiently large positive quantity V, be made less than any arbitrarily assigned positive quantity η;*

(4) that, if both u and $|v|$ be large and positive and $\epsilon > 0$,

$$|{}_pS_p(s)| < \eta_u \exp\{-\tfrac{3}{2}\pi+\epsilon)|v|\}$$

where, for any finite positive value of k, $\eta_u \exp\{ku\}$ can be made as small as we please by taking u sufficiently large, if s be not in the immediate vicinity of one of the poles of ${}_pS_p(s)$.

From these results the asymptotic expansion of that combination of generalised hypergeometric integral functions ($\mu = 1$) which is exponentially infinite at infinity is deduced.

6. *The case when μ is an integer* > 1 is discussed in Part III. There I construct the function ${}_pS_q(s)$ defined when

$$R(s) > R[\Sigma\alpha - \Sigma\rho + \tfrac{1}{2}(\mu-1)]/\mu$$

by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^{q} \Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^{\mu} \Gamma(t/\mu+r/\mu) \prod_{r=1}^{p} \Gamma(1-\alpha_r+t/\mu-s)},$$

and by more elaborate methods shew that it possesses similar properties:—

(1) ${}_pS_q(s)$ is an analytic function over the whole s-plane.

(2) Apart from the obvious poles which arise from $\Gamma(-\mu s)$ and $\Gamma(\mu-\mu\rho_r-\mu s)$, it has for its sole finite singularities simple poles at

* In this case I say that $|{}_pS_p(s)| \exp\{(\frac{3}{2}\pi-\epsilon)|v|\}$ tends uniformly to zero as $|v|$ tends to infinity.

the points $[\Sigma a - \Sigma \rho + \frac{1}{2}(\mu-1) - r]/\mu \quad (r = 0, 1, 2, \ldots, \infty)$

at which the residues can be calculated with sufficient labour.

(3) The expression $|{}_pS_q(s)| \exp\{[(\mu+1)\pi - \epsilon]|v|\}$ tends uniformly to zero as $|v|$ tends to infinity, if $\epsilon > 0$.

(4) $|{}_pS_q(s)| < \eta_u \exp\{[-(\mu+1)\pi - \epsilon]|v|\}$ when u and $|v|$ are large and positive, where $\eta_u \exp\{ku\}$ can be made as small as we please by taking u sufficiently large and independent of $|v|$.

From these results we deduce the complete series of asymptotic ex-expansions (B) given in § 3.

I do not in the present paper attempt to apply the methods used to more general types of factorial series (*e.g.*, when μ is not an integer), nor do I consider the associated problem of the nature of the finite singularities of generalised hypergeometric series of finite radius of convergence. These questions I hope to consider on a future occasion.

PART I.

The Asymptotic Expansion of Linear Combinations of Generalised Hypergeometric Integral Functions which are not exponentially Infinite at Infinity.

7. *Let $p \leqslant q$ and $\mu = q+1-p$. We proceed to shew that there are p asymptotic equalities of the type*

$$\frac{\Gamma(a_1)\Gamma(1-\rho_1)\ldots\Gamma(1-\rho_q)}{\Gamma(1-a_2)\ldots\Gamma(1-a_p)}\,{}_pF_q\{a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_q;\ (-)^{p-q}x\}$$

$$+\sum_{r=1}^{q}\frac{\Gamma(a_1-\rho_r+1)\Gamma(\rho_r-1)\prod_{t=1}^{q}{}'\,\Gamma(\rho_r-\rho_t)}{\prod_{t=2}^{p}\Gamma(\rho_r-a_t)}$$

$$\times x^{1-\rho_r}\,{}_pF_q\{a_1-\rho_r+1, \ldots, a_p-\rho_r+1;$$

$$2-\rho_r, 1-\rho_r+\rho_1, \ldots, 1-\rho_r+\rho_q;\ (-)^{p-q}x\}$$

$$= \frac{\Gamma(a_1)\prod_{t=1}^{q}\Gamma(a_1-\rho_t+1)}{\prod_{t=2}^{p}\Gamma(1+a_1-a_t)}\,x^{-a_1}\,{}_{q+1}F_{p-1}\{a_1, a_1-\rho_1+1, \ldots, a_1-\rho_q+1;$$

$$a_1-a_2+1, \ldots, a_1-a_p+1;\ -1/x\}, \qquad \text{(A)}$$

each valid when $|\arg x| < \pi(1+\frac{1}{2}\mu)$. In the asymptotic equality the error which results from stopping at the $(k+1)$-th term of the series for ${}_{q+1}F_{p+1}\{-1/x\}$ is a quantity J_k where $|J_k x^{a_1+k}|$ can be made as small as we please by taking $|x|$ sufficiently large. The other $(p-1)$ asymptotic equalities are obtained by interchanging a_1 with $a_2, a_3, \ldots, a_p$ respectively.

Let us denote the expression on the left-hand side of the asymptotic equality (A) by Π.

We will first shew that

$$\Pi = -\frac{1}{2\pi\iota}\int \frac{\Gamma(-s)\,\Gamma(a_1+s)\prod_{t=1}^{q}\Gamma(1-\rho_t-s)}{\prod_{t=2}^{p}\Gamma(1-a_t-s)}\,x^s\,ds, \tag{B}$$

the integral being taken round a contour which embraces the positive half of the real axis and encloses all the poles of the subject of integration except those of $\Gamma(a_1+s)$.

In the first place, since $q+1 > p$, the integral is absolutely convergent. By Cauchy's theory of residues it is therefore equal to the sum of the residues of the subject of integration inside the contour.

Now the residue of $\Gamma(n-s)$ at its pole $s = n+m$ is $(-)^{m-1}/m!$. Hence the integral is equal to

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,\frac{\Gamma(a_1+n)\prod_{t=1}^{q}\Gamma(1-\rho_t-n)}{\prod_{t=2}^{p}\Gamma(1-a_t-n)}$$

$$+\sum_{r=1}^{q} x^{1-\rho_r}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,\frac{\Gamma(\rho_r-1-n)\prod_{t=1}^{q}{}'\,\Gamma(\rho_r-\rho_t-n)\,\Gamma(a_1+1-\rho_r+n)}{\prod_{t=2}^{p}\Gamma(\rho_r-a_t-n)} = \Pi.$$

8. *We may next alter the contour of integration so as to give us the asymptotic expansion* (A).

By the asymptotic expansion of the gamma function we know that, if $|\arg s| < \pi$ and $|s|$ be large,

$$\Gamma(s+a) = s^{s-\frac{1}{2}+a}e^{-s}\sqrt{2\pi}\,J$$

where $|J|$ tends uniformly to unity as $|s|$ tends to infinity.

Hence, if $s = u+\iota v$ where u is finite and $|v|$ is large,

$$|\Gamma(s+a)| = \exp\{-\tfrac{1}{2}\pi|v|\}\,|v^{a+u-\frac{1}{2}}e^{\pm\frac{1}{2}\iota\pi a}|\sqrt{2\pi}\,J,$$

the + or − sign being taken as v positive or negative.

Hence the contour integral (B) will vanish when taken round that part of a great circle at infinity for which $R(s) > -k$ where k is any finite positive quantity, provided

$$|\arg x| < (q+3-p)\tfrac{1}{2}\pi,$$

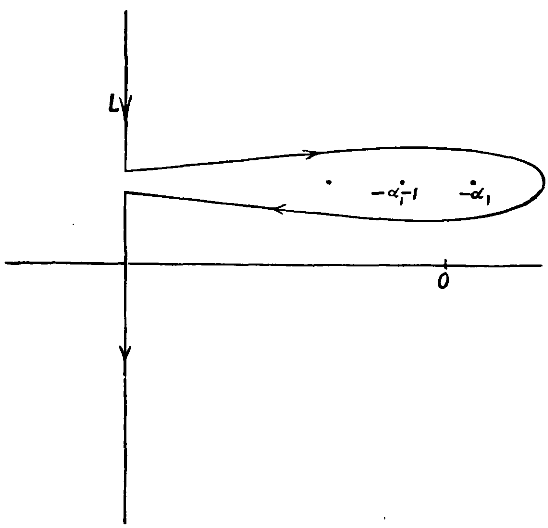
i.e.,

$$|\arg x| < \pi(1+\tfrac{1}{2}\mu).$$

If, then, L is a line parallel to the imaginary axis which passes between the points $-a_1-k$ and $-a_1-k-1$, we shall have $\Pi = -\frac{1}{2\pi\iota}\int_L$ together with the sum of the residues of

$$\frac{\Gamma(-s)\,\Gamma(a_1+s)\prod_{t=1}^{q}\Gamma(1-\rho_t-s)\,x^s}{\prod_{t=2}^{p}\Gamma(1-a_t-s)}$$

at its poles $-a_1, -a_1-1, \ldots, -a_1-k$.



Thus $$\Pi = x^{-a_1}\sum_{n=0}^{k}\frac{(-x)^{-n}}{n!}\,\frac{\Gamma(a_1+n)\prod_{t=1}^{q}\Gamma(1-\rho_t+a_1+n)}{\prod_{t=2}^{p}\Gamma(1-a_t+a_1+n)}+J_k,$$

and it is evident that $|J_k x^{a_1+k}|$ for any finite value of k can be made as small as we please by taking $|x|$ greater than an assignable positive large quantity.

We thus have the given asymptotic expansion.

9. Let us consider the set of expansions of the type (A) in detail.

When $p = q = 1$, so that $\mu = 1$, we have one asymptotic expansion of the sum of two hypergeometric integral functions, and this asymptotic equality is valid when $|\arg x| < \frac{3}{2}\pi$.

The single expansion is therefore equal to two separate expansions when $R(x) < 0$.† From these two we may eliminate one of the hypergeometric integral functions and obtain an asymptotic expansion for the other when $R(x) < 0$ whose dominant term is only algebraically infinite (or zero). The work has been carried out in my previous paper,‡ where it has been shewn that we get the asymptotic equality

$$ {}_1F_1\{a;\,\rho;\,x\} = \frac{\Gamma(\rho)}{\Gamma(\rho-a)}(-x)^{-a}\,{}_2F_0\{a,\,1+a-\rho;\;-1/x\}. $$

More generally, when $p = q > 1$, so that $\mu = 1$, we have p asymptotic expansions of the sum of $(p+1)$ hypergeometric integral functions, and these asymptotic equalities are valid when $|\arg x| < \frac{3}{2}\pi$. There are thus in all $2p$ relations when $R(x) < 0$. At first sight we might be tempted to say that by eliminating p of the hypergeometric integral functions we can obtain p different asymptotic expansions, when $R(x) < 0$, for the remaining hypergeometric function. This, however, is evidently absurd, and, in point of fact, of the $2p$ asymptotic equalities only $p+1$ are independent, as may be shewn by the somewhat laborious algebra which results from writing down the equations and actually performing the elimination.

The single asymptotic equality which results from eliminating all the hypergeometric integral functions in the expression II except the first is actually the relation, valid when $R(x)<0$,

$$ \prod_{t=1}^{p}\left\{\frac{\Gamma(a_t)}{\Gamma(\rho_t)}\right\}\,{}_pF_p\{a_1,\,\ldots,\,a_p;\,\rho_1,\,\ldots,\,\rho_p;\,x\} $$

$$ = \sum_{r=1}^{p}(-x)^{-a_r}\,\frac{\Gamma(a_r)\prod_{t=1}^{p}{}'\,\Gamma(a_t-a_r)}{\prod_{t=1}^{p}\Gamma(\rho_t-a_r)}\,{}_{p+1}F_{p-1}\left\{\begin{matrix}a_r,\,1+a_r-\rho_1,\,\ldots,\,1+a_r-\rho_p;\\ 1+a_r-a_1,\,\ldots,\,*\,\ldots,\,1+a_r-a_p;\;-1/x\end{matrix}\right\}, $$

the star denoting the omission of the term $1+a_r-a_r$.

10. This may be proved directly by contour integration as follows:—

We consider the integral

$$ -\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\prod_{t=1}^{p}\Gamma(a_t+s)}{\prod_{t=1}^{p}\Gamma(\rho_t+s)}(-x)^s\,ds $$

taken round a contour which embraces the positive half of the real axis

† The reader may compare the more complex case of § 48, where the various ranges of arg x are given in detail.

‡ *Cambridge Philosophical Transactions*, Vol. xx., pp. 253–279, § 12.

and encloses only the poles $s = 0, 1, 2, \ldots$ of the subject of integration. It is evidently, by Cauchy's theory of residues, equal to

$$\prod_{t=1}^{p} \left\{ \frac{\Gamma(a_t)}{\Gamma(\rho_t)} \right\} {}_pF_p \{ a_1, \ldots, a_p; \rho_1, \ldots, \rho_p; x \}.$$

But, if $|\arg(-x)| < \frac{1}{2}\pi$, we may swing the integral back as in § 8. We thus see that it is asymptotically equal to

$$\sum_{r=1}^{p} (-x)^{-a_r} \sum_{n=0}^{\infty} \frac{x^{-n}}{n!} \frac{\Gamma(a_r+n) \prod_{t=1}^{p}{}' \Gamma(a_t - a_r - n)}{\prod_{t=1}^{p} \Gamma(\rho_t - a_r - n)}.$$

We thus have the given result.

11. *When $\mu > 1$ there is no direct analogue of the asymptotic equality just obtained: in fact, there is no region of the plane at infinity over which ${}_pF_q\{x\}$ can be represented by an asymptotic expansion whose dominant term is algebraically infinite (or zero).*†

To take the simplest case, suppose that $\mu = 2$ so that $q = p+1$. Then we have p equalities of the type (A), § 7, each of which gives the asymptotic expansion of a sum of $(p+2)$ hypergeometric integral functions valid when $|\arg x| < 2\pi$. These are equivalent to $2p$ relations between the $(p+2)$ functions when $|\arg x| < \pi$. But only $(p+1)$ of these relations are independent. Therefore the sum of suitable multiples of two hypergeometric integral functions $(p = q-1 > 0)$ can be expressed as an asymptotic expansion with its dominant term algebraic, and the equality will hold all round infinity; but a single hypergeometric integral function $(p = q-1)$ does not admit of such an expansion for any portion of the plane at infinity.

In the case $p = 0$, $q = 1$, discussed in my previous memoir, no asymptotic expansion exists whose dominant term is algebraic.

More generally, when $\mu \geqslant 2$, the p equalities of the type (A), § 7, give rise to $(p+1)$ independent asymptotic equalities, each involving $(q+1)$ hypergeometric integral functions. Hence the sum of suitable multiples of $q+1-p = \mu$ such functions can be represented by an asymptotic expansion whose dominant term is algebraic, but no combination of less than μ is capable of such representation.

† A reference to Parts II. and III. will shew the connection between this theorem and the fact that e^x tends to zero as $|x|$ tends to infinity when $|\arg(-x)| < \frac{1}{2}\pi$, while $\exp(\mu/x^{1/\mu})$ where μ is an integer > 1 does not possess a similar property when $|\arg x| < \pi$.

12. The fundamental combination of this type may be obtained by considering the integral

$$-\frac{1}{2\pi\iota}\int \frac{\Gamma(-s)\prod_{r=1}^{p}\Gamma(a_r+s)\prod_{r=p+1}^{q}\Gamma(1-\rho_r-s)}{\prod_{r=1}^{p}\Gamma(\rho_r+s)}(-x)^s\,ds.$$

If the integral be taken round a contour which embraces the positive half of the real axis and excludes the poles of $\prod_{r=1}^{p}\Gamma(a_r+s)$, it will be equal to

$$\frac{\prod_{r=1}^{p}\Gamma(a_r)}{\prod_{r=1}^{p}\Gamma(\rho_r)}\prod_{r=p+1}^{q}\Gamma(1-\rho_r)\,{}_pF_q\{a_1,\ldots,a_p;\ \rho_1,\ldots,\rho_q;\ (-)^{q-p}x\}$$

$$+\sum_{r=p+1}^{q}(-x)^{1-\rho_r}\frac{\Gamma(\rho_r-1)\prod_{t=1}^{p}\Gamma(1+a_t-\rho_r)\prod_{t=p+1}^{q}{}'\,\Gamma(\rho_r-\rho_t)}{\prod_{t=1}^{p}\Gamma(\rho_t+1-\rho_r)}$$

$$\times{}_pF_q\{1+a_1-\rho_r,\ \ldots,\ 1+a_p-\rho_r;$$

$$2-\rho_r,\ 1+\rho_1-\rho_r,\ \ldots,\ ^{*}\ldots,\ 1+\rho_q-\rho_r;\ (-)^{q-p}x\}.$$

If $|\arg(-x)|<\frac{1}{2}\mu\pi$, the integral may be swung back in the previous manner and we shall obtain, as the asymptotic expansion of the linear combination of the μ generalised hypergeometric integral functions just written down, the series

$$\sum_{r=1}^{p}(-x)^{-a_r}\frac{\Gamma(a_r)\prod_{t=1}^{p}{}'\,\Gamma(a_t-a_r)\prod_{t=p+1}^{q}\Gamma(1+a_r-\rho_t)}{\prod_{t=1}^{p}\Gamma(\rho_t-a_r)}$$

$$\times{}_{p+1}F_{q-1}\{a_r,\ a_r+1-\rho_1,\ \ldots,\ a_r+1-\rho_q;$$

$$1+a_r-a_1,\ \ldots,\ ^{*}\ldots,\ 1+a_r-a_p;\ -1/x\}.$$

The sum of these p series is asymptotic in Poincaré's sense.

In the same way, by considering other contour integrals which may with ease be written down, we can obtain suitable combinations of any number [less than $(q+2)$ and greater than $(\mu-1)$] of hypergeometric integral functions whose asymptotic expansions have their dominant terms algebraic.

We thus get all the results which can be derived from the $(p+1)$ independent asymptotic expansions contained in the formula (A) of § 7 by eliminating any possible number of the hypergeometric integral functions involved.

Other combinations of the hypergeometric integral functions involved have their dominant terms exponentially infinite : to the development of such expansions we now proceed.

PART II.

The Exponentially Infinite Asymptotic Expansions of Linear Combinations of Generalised Hypergeometric Integral Functions for which $p = q$.

13. In the ensuing investigation we limit ourselves to the case $p = q$, $\mu = 1$, partly because the theory may be developed in a more simple manner than the most general theory when $\mu > 1$, partly because the results differ somewhat in character from those of that theory.

The differential equation of the functions considered is now

$$[(\vartheta+a_1)\ \dots\ (\vartheta+a_p)-\frac{d}{dx}(\vartheta+\rho_1-1)\ \dots\ (\vartheta+\rho_p-1)]\,y = 0.$$

It is of order $(p+1)$, and the independent primitives are

$$_pF_p\{a_1, \dots, a_p;\ \rho_1, \dots, \rho_p;\ x\},$$

$$x^{1-\rho_m}\,{}_pF_p\{a_1-\rho_m+1, \dots, a_p-\rho_m+1;$$
$$2-\rho_m,\ \rho_1-\rho_m+1, \dots, * \dots, \rho_p-\rho_m+1;\ x\}$$
$$(m = 1, 2, \dots, p).$$

We have seen in §§ 9 and 10 that when $R(x) < 0$ each of these functions can, near $|x| = \infty$, be represented by an asymptotic expansion whose dominant term is algebraic. We now proceed to find asymptotic expansions of these functions, valid when $R(x) > 0$, and to find Orr's linear combination of the functions which has an asymptotic expansion of the same exponential type.

For this purpose we construct the function $_pS_p(s;\ a_1, \dots, a_p;\ \rho_1, \dots, \rho_p)$ or, as we shall briefly write it, $_pS_p(s)$, which, when $R(s) > R(\Sigma a-\Sigma\rho)$, is represented by the convergent series of gamma functions

$$\sum_{t=0}^{\infty}\frac{\Gamma(-s+t)}{\Gamma(t+1)}\prod_{r=1}^{p}\left\{\frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)}\right\}.$$

I have previously discussed the corresponding function for which $p = 1$, and have shewn (*loc. cit.*, p. 263) that

$$_1S_1(s;\ a;\ \rho) = \frac{\Gamma(-s)\,\Gamma(-s+1-\rho)\,\Gamma(s+\rho-a)}{\Gamma(1-a)\,\Gamma(\rho-a)}.$$

This equality corresponds to Gauss's well known theorem. It shews that ${}_1S_1(s)$ admits of analytic continuation over the whole finite portion of the s-plane; that ${}_1S_1(s)$ has simple poles at $s = a-\rho-r$ $(r = 0, 1, 2, \ldots, \infty)$, and no other finite singularities except those of $\Gamma(-s)$ and $\Gamma(1-s-\rho)$; and that, if $s = u+\iota v$ and u be finite, $|{}_1S_1(s)| \exp\{(\frac{3}{2}\pi-\epsilon)|v|\}$ tends to zero as $|v|$ tends to infinity, if $\epsilon > 0$. Unfortunately, it does not seem possible to obtain any such simple expression for ${}_pS_p(s)$ when $p > 1$. We must therefore employ other methods to obtain the corresponding properties of ${}_pS_p(s)$ which have been already stated in § 5.

14. *Let ${}_p^kS_p(s)$ denote the sum of all terms of the series by which ${}_pS_p(s)$ has been defined when $R(s) > R(\Sigma a - \Sigma\rho)$ except the first k of such terms. Choose the positive integer k so that $R(-s+k+1-\rho_r) > 0$ $(r = 1, 2, \ldots, p)$. Let l be a straight contour parallel to the imaginary axis passing to the left of the point $-s+k$ and to the right of the points $\rho_r - 1$. Then I say that*

$$\frac{\sin \pi s}{\pi}\, {}_p^kS_p(s) = \frac{1}{2\pi\iota}\int_l \frac{\Gamma(-\phi-s)}{\Gamma(1-\phi)} \prod_{r=1}^{p} \frac{\Gamma(1-\rho_r+\phi)}{\Gamma(1-a_r+\phi)}\, d\phi,$$

provided $R(s) > R(\Sigma a - \Sigma\rho)$.

For under this limitation the integral is convergent, since the subject of integration behaves at infinity like $(-\phi)^{-s-1}\phi^{\Sigma a_r - \Sigma\rho_r}$.

Also the integral will vanish when taken round an infinite circular contour to the right of the axis l if this contour pass between the poles of $\Gamma(-\phi-s)$.

Hence, by Cauchy's theory of residues, the integral is equal to

$$\sum_{t=k}^{\infty} \prod_{r=1}^{p} \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)} \frac{(-)^{t-1}}{\Gamma(t+1)\Gamma(1+s-t)}$$

$$= \sum_{t=k}^{\infty} \prod_{r=1}^{p} \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)} \frac{\Gamma(t-s)}{\Gamma(t+1)} \frac{\sin \pi s}{\pi}$$

$$= \frac{\sin \pi s}{\pi}\, {}_p^kS_p(s).$$

15. *We will next shew that $\frac{\sin \pi s}{\pi}\, {}_pS_p(s)$ can be expressed as the sum of multiples of p series of the form*

$$\sum_{n=0}^{\infty} \frac{\Gamma(1-\rho_r+n-s)}{\Gamma(n+1)} \prod_{t=1}^{p} \frac{\Gamma({}_rb_t+n)}{\Gamma({}_rc_t+n)} \quad (r = 1, 2, \ldots, p),$$

the b's and c's being linear combinations of the a's and ρ's which differ for different values of r.

Provided $R(s) > R(\Sigma a - \Sigma\rho)$, the previous integral vanishes when taken round an infinite circular contour to the left of the axis l if this contour pass between the poles of $\Gamma(1-\rho_r+\phi)$.

Hence $\pi^{-1}\sin\pi s\,{}_p^k S_p(s)$ is equal to *minus* the sum of the residues of $\frac{\Gamma(-\phi-s)}{\Gamma(1-\phi)}\prod_{r=1}^{p}\frac{\Gamma(1-\rho_r+\phi)}{\Gamma(1-a_r+\phi)}$ at its poles which lie to the left of the axis l.

We thus have, if $R(s) > R(\Sigma a - \Sigma\rho)$,

$$\pi^{-1}\sin\pi s\,{}_pS_p(s) = \sum_{r=1}^{p} T_r(s)$$

where
$$T_r(s) = \sum_{n=0}^{\infty}\frac{\prod_{t=1}^{p}{}'\,\Gamma(\rho_r-\rho_t-n)}{\prod_{t=1}^{p}\Gamma(\rho_r-a_t-n)}\,\frac{\Gamma(n+1-\rho_r-s)}{\Gamma(2-\rho_r+n)}\,\frac{(-)^{n-1}}{\Gamma(n+1)},$$

the accent denoting that the term corresponding to $t = r$ is to be omitted in the product.

Thus

$$T_r(s) = \sum_{n=0}^{\infty}\frac{\prod_{t=1}^{p}\Gamma(1-\rho_r+a_t+n)}{\prod_{t=1}^{p}{}'\,\Gamma(1-\rho_r+\rho_t+n)\,\Gamma(2-\rho_r+n)} \times\frac{\Gamma(1-\rho_r+n-s)}{\Gamma(n+1)}\,\frac{\prod_{t=1}^{p}\sin\pi(a_t-\rho_r)}{\pi\prod_{t=1}^{p}{}'\sin\pi(\rho_t-\rho_r)}.$$

Now this series for $T_r(s)$ is convergent if

$$R(s+\Sigma\rho-\Sigma a) > 0,$$

that is to say, under precisely the same limitation as the series by which ${}_pS_p(s)$ was defined.

Therefore, if $T_r(s)$ denote the analytic continuation of the corresponding series, we have, over the whole of the s-plane, the equality

$$\pi^{-1}\sin\pi s\,{}_pS_p(s) = \sum_{r=1}^{p} T_r(s).$$

We have therefore reduced the consideration of the function ${}_pS_p(s)$ to the consideration of functions of the type defined, when $R(s+\Sigma\rho-\Sigma a) > 0$, by the series
$$\sum_{n=0}^{\infty}\frac{\Gamma(1-\rho_r+n-s)}{\Gamma(n+1)}\prod_{t=1}^{p}\frac{\Gamma({}_rb_t+n)}{\Gamma({}_rc_t+n)}.$$

Our typical series we shall write

$$T(s) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n-s)}{\Gamma(n+1)}\prod_{t=1}^{p}\frac{\Gamma(b_t+n)}{\Gamma(c_t+n)}.$$

It is convergent if $R(s-a+\Sigma c-\Sigma b) > 0$.

The relation of the present paragraph is readily verified when $p = 1$. In this case we have, if $R(s+b-a) > 0$,

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t)}{\Gamma(t+1)} \frac{\Gamma(a+t-s)}{\Gamma(b+t-s)} = \frac{\sin \pi(a-b)}{\sin \pi s} \sum_{t=0}^{\infty} \frac{\Gamma(a+t-s)\,\Gamma(a-b+1+t)}{\Gamma(t+1)\,\Gamma(a+1+t)}$$

or
$$\frac{\Gamma(-s)}{\Gamma(b-s)}\, {}_2F_1(a-s, -s;\; b-s;\; 1)$$
$$= \frac{\sin \pi(a-b)}{\sin \pi s} \frac{\Gamma(a-b+1)}{\Gamma(a+1)}\, {}_2F_1(a-s,\; a-b+1;\; a+1;\; 1).$$

By Gauss's theorem this equality may be written

$$\frac{\Gamma(-s)\,\Gamma(b-a+s)}{\Gamma(b-a)\,\Gamma(b)} = \frac{\sin \pi(a-b)}{\sin \pi s} \frac{\Gamma(a-b+1)\,\Gamma(b-a+s)}{\Gamma(1+s)\,\Gamma(b)}.$$

It is therefore obviously true.

16. *Let* λ *be a contour parallel to the imaginary axis passing to the left of the point* k *and to the right of the points* $-b_t$ $(t = 1, 2, \ldots, p)$, *where* k *is a positive integer so chosen that* $k+R(b_t) > 0$.

Then, if ${}_kT(s)$ *denote the sum of all terms after the first* k *of the series by which* $T(s)$ *has been defined when* $R(s-a+\Sigma c-\Sigma b) > 0$, *we will shew that*

$$\frac{\sin \pi(s-a)}{\pi}\, {}_kT(s) = \frac{1}{2\pi \iota} \int_\lambda \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^{p} \frac{\Gamma(b_t+\phi)}{\Gamma(c_t+\phi)}\, dy.$$

For, as before, the integral is equal to the sum of the residues of the subject of integration to the right of the axis λ. It is therefore equal to

$$\sum_{n=k}^{\infty} \frac{(-)^{n-1}}{\Gamma(n+1)\,\Gamma(1-a-n+s)} \prod_{t=1}^{p} \frac{\Gamma(b_t+n)}{\Gamma(c_t+n)}$$
$$= \frac{\sin \pi(s-a)}{\pi} \sum_{n=k}^{\infty} \frac{\Gamma(a+n-s)}{\Gamma(n+1)} \prod_{t=1}^{p} \frac{\Gamma(b_t+n)}{\Gamma(c_t+n)}.$$

We thus have the given theorem.

17. *The Analytic Continuation of* ${}_kT(s)$.—*We proceed now to shew that, if* k_t *be the smallest integer such that* $R(k_t+b_t) > 0$,

$$\frac{\sin \pi(s-a)}{\pi}\, {}_kT(s) = -\sum_{t=1}^{p} \sum_{n=0}^{k_t-1} \frac{\Gamma(b_t+n)}{\Gamma(1-a+b_t+n-s)} \frac{\prod_{m=1}^{p}{}' \,\Gamma(b_m-b_t-n)}{\prod_{m=1}^{p} \Gamma(c_m-b_t-n)} \frac{(-)^n}{\Gamma(n+1)}$$
$$- \sum_{r=0}^{k} \frac{1}{\pi} \sum_{t=1}^{p} \frac{\prod_{m=1}^{p} \sin \pi(c_m-b_t)}{\prod_{m=1}^{p}{}' \sin \pi(b_m-b_t)}$$
$$\times\, \zeta(s-a+\Sigma c-\Sigma b+r+1,\; k_t+b_t)+I$$

where

$$I = \frac{1}{2\pi\iota}\int_{\lambda'}\left\{\frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)}\prod_{t=1}^{p}\frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} - \sum_{r=0}^{R}\frac{f_r(s)}{(-\phi)^{\sigma+r}}\right\}$$

$$\times\prod_{t=1}^{p}\frac{\sin\pi(c_t+\phi)}{\sin\pi(b_t+\phi)}\,d\phi,$$

$\sigma = s-a+\Sigma c-\Sigma b+1$, *and where* λ' *is a contour parallel to the imaginary axis which cuts the real axis just to the left of the origin. The equality is valid for all values of* s *such that*

$$R(s-a+\Sigma c-\Sigma b+R) > 0\,;$$

it is therefore valid for any finite value of u *if we take* R *sufficiently large.*

By the asymptotic expansion of the gamma function we know that, if

$$|\arg(-\phi)| < \pi,$$

$$\frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)}\prod_{t=1}^{p}\frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} = \sum_{r=0}^{R}\frac{f_r(s)}{(-\phi)^{\sigma+r}} + \frac{J_R(s,\phi)}{(-\phi)^{\sigma+R}}$$

where $f_0(s) = 1$, $f_r(s)$ is a polynomial in s, and $|J_R(s,\phi)|$ tends uniformly to zero as $|\phi|$ tends to infinity.

Hence the integral I is convergent provided $R(\sigma+R-1) > 0$. Consider now the integral

$$I_r = \frac{1}{2\pi\iota}\int_{\lambda'}\frac{1}{(-\phi)^{\sigma+r}}\prod_{t=1}^{p}\frac{\sin\pi(c_t+\phi)}{\sin\pi(b_t+\phi)}\,d\phi.$$

That value of $(-\phi)^{\sigma+r}$ is to be taken which is equal to

$$\exp\{(\sigma+r)\log(-\phi)\},$$

where $\log(-\phi)$ is real when ϕ is real and negative, and has a cross-cut along the positive half of the real axis. Hence the subject of integration is one-valued in the area to the left of λ'. Also, if $R(\sigma)+r > 1$, the integral vanishes when taken round an infinite contour enclosing this area. Hence, if $R(\sigma)+r > 1$,

$$I_r = \sum_{t=1}^{p}\frac{1}{\pi}\sum_{n=k_t}^{\infty}\frac{(-)^{n-1}}{(b_t+n)^{\sigma+r}}\,\frac{\prod_{m=1}^{p}\sin\pi(c_m-b_t-n)}{\prod_{m=1}^{p}{}'\sin\pi(b_m-b_t-n)},$$

the accent denoting that the term corresponding to $m = t$ is to be omitted. Thus

$$I_r = -\sum_{t=1}^{p}\frac{1}{\pi}\,\zeta(\sigma+r,\,b_t+k_t)\,\frac{\prod_{m=1}^{p}\sin\pi(c_m-b_t)}{\prod_{m=1}^{p}{}'\sin\pi(b_m-b_t)}$$

where $\zeta(s, a)$ denotes the Riemann ζ function defined, when $R(s) > 1$, by the equality

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}.$$

Now, by the result of § 16, we see that

$$\frac{\sin \pi (s-a)}{\pi}\, {}_kT(s)$$
$$= \frac{1}{2\pi\iota} \int_\lambda \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^{p} \left\{ \frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} \frac{\sin \pi (c_t+\phi)}{\sin \pi (b_t+\phi)} \right\} dy.$$

Hence, if we take account of the poles of the subject of integration between the contours λ and λ', we have the equality given in the enunciation, provided $R(\sigma) > 1$. But all the quantities on the right-hand side of the equality are continuous functions (except for isolated poles), analytic if $R(\sigma+R) > 1$.

We thus have the given theorem.

18. *We may now shew that, if $s = u+\iota v$, and u have any finite value,*

$$|T(s)| \exp \{(\tfrac{1}{2}\pi - \epsilon) |v|\},$$

where $\epsilon > 0$, can be made as small as we please by taking $|v|$ greater than an assignable large positive quantity V.

We take the result of the preceding paragraph.

Then, in the first place, the integral I is such that

$$|I| \exp \{(-\tfrac{1}{2}\pi - \epsilon) |v|\}$$

tends uniformly to zero as $|v|$ tends to infinity. For $(-\phi)^{\sigma+r}$ is such that the argument of $-\phi$ has its modulus $\leqslant \frac{1}{2}\pi$ on the line of integration. Also for any finite value of $|\phi|$ the expression

$$|\Gamma(1-a-\phi+s)| \exp \{(\tfrac{1}{2}\pi - \epsilon) |v|\}$$

tends uniformly to zero as $|v|$ increases; and therefore

$$\left| \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \right| \exp \{(-\tfrac{1}{2}\pi - \epsilon) |v|\}$$

will tend uniformly to zero for all values of ϕ, finite or infinite (including even values for which $\phi - s$ is finite when $|v|$ is very large) on the line of integration. Hence, since the integral is uniformly convergent with respect to s, we have the given statement.

In the second place, the modulus of each term of the finite double series of gamma functions of s which occurs first on the right-hand side of the equality of § 17 will evidently tend uniformly to zero when multiplied by $\{\exp(-\frac{1}{2}\pi-\epsilon)\,|\,v\,|\}$.

Finally, so far as the expressions I_r depend upon s, they depend upon the Riemann ζ functions $\zeta(\sigma+r,\ k_t+b_t)$. Now the integer k_t was so chosen that $|\arg(k_t+b_t)| < \frac{1}{2}\pi$. Hence, by the theory of the Riemann ζ function,† $|\zeta(\sigma+r,\ k_t+b_t)|\exp\{(-\frac{1}{2}\pi-\epsilon)\,|\,v\,|\}$ tends uniformly to zero as $|\,v\,|$ tends to infinity.

Hence, by the equality of § 17, we see that, for all finite values of u, $|\,\pi^{-1}\sin\pi(s-a)\,{}_kT(s)\,|\exp\{-\frac{1}{2}\pi-\epsilon\,|\,v\,|\}$ will tend uniformly to zero as $|v|$ tends to infinity. Therefore $|\,{}_kT(s)|\exp\{(\frac{1}{2}\pi-\epsilon)\,|\,v\,|\}$ possesses the same property. Now ${}_kT(s)$ only differs from $T(s)$ by the sum of constant multiples of a finite number of terms of the type $\Gamma(n+a-s)$; therefore $|\,T(s)\,|\exp\{(\frac{1}{2}\pi-\epsilon)\,|\,v\,|\}$ will tend uniformly to zero as $|v|$ tends to infinity.

We thus have the given theorem.

19.‡ *We may now shew that, if u is large and positive, and $|v|$ be large or finite,*

$$|\,T(s)\,| < K\exp\{(-\tfrac{1}{2}\pi+\epsilon)\,|\,v\,|-ku\},$$

where k is any finite positive quantity, $\epsilon > 0$, and K is a finite quantity independent of u and $|v|$. We assume that s is not in the immediate vicinity of the large poles $a+n$, n a positive integer, of $T(s)$.

We have seen in § 16 that, when u is sufficiently large,

$$\frac{\sin\pi(s-a)}{\pi}\,{}_mT(s) = \frac{1}{2\pi\iota}\int_\lambda \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)}\prod_{t=1}^{p}\frac{\Gamma(b_t+\phi)}{\Gamma(c_t+\phi)}\,d\phi,$$

where λ passes to the left of the point m, and to the right of the points $-b_t$ $(t = 1, 2, \ldots, p)$, and m is the smallest positive integer which makes

$$m+R(b_t) > 0.$$

Now, if $\quad R(u-m-a) > 0 \quad$ and $\quad R(m-\phi) > -1$,

$$|\,\Gamma(u-m-a)\,|\;|\,\Gamma(1+\iota v+m-\phi)\,|$$
$$= |\,\Gamma(u-a+1+\iota v-\phi\,|\;\left|\int_0^1 x^{u-a-m-1}(1-x)^{m+\iota v-\phi}\,dx\right|.$$

† See, for instance, a paper by the author, "The Maclaurin Sum Formula," *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 256.

‡ The author expresses his thanks to the referees for pointing out an arithmetical error in the original statement of this theorem.

Also, if u is sufficiently large, and $R(m-\phi)>0$,

$$\left|\int_0^1 x^{u-a-m-1}(1-x)^{m+\iota v-\phi}\,dx\right| \leqslant \int_0^1 x^{u-R(a)-m-1}(1-x)^{m-R(\phi)}\,dx < 1.$$

Hence $|1/\Gamma(1-a-\phi+s)| < 1/\{|\Gamma(u-m-a)|\;|\Gamma(1+m+\iota v-\phi)|\}$.

Therefore

$$\left|\frac{\sin\pi(s-a)}{\pi}\,{}_mT(s)\right|$$

$$< \frac{1}{2\pi|\Gamma(u-m-a)|}\int_\lambda \left|\frac{\Gamma(-\phi)}{\Gamma(1+m+\iota v-\phi)}\prod_{t=1}^{p}\frac{\Gamma(b_t+\phi)}{\Gamma(c_t+\phi)}\right|\,|d\phi|$$

$$< K\exp\{\tfrac{1}{2}\pi(1+\epsilon)|v| - ku\},$$

as in § 18, where $\epsilon>0$, K is finite and independent of u and $|v|$, and k is any finite positive quantity, however large.

Hence, if s be not in the immediate vicinity of one of the poles $a+n$ of $T(s)$,

$$|{}_mT(s)| < K\exp\{(-\tfrac{1}{2}\pi+\epsilon)|v| - ku\}.$$

The same inequality is evidently true, under the same limitation, of the finite number of terms by which $T(s)$ differs from ${}_mT(s)$.

We therefore have the given theorem.

20. *We can now shew that the function* ${}_pS_p(s)$ *is such that, if* $s=u+\iota v$, *and u be finite,*

$$|{}_pS_p(s)|\,e^{(\frac{1}{2}\pi-\epsilon)|v|},$$

where $\epsilon>0$, *tends uniformly to zero as* $|v|$ *tends to infinity.*

For we have seen in § 15 that

$$\pi^{-1}\sin\pi s\;{}_pS_p(s) = \sum_{r=1}^{p} T_r(s).$$

Therefore $$|\pi^{-1}\sin\pi s|\,e^{(\frac{1}{2}\pi-\epsilon)|v|}\,|{}_pS_p(s)| < \sum_{r=1}^{p}|T_r(s)|\,e^{(\frac{1}{2}\pi-\epsilon)|v|},$$

that is to say, is less than a quantity which tends uniformly to zero.

We therefore have the given theorem.

Again, we can show that ${}_pS_p(s)$ *admits of analytic continuation over the whole of the finite portion of the s-plane.*

For we have seen in § 17 that this is true of the functions $T_r(s)$; it is therefore true of the function ${}_pS_p(s)$.

It is similarly evident *that, if u be large and positive and $|v|$ be very large or finite,*

$$|{}_pS_p(s)| < K\exp\{(-\tfrac{3}{2}\pi+\epsilon)|v| - ku\}$$

where K is finite and independent of u and $|v|$, and k is any finite positive quantity, however large, provided s be not in the immediate vicinity of one of the poles $s = n$ or $1-\rho_r+n$ $(r = 1, 2, \ldots, p)$ of ${}_pS_p(s)$.

21. *We proceed now to shew that ${}_pS_p(s)$ has, except for those poles which are poles of $\Gamma(-s)$ or $\Gamma(1-\rho_r+s)$ $(r = 1, 2, \ldots, p)$, as its sole finite singularities poles at the points*

$$s = a_1+\ldots+a_p-\rho_1-\ldots-\rho_p-r \quad (r = 0, 1, 2, \ldots, \infty),$$

and that the residues at these poles may be obtained with sufficient labour.

By the definition of § 13, ${}_pS_p(s)$ is represented, when $R(s) > R(\Sigma a-\Sigma\rho)$, by the convergent series of gamma functions

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t)}{\Gamma(t+1)} \prod_{r=1}^{p} \left\{\frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)}\right\}.$$

By the asymptotic expansion of the gamma function we know that, when t is large,

$$\frac{\Gamma(-s+t)}{\Gamma(1+t)} \prod_{r=1}^{p} \left\{\frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)}\right\} = \frac{1}{t^{s+1+\Sigma\rho-\Sigma a}} \exp\left\{\sum_{m=1}^{\mu} \frac{(-)^{m-1}}{mt^m} V_m + \frac{J_\mu}{t^\mu}\right\}$$

where, for any finite value of μ, $|J_\mu|$ can be made as small as we please by taking t sufficiently large, and where

$$V_m = \sum_{r=1}^{p} \{S_m(1-\rho_r-s)-S_m(1-a_r-s)\} + S_m(-s)-S_m(1).$$

$S_m(x)$ denotes the m-th simple Bernoullian function of x, and is a polynomial of degree $(m+1)$ in x.

When t is large we see, then, that the above term may be asymptotically written

$$\frac{1}{t^{s+1+\Sigma\rho-\Sigma a}} \left\{\sum_{r=0}^{R} \frac{f_r(s)}{t^r} + \frac{J_R(s)}{t^R}\right\}$$

where $f_0(s) = 1$, and $f_r(s)$ is a polynomial in s whose value for any assigned value of r can be determined with sufficient labour.

Consider now the function

$${}_pS_p(s) - \sum_{r=0}^{R} f_r(s)\, \zeta(s+r+1+\Sigma\rho-\Sigma a).$$

When $R(s+\Sigma\rho-\Sigma a) > 0$ this function may be expressed by the series

$$\sum_{t=0}^{\infty}{}'' \left\{\frac{\Gamma(-s+t)}{\Gamma(1+t)} \prod_{r=1}^{p} \frac{\Gamma(1-\rho_r-s+t)}{\Gamma(1-a_r-s+t)} - \sum_{r=0}^{R} \frac{f_r(s)}{t^{r+s+1+\Sigma\rho-\Sigma a}}\right\}$$

(the double accent denoting that the summation $\sum_{r=0}^{R}$ does not exist when $t=0$), and this series is equal to

$$\sum_{t=0}^{\infty} \frac{J_R(s)}{t^{s+1+R+\Sigma\rho-\Sigma a}}.$$

But this series is convergent, provided

$$R(s+R+\Sigma\rho-\Sigma a) > 0.$$

Hence the function

$${}_pS_p(s) - \sum_{r=0}^{R} f_r(s)\,\zeta(s+r+1+\Sigma\rho-\Sigma a)$$

has no singularities when

$$R(s) > \Sigma a - \Sigma\rho - R.$$

Now the sole finite singularity of $\zeta(s)$ is a pole at the point $s=1$ at which the residue is unity.

Hence the sole finite singularities of ${}_pS_p(s)$ apart from those poles which are poles of $\Gamma(-s)$ or $\Gamma(1-\rho_r+s)$ $(r=1, 2, \ldots, p)$ are poles at the points

$$s = \Sigma a - \Sigma\rho - r \quad (r = 0, 1, 2, \ldots, \infty),$$

At the point $s=\Sigma a-\Sigma\rho-r$ the residue is $f_r(\Sigma a-\Sigma\rho-r)$, and this quantity can be calculated with sufficient labour, should necessity arise.

22. *We may now prove that, if* $|\arg(-x)| < \frac{3}{2}\pi$,

$$\prod_{r=1}^{p}\left\{\frac{\Gamma(1-\rho_r)}{\Gamma(1-a_r)}\right\} {}_pF_p\{a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_p;\ x\}$$

$$+\sum_{r=1}^{p} (-x)^{1-\rho_r} \frac{\Gamma(\rho_r-1)\prod_{m=1}^{p}\Gamma(\rho_r-\rho_m)}{\prod_{m=1}^{p}\Gamma(\rho_r-a_m)}$$

$$\times {}_pF_p\{1+a_1-\rho_r, \ldots, 1+a_p-\rho_r;$$

$$2-\rho_r, 1+\rho_1-\rho_r, \ldots, *, \ldots, 1+\rho_p-\rho_r;\ x\}$$

$$= e^x(-x)^{\Sigma a-\Sigma\rho}\left\{\sum_{r=0}^{R}\frac{P_r}{x^r}+\frac{J_R}{x^R}\right\}$$

where $|J_R|$ *tends uniformly to zero as* $|x|$ *tends to infinity. The quantity* $(-)^r P_r$ *is the residue of* ${}_pS_p(s)$ *at its pole* $s=\Sigma a-\Sigma\rho-r$.

We consider the integral

$$-\frac{1}{2\pi i}\int {}_pS_p(s)\,x^s\,ds,$$

taken round a contour which encloses all the poles of ${}_pS_p(s)$ except the poles

$$s = \Sigma a - \Sigma \rho - r \quad (r = 0, 1, 2, \ldots, \infty),$$

and embraces the positive half of the real axis. The integral is convergent by § 20 for all finite values of $|x|$, and by Cauchy's theory of residues is equal to

$$\sum_{n=0}^{\infty} \sum_{=0}^{\infty} \frac{(-)^n}{n!\,\Gamma(t+1)} \prod_{r=1}^{p} \left\{ \frac{\Gamma(1-\rho_r-n)}{\Gamma(1-a_r-n)} \right\} x^{t+n}$$

$$+\sum_{r=1}^{n} x^{1-\rho_r} \sum_{n=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-)^n}{n!\,\Gamma(t+1)} \frac{\Gamma(\rho_r-1-n) \prod\limits_{m=1}^{p}{}' \Gamma(\rho_r-\rho_m-n)}{\prod\limits_{m=1}^{p} \Gamma(\rho_r-a_m-n)} x^{t+n}$$

$$= e^x \prod_{r=1}^{p} \left\{ \frac{\Gamma(1-\rho_r)}{\Gamma(1-a_r)} \right\} {}_pF_p \{ a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_p;\ -x \}$$

$$+e^x \sum_{r=1}^{p} x^{1-\rho_r} \frac{\Gamma(\rho_r-1) \prod\limits_{m=1}^{p}{}' \Gamma(\rho_r-\rho_m)}{\prod\limits_{m=1}^{p} \Gamma(\rho_r-a_m)}$$

$$\times {}_pF_p \{ 1+a_1-\rho_r, \ldots, 1+a_p-\rho_r;$$

$$2-\rho_r, 1+\rho_1-\rho_r, \ldots, *, \ldots, 1+\rho_p-\rho_r;\ -x \}.$$

Again, by the results of § 20, if $|\arg x| < \frac{3}{2}\pi$, the contour of integration may be swung back as in § 8, and we get for the value of the integral the asymptotic series

$$\sum_{r=0}^{R} (-)^r P_r x^{\Sigma a - \Sigma \rho - r} + J_R / x^{\Sigma a - \Sigma \rho + R}$$

where $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity. Changing x into $(-x)$, we have the given result.

The result of this paragraph is a generalisation of that previously obtained† for the case $p = 1$. In that case the coefficients P_r admit of a simple expression, and we have the theorem that, if $|\arg(-x)| < \frac{3}{2}\pi$,

$$\frac{\Gamma(1-\rho)}{\Gamma(1-a)} {}_1F_1 \{a;\ \rho;\ x\} + \frac{\Gamma(\rho-1)}{\Gamma(\rho-a)} (-x)^{1-\rho} {}_1F_1 \{a-\rho+1;\ 2-\rho;\ x\}$$

admits the asymptotic expansion

$$e^x (-x)^{a-\rho} {}_2F_0 \{\rho-a;\ 1-a;\ 1/x\}.$$

† *Loc. cit.*, § 1, Paper η, § 15.

23. We now see that the $(p+1)$ asymptotic solutions of the differential equation of § 13,

$$\left[(\vartheta+a_1)\dots(\vartheta+a_p)-\frac{d}{dx}(\vartheta+\rho_1-1)\dots(\vartheta+\rho_p-1)\right]y=0, \qquad (1)$$

are the series which have been obtained in § 7,

$$(-x)^{-a_r}\,{}_{p+1}F_{p-1}\{a_r,\ 1+a_r-\rho_1,\ \dots,\ 1+a_r-\rho_p;$$
$$1+a_r-a_1,\ \dots,\ *,\ \dots,\ 1+a_r-a_p;\ -1/x\} \qquad (A)$$

when $r=1, 2, \dots, p$, and the series

$$e^x(-x)^{\Sigma a-\Sigma\rho}\sum_{r=0}^{\infty}\frac{P_r}{x^r}, \qquad (B)$$

which is obtained in § 22.

In § 7 we have connected each of the first series with the hypergeometric integral functions which exist as principal solutions of the differential equation in the finite part of the plane, and in § 22 we have connected the last series with the same solutions.

If $R(x)>0$, the result of § 22, which is valid if $|\arg(-x)|<\frac{3}{2}\pi$, is therefore equivalent to two relations. But in this case the series (B) outweighs all the series (A), and therefore each hypergeometric integral function which is a principal solution of the differential equation (1) can be expressed, when $R(x)>0$, as a multiple of the series (B). The different principal solutions are therefore expressible as multiples of the same asymptotic series, though, of course, they are not corresponding multiples of one another. We have an illustration of Poincaré's axiom that different functions may admit the same asymptotic expansion.

24. When $R(x)>0$ the actual expression of, let us say, ${}_pF_p\{x\}$ as such an asymptotic expansion is not best obtained by elimination between the results of §§ 7 and 22. It may be obtained directly by considering the contour integral

$$-\frac{1}{2\pi\iota}\int T(s)\,x^s\,ds,$$

where $T(s)$ denotes the series defined when $R(s)>R(\Sigma a-\Sigma\rho)$ by the convergent series of gamma functions

$$\sum_{n=0}^{\infty}\frac{\Gamma(n-s)}{\Gamma(n+1)}\prod_{r=1}^{p}\left\{\frac{\Gamma(a_r+n)}{\Gamma(\rho_r+n)}\right\}.$$

It is evident that this function is a particular case of the function $T(s)$ considered in §§ 15–19 for which $a=0$.

Hence $T(s)$ has poles when $s = n$ $(n = 0, 1, \ldots, \infty)$, and when

$$s = \Sigma a - \Sigma \rho - n.$$

If the integral be taken round a contour C which embraces the positive half of the real axis and encloses all the poles $s = n$, but none of the poles $s = \Sigma a - \Sigma \rho - n$, it is evidently convergent for all finite values of $|x|$, and equal, by Cauchy's theory of residues, to

$$\sum_{n=0}^{\infty} \sum_{t=0}^{\infty} \prod_{r=1}^{p} \left\{ \frac{\Gamma(a_r+n)}{\Gamma(\rho_r+n)} \right\} \frac{(-)^t x^{n+t}}{t!\,\Gamma(n+1)} = \prod_{r=1}^{p} \left\{ \frac{\Gamma(a_r)}{\Gamma(\rho_r)} \right\} e^{-x} {}_pF_p \{a;\ \rho;\ x\}.$$

Now the residue at $s = \Sigma a - \Sigma \rho$ is unity; and the residue at

$$s = \Sigma a - \Sigma \rho - r$$

can be calculated with sufficient labour from the asymptotic expansion of the gamma function: let it be M_r.

If $|\arg x| < \frac{1}{2}\pi$, we see by § 18 that the contour of the integral can be swung back as in § 8, and we get for the value of the integral the asymptotic series

$$x^{\Sigma a - \Sigma \rho} \left\{ 1 + \sum_{r=1}^{R} \frac{M_r}{x^r} + \frac{J_R}{x^R} \right\}$$

where, for all finite values of R, $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity.

Therefore, if $|\arg x| < \frac{1}{2}\pi$, we have the asymptotic equality

$$\prod_{r=1}^{p} \left\{ \frac{\Gamma(a_r)}{\Gamma(\rho_r)} \right\} {}_pF_p \{x\} = e^x x^{\Sigma a - \Sigma \rho} \left\{ 1 + \sum_{r=1}^{R} \frac{M_r}{x^r} + \frac{J_R}{x^R} \right\}.$$

This result evidently includes that obtained by Stokes for real positive values of the variable (*vide* § 3).

Evidently we can obtain similar exponentially infinite asymptotic expansions, when $R(x) > 0$, for the other principal solutions

$$x^{1-\rho_r} {}_pF_p \{a_1 - \rho_r + 1, \ldots, a_p - \rho_r + 1;$$
$$2 - \rho_r, \rho_1 - \rho_r + 1, \ldots, *, \ldots, \rho_p - \rho_r + 1;\ x\}$$

of the differential equation (1) of § 23.

25. When $R(x) < 0$ the result of § 22 is equivalent to but one identity.

We thus see that, when $R(x) < 0$, although, as in §§ 9 and 10, each hypergeometric integral function which is a principal solution in the finite part of the plane of the differential equation (1) of § 23 is represented near $|x| = \infty$ by linear combinations of the p asymptotic series whose dominant terms are of the type x^{-a_r} $(r = 1, 2, \ldots, p)$, yet one, and only one, particular linear combination of the principal solutions exists which is exponentially zero when $|x|$ is very large. This combination is given in § 22.

The complete asymptotic theory of the solutions of the differential equation (1) of § 23 is now concluded. We next proceed to discuss the more difficult case when $p \neq q$.

PART III.

The Exponentially Infinite Asymptotic Expansions of Linear Combinations of Generalised Hypergeometric Integral Functions for which $p \neq q$.

26. We proceed now to consider the further theory of those hypergeometric integral functions for which $p < q$, and consequently $\mu > 1$.

We shall in the first place investigate the function

$$ {}_pS_q \quad (s\,;\; a_1,\,\ldots,\,a_p\,;\; \rho_1,\,\ldots,\,\rho_q), $$

or, as we shall briefly write it, ${}_pS_q(s)$, which, when

$$ R(s) > R\left\{a_1+\ldots+a_p-\rho_1-\ldots-\rho_q+\tfrac{1}{2}(\mu-1)\right\}/\mu, \tag{1} $$

is represented by the convergent series of gamma functions

$$ \sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu)\prod\limits_{r=1}^{q}\Gamma(1-\rho_r+t/\mu-s)}{\prod\limits_{r=1}^{\mu}\Gamma\left(\dfrac{t+r}{\mu}\right)\prod\limits_{r=1}^{p}\Gamma(1-a_r+t/\mu-s)}. $$

The number $\mu = q+1-p$ is an integer.

By the multiplication formula for the gamma function, we have equally under the fundamental restriction (1)

$$ {}_pS_q(s)\,(2\pi)^{\frac{1}{2}(\mu-1)}\,\mu^{-\frac{1}{2}} = \sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu)\prod\limits_{r=1}^{q}\Gamma(1-\rho_r+t/\mu-s)}{\Gamma(t+1)\prod\limits_{r=1}^{p}\Gamma(1-a_r+t/\mu-s)}\,\mu^{t}. $$

We shall shew that ${}_pS_q(s)$ possesses properties which are analogous to those of ${}_pS_p(s)$ obtained in Part II., and which have already been stated in § 6.

These properties are direct generalisations of the properties of the function ${}_0S_1(x)$ defined when $R(2s+\rho-\frac{1}{2}) > 0$ by the series

$$ \sum_{t=0}^{\infty} \frac{\Gamma(-s+\frac{1}{2}t)\,\Gamma(1-\rho-s+\frac{1}{2}t)}{\Gamma(\frac{1}{2}t+1)\,\Gamma(\frac{1}{2}t+\frac{1}{2})}. $$

It is easy to see that, by an application of Gauss's theorem, ${}_0S_1(s)$ could be written in the form

$$\pi^{-1}\sin\pi(\rho-\tfrac{1}{2})\,2^{2\rho+4s}\,\Gamma(-2s)\,\Gamma(2-2\rho-2s)\,\Gamma(2s+\rho-\tfrac{1}{2}).$$

27. *We note in the first place that* ${}_pS_q(s)$ *can be written as the sum of* μ *series in the form*

$$\sum_{n=0}^{\mu-1}\sum_{t=0}^{\infty}\frac{\Gamma(-s+t+n/\mu)\prod_{r=1}^{q}\Gamma(1-\rho_r+t+n/\mu-s)}{\Gamma(t+1)\prod_{m=1}^{\mu}{}''\,\Gamma\left(t+\dfrac{m+n}{\mu}\right)\prod_{r=1}^{p}\Gamma(1-\alpha_r+t+n/\mu-s)},$$

the double accent denoting that the term corresponding to $m=\mu-n$ *is to be omitted.*

Each of these series is of the same type as the series ${}_pS_p(s)$ previously considered in Part II., with the exception that s does not occur in μ gamma functions in the denominator of the typical term of the series.

Let, now, ${}_p^kS_q(s)$ denote the sum of all terms of the series by which ${}_pS_q(s)$ has been defined under the condition (1) except the first k terms. Choose the positive integer k so that

$$R(-s+k/\mu+1-\rho_r)>0\quad(r=1,2,\dots,q).$$

Let l be a straight contour parallel to the imaginary axis passing to the left of the point $k/\mu-s$ and to the right of the points ρ_r-1 $(r=1,2,\dots,q)$, and also of the point $(k-1)/\mu-s$.

Then, exactly as in § 14, we see that

$$\begin{aligned}&{}_p^kS_q(s)\\&=\sum_{n=0}^{\mu-1}\frac{\pi}{\sin\pi(s-n/\mu)}\frac{1}{2\pi\iota}\int_l\frac{\Gamma(-\phi-s+n/\mu)}{\Gamma(1-\phi)\prod_{m=1}^{\mu}{}''\,\Gamma(m/\mu+\phi+s)}\,\frac{\prod_{r=1}^{q}\Gamma(1-\rho_r+\phi)}{\prod_{r=1}^{p}\Gamma(1-\alpha_r+\phi)}\,d\phi;\end{aligned}\tag{A}$$

for the terms of the series ${}_p^kS_q(s)$ which arise from poles of $\Gamma(-\phi-s+n/\mu)$ occur when $\phi+s-n/\mu=t$ or $\phi+s=t+n/\mu$, when these points belong to the series k/μ, $(k+1)/\mu$, ... ; that is to say, when

$$\mu t+n \geqslant k.$$

28. *We may now shew that, under the condition* (1) *of* § 26,

$$\begin{aligned}&{}_p^kS_q(s)\\&=-\frac{1}{2\pi\iota}\int_l\frac{\pi\mu\sin\pi\mu\phi\,\Gamma(\phi)}{\sin\pi\mu s\sin\pi\mu(s+\phi)}\,\frac{\prod_{r=1}^{q}\Gamma(1-\rho_r+\phi)}{\prod_{m=1}^{\mu}\Gamma(m/\mu+\phi+s)\prod_{r=1}^{p}\Gamma(1-\alpha_r+\phi)}\,d\phi.\end{aligned}$$

For the previous integral (A) is equal to

$$\frac{1}{2\pi\iota}\int_l \frac{\prod_{r=1}^{q}(1-\rho_r+\phi)}{\prod_{r=1}^{p}\Gamma(1-a_r+\phi)\prod_{m=1}^{\mu}\Gamma(m/\mu+\phi+s)}\,\Theta\, d\phi$$

where $\Theta = \sum_{n=0}^{\mu-1} \frac{\pi}{\sin\pi(s-n/\mu)} \frac{\Gamma(1-n/\mu+\phi+s)\,\Gamma(-\phi-s+n/\mu)}{\Gamma(1-\phi)}$

$$= -\frac{1}{\Gamma(1-\phi)}\sum_{n=0}^{\mu-1}\frac{\pi^2}{\sin\pi(s-n/\mu)\sin(\pi\phi+s-n/\mu)}$$

$$= -\frac{\pi^2}{\Gamma(1-\phi)\sin\pi\phi}\sum_{n=0}^{\mu-1}\{\cot\pi(s-n/\mu)-\cot\pi(\phi+s-n/\mu)\}$$

$$= -\frac{\pi^2\mu}{\Gamma(1-\phi)\sin\pi\phi}\{\cot\mu s\pi-\cot\mu\pi(s+\phi)\}$$

$$= -\frac{\mu\pi\sin\pi\mu\phi\,\Gamma(\phi)}{\sin\pi\mu s\sin\pi\mu(s+\phi)}.$$

The integral is evidently convergent if

$$R\left(\mu s+\frac{\mu+1}{2}+\Sigma\rho_r-\Sigma a_r+p-q\right) > 1,$$

i.e., if $\qquad R(s) > R\{\tfrac{1}{2}(\mu-1)+\Sigma a_r-\Sigma\rho_r\}/\mu,$

which is the condition (1) of § 26.

We note that the integral just obtained may equally be written

$$ {}^k_pS_q(s) = \frac{1}{2\pi\iota}\int_l \frac{\prod_{r=1}^{q}\Gamma(1-\rho_r+\phi)\,\Gamma(\phi)\,\Gamma(-\mu s-\mu\phi)}{\prod_{r=1}^{p}\Gamma(1-a_r+\phi)}\,\frac{\sin\pi\mu\phi}{\sin\pi\mu s}\,\frac{\mu^{\mu\phi+\mu s+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}}\,d\phi; \quad (1)$$

for by the multiplication formula for the gamma function

$$\prod_{m=1}^{\mu}\Gamma(m/\mu+\phi+s) = \frac{\Gamma(\mu\phi+\mu s+1)\,(2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\mu\phi+\mu s+\frac{1}{2}}}.$$

The equality (1) is, of course, obvious when the integral is once written down: the process adopted has been employed to shew how such an integral may be built up from others of a more elementary character.

29. *We now proceed to shew that*

$$\sin\pi\mu s\,{}_pS_q(s) = -\sum_{r=1}^{q}\frac{\prod_{t=1}^{p}\sin\pi(\rho_r-a_t)}{\prod_{t=1}^{q}{}'\sin\pi(\rho_r-\rho_t)}\,\frac{\sin\pi\mu(\rho_r-1)}{\sin\pi(\rho_r-1)}\,\frac{\mu^{\mu s+\mu+1}}{2^{\mu-1}}\,U_r(s)$$

where $U_r(s)$ is the function defined when

$$R\{\mu s+\Sigma\rho-\Sigma a-\tfrac{1}{2}(\mu-1)\}>0$$

by the series

$$\sum_{n=0}^{\infty}\frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)}\frac{\prod_{t=1}^{p}\Gamma(1+a_t-\rho_r+n)\prod_{t=1}^{\mu-1}\Gamma(2+t/\mu-\rho_r+n)}{\prod_{t=1}^{q}\Gamma(1+\rho_t-\rho_r+n)}.$$

Since $\mu=q+1-p$, we may put

$$\begin{aligned} {}_ra_t &= 2+t/\mu-\rho_r \quad (t=1,2,\ldots,\mu-1)\\ &= 1+a_t-\rho_r \quad (t=\mu,\mu+1,\ldots,q),\\ {}_rb_t &= 1+\rho_t-\rho_r \quad (t=1,2,\ldots,q);\end{aligned}$$

and now we may say that $U_r(s)$ is the function defined when

$$R(\mu s+\mu+\Sigma b-\Sigma a)>1$$

by the series $$\sum_{n=0}^{\infty}\frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)}\prod_{t=1}^{q}\frac{\Gamma({}_ra_t+n)}{\Gamma({}_rb_t+n)}$$

where one of the quantities ${}_rb_t$ is unity.

The theorem therefore enables us to reduce the consideration of the function ${}_pS_q(s)$ to that of the more simple function $U(s)$.

Under the condition

$$R(s)>R\{\tfrac{1}{2}(\mu-1)+\Sigma a-\Sigma\rho\}/\mu,$$

the integral of the previous paragraph will vanish when taken round that part of an infinite circular contour which lies to the left of the axis l. If, then, we apply Cauchy's theory of residues, we have

$$\sin\pi\mu s\,{}_pS_q(s)=\sum_{r=1}^{q}\sum_{n=0}^{\infty}\frac{\prod_{t=1}^{q}{}'\,\Gamma(\rho_r-\rho_t-n)}{\prod_{t=1}^{p}\Gamma(\rho_r-a_t-n)}\frac{(-)^{n-1}\sin\pi\mu(\rho_r-1-n)}{\Gamma(n+1)(2\pi)^{\frac{1}{2}(\mu-1)}}$$
$$\times\Gamma(\rho_r-1-n)\,\Gamma(-\mu s-\mu\rho_r+\mu+\mu n)\,\mu^{\mu\rho_r-\mu-\mu n+\mu s+\frac{1}{2}},$$

the accent denoting that the term corresponding to $t=r$ is to be omitted. Thus

$$\sin\pi\mu s\;{}_pS_q(s)=-\sum_{r=1}^{q}\frac{\prod_{t=1}^{p}\sin\pi(\rho_r-a_t)}{\prod_{t=1}^{q}{}'\sin\pi(\rho_r-\rho_t)}\frac{\sin\pi\mu(\rho_r-1)}{\sin\pi(\rho_r-1)}\frac{\mu^{\mu s+\mu+1}}{2^{\mu-1}}U_r(s)$$

where

$$U_r(s)=(2\pi)^{\frac{1}{2}(\mu-1)}\mu^{\mu\rho_r-2\mu+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\prod_{t=1}^{p}\Gamma(1+a_t-\rho_r+n)}{\prod_{t=1}^{q}\Gamma(1+\rho_r-\rho_t+n)}\mu^{-\mu n}\frac{\Gamma(\mu n-\mu\rho_r-\mu s+\mu)}{\Gamma(2-\rho_r+n)}$$

Now by the multiplication formula for the gamma function

$$\Gamma(\mu n-\mu\rho_r+2\mu) = \frac{\mu^{\mu n-\mu\rho_r+2\mu-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}} \prod_{t=0}^{\mu-1} \Gamma(n-\rho_r+2+t/\mu).$$

Hence

$$U_r(s) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)} \frac{\prod_{t=1}^{p} \Gamma(1+a_t-\rho_r+n) \prod_{t=1}^{\mu-1} \Gamma(2+t/\mu-\rho_r+n)}{\prod_{t=1}^{q} \Gamma(1+\rho_t-\rho_r+n)}.$$

Evidently the series $U_r(s)$ converges if $R\{\mu s+\Sigma\rho-\Sigma a-\frac{1}{2}(\mu-1)\} > 0$, which is the limitation under which the series ${}_pS_q(s)$ was defined. Therefore the sum of the continuations of the specified multiples of the series $U_r(s)$ will give the function $\sin\pi\mu s\,{}_pS_q(s)$.

We thus have the given theorem.

30. *We can readily verify the preceding transformation when $p = 0$, $q = 1$, and therefore $\mu = 2$.*

We have

$${}_0S_1(s) = \frac{1}{2\pi\iota}\int_l \Gamma(1-\rho+\phi)\,\Gamma(\phi)\,\Gamma(-2s-2\phi)\,\frac{\sin 2\pi\phi}{\sin 2\pi s}\,\frac{2^{2\phi+2s+\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}}\,d\phi,$$

and the transformation shews that

$$\sin 2\pi s\,{}_0S_1(s) = -\frac{2^{2\rho+2s-\frac{1}{2}}}{(2/\pi)^{\frac{1}{2}}}\,\frac{\sin 2\pi(\rho-1)}{\sin\pi(\rho-1)}\,U(s) \tag{1}$$

where $$U(s) = \sum_{n=0}^{\infty} \frac{2^{-2n}\,\Gamma(2n-2\rho-2s+2)}{\Gamma(1+n)\,\Gamma(2-\rho+n)},$$

the series being convergent if $R(2s+\rho) > \frac{1}{2}$.

Now $$U(s) = \frac{2^{-2\rho-2s+\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n-\rho-s+1)\,\Gamma(n-\rho-s+\frac{3}{2})}{\Gamma(1+n)\,\Gamma(2-\rho+n)},$$

and therefore, by Gauss's theorem, is equal to

$$\frac{2^{-2\rho-2s+\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}}\,\frac{\Gamma(1-s-\rho)\,\Gamma(\frac{3}{2}-s-\rho)\,\Gamma(-\frac{1}{2}+\rho+2s)}{\Gamma(1+s)\,\Gamma(\frac{1}{2}+s)}$$
$$= \frac{2^{2s}}{\pi^{\frac{1}{2}}}\,\frac{\Gamma(2-2\rho-2s)\,\Gamma(\rho-\frac{1}{2}+2s)}{\Gamma(1+2s)}.$$

Now, by definition,

$${}_0S_1(s)\,\pi^{\frac{1}{2}} = \sum_{t=0}^{\infty} \frac{\Gamma(-s+\frac{1}{2}t)\,\Gamma(1-\rho+\frac{1}{2}t-s)}{\Gamma(t+1)}\,2^t;$$

and therefore by a previous investigation (§ 26)

$${}_0S_1(s) = \pi^{-1}\sin\pi(\rho-\tfrac{1}{2})\,2^{2\rho+4s}\,\Gamma(-2s)\,\Gamma(2-2\rho-2s)\,\Gamma(2s+\rho-\tfrac{1}{2}).$$

Thus the identity (1) is immediately verified.

31. We have now to consider the nature of factorial series of the type $U_r(s)$, and especially the nature of their behaviour when $s = u + \iota v$, and u and $|v|$ either or both tend to positive infinity. With this object we shall first consider more elementary series from whose properties we can deduce inequalities and relations of the type which we desire to obtain.

As a simple example of the factorial series which we proceed to consider, let us take the function $S(s)$, defined when $R(s) > 0$ by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + 1)}$$

where μ is an integer.

By the investigation of the binomial theorem due to Abel, we know that for all values of x and s for which the series is convergent *

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} x^n = \Gamma(-s)(1-x)^s$$

where $(1-x)^s = \exp\{s \log(1-x)\}$ and $|\arg(1-x)| < \pi$. When $x = 1$ the value of the series is zero.

Let, now, w be a special root of the equation $y^\mu = 1$, so that $1, w, w^2, \ldots, w^{\mu-1}$ are all the roots of the equation. Then we have

$$\sum_{r=0}^{\mu-1} (w^n)^r = 0, \quad \text{unless } n = M(\mu)$$

$$= \mu, \quad \text{if} \quad n = M(\mu).$$

Hence
$$\mu \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + 1)} = \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} (w^n)^r.$$

Therefore
$$S(s) = \mu^{-1} \Gamma(-s) \sum_{r=0}^{\mu-1} (1-w^r)^s.$$

This equality represents the function $S(s)$ over the whole of the s-plane.

The term corresponding to $r = 0$ is zero; therefore

$$S(s) = \mu^{-1} \Gamma(-s) \sum_{r=1}^{\mu-1} (1-w^r)^s.$$

In this expression we have to take such a value of $(1-w^r)$ that the modulus of its argument is $< \frac{1}{2}\pi$.

Now, if $w = e^{2\pi\iota/\mu}$, this being the simplest special root of $y^\mu = 1$, we have, as is readily seen from a figure,

$$\arg(1-w^r) = \pi r/\mu - \tfrac{1}{2}\pi, \quad \text{if} \quad 2r < \mu.$$

* A full discussion of the theorem appears in a paper by the author, *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 108–116.

And the maximum value of $|\arg(1-w^r)|$

$$= \tfrac{1}{2}\pi - \pi/\mu$$

if $\mu > 1$.

Hence, if $s = u + \iota v$ *and* v *tends to infinity, the expression*

$$|S(s)|\exp\{(\pi/\mu - \epsilon)|v|\}$$

where $\epsilon > 0$ *behaves like*

$$\Gamma(-s)\, e^{(\frac{1}{2}\pi-\epsilon)|v|} \sum_{r=1}^{\mu-1} (1-w^r)^s e^{-(\frac{1}{2}\pi-\pi/\mu)|v|},$$

and therefore tends exponentially to zero as $|v|$ *tends to infinity for all finite values of* u.

A fortiori it is evident that, except at the poles of $\Gamma(-s)$, $|S(s)|$ will tend exponentially to zero for all values of $|v|$ if u tend to positive infinity.

32. It is evident that the previous discussion is only suggestive of similar results for more complex series, and that for such series other methods of proof must be discovered.

We proceed to consider the behaviour, as $|v|$ *tends to infinity, of the more general function defined when* $R(s+c) > 1$ *by the series*

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)},$$

μ *being an integer* > 1.

As before, this series may be written

$$\frac{1}{\mu} \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+c)} (w^r)^n.$$

Suppose now that the coefficients $e_1, \ldots, e_m$ are so chosen that

$$(1-wx)^{-1} - 1 - wx - \ldots - w^n x^n - e_1 x^{n+1} - \ldots - e_m x^{n+m} = (1-x)^m Q(x)$$

where $Q(x)$ is finite near $x = 1$. Evidently this is always possible.

Consider now the integral

$$\int_0^1 (1-x)^{s+c-1} x^{-s-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - e_1 x^{n+1} - \ldots - e_m x^{n+m} \right\} dx.$$

It is convergent at $x = 0$ if $R(n-s) > -1$ and at $x = 1$ if $R(s+c+m) > 0$, provided $w \neq 1$.

Thus it is convergent at both limits if $n > R(s) > -m - R(c)$.

If $n > R(s) > -R(c)$, the integral is equal to

$$\sum_{t=n+1}^{\infty} w^t \frac{\Gamma(t-s)\Gamma(s+c)}{\Gamma(t+c)} - \sum_{t=1}^{m} e_t \frac{\Gamma(n+t-s)\Gamma(s+c)}{\Gamma(n+t+c)}$$

$$= \Gamma(s+c) \left\{ \sum_{t=}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t - \sum_{t=0}^{n} \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t - \sum_{t=1}^{m} e_t \frac{\Gamma(n+t-s)}{\Gamma(n+t+c)} \right\}.$$

Evidently, if $w \neq 1$, when $\sum_{t=0}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t$ is replaced by the function which it defines when $R(s+c) > 0$, the equality holds for all values of s such that

$$n > R(s) > -m - R(c).$$

Thus, if n and m be sufficiently large, it holds for all values of $R(s)$ which are finite.

The integral may be written

$$\int_0^1 \left(\frac{1-x}{x}\right)^s x^{-1}(1-x)^{c-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - \sum_{t=1}^{m} e_t x^{n+t} \right\} dx.$$

Instead of taking the integral along the real axis from 0 to 1, we may, since the subject of integration is one-valued, take it along any circular arc from 0 to 1 which does not enclose singularities of the subject of integration.* On such arcs $\frac{1-x}{x}$ has a constant argument.

If $w = e^{2r\pi\iota/\mu}$, it is evident that on the arc which passes through $1/w$, which is the sole singularity of the subject of integration other than the terminal points 0 and 1,

$$\arg \frac{1-x}{x} = \arg \frac{1-e^{-2r\pi\iota/\mu}}{e^{-2r\pi\iota/\mu}} = \arg \left\{ 2 \sin \frac{r\pi}{\mu}\, e^{(\frac{1}{2}\pi + r\pi/\mu)\iota} \right\} = \tfrac{1}{2}\pi + \frac{r\pi}{\mu}$$

if $2r < \mu$.

If $2r = \mu$, $\quad \arg\left(\frac{1-x}{x}\right) = \pm \pi.$

When $2\mu > 2r > \mu$, $\quad \arg\left(\frac{1-x}{x}\right) = -\tfrac{1}{2}\pi - \pi\frac{\mu - r}{\mu}.$

We see then that, if $w \neq 1$ and therefore $r \neq 0$, the modulus of the integral

$$\frac{1}{\Gamma(s+c)} \int_0^1 \left(\frac{1-x}{x}\right)^s x^{-1}(1-x)^{c-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - \sum_{t=1}^{m} e_t x^{n+t} \right\} dx,$$

for all finite values of u as $|v|$ tends to infinity, behaves like

$$e^{-(r\pi/\mu - \epsilon)|v|} F(v), \quad \text{if} \quad 2r \leqslant \mu,$$

or like

$$e^{-[(\mu-r)\pi/\mu - \epsilon]|v|} F(v), \quad \text{if} \quad 2\mu > 2r \geqslant \mu,$$

where $F(v)$ tends exponentially to zero as $|v|$ tends to infinity if $\epsilon > 0$.

If $w = 1$, we have the function defined when $R(s+c) > 0$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+c)}$, which, for all finite values of u as $|v|$ tends to infinity, behaves like

$$e^{-(\frac{1}{2}\pi - \epsilon)|v|} F(v).$$

---

* A figure corresponding to this transformation and further details will be found in § 24 of the paper by the author quoted in § 1 (Paper ϵ).

Finally, therefore, the modulus of the function defined when $R(s) > -R(c)$ *by the series* $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)}$, μ *an integer* > 1, *behaves for all finite values of* u *as* $|v|$ *tends to infinity like* $e^{-(\pi/\mu - \epsilon)|v|} F(v)$ *where* $F(v)$ *tends exponentially to zero as* $|v|$ *tends to infinity if* $\epsilon > 0$.

33. *We will further show that, if* $s = R + \iota v$ *where* R *is large and positive,*

$$\left| \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)} \right| < \eta_R \exp\{(-\pi/\mu + \epsilon)|v|\}$$

where $\epsilon > 0$, *and* $\eta_R e^{kR}$, $k > 0$ *but finite, is independent of* v *and can be made as small as we please by taking* R *sufficiently large, if* s *is not in the immediate vicinity of one of the points* n.

We have, if R be large and positive, u be finite, and n be just so large that $n - R > R(s)$,

$$I = \frac{1}{\Gamma(R+s+c)} \int_0^1 \left(\frac{1-x}{x}\right)^s (1-x)^{R+c-1} \frac{w^{n+1} x^{n-R}}{1-wx}\, dx$$

$$= \sum_{t=0}^{\infty} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t - \sum_{t=0}^{n} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t.$$

The integral can be taken along a circular arc as before, and its modulus is

$$< e^{-(\frac{1}{2}\pi + \pi/\mu - \epsilon)|v|} \left| w^{n+1} \right| \mu^R K$$

where μ is the maximum value of $|1-x|$ on the arc and K is independent of v and R.

Also, if R and $R(s+c)$ be positive and greater than unity,

$$|\Gamma(R)|\, |\Gamma(s+c)| < |\Gamma(R+s+c)|.$$

Hence $$|I| < e^{(-\pi/\mu + \epsilon)|v|} \frac{\mu^R}{\Gamma(R)} K.$$

Again, $\left| \sum_{t=0}^{n} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t \right| < \theta R$ times the modulus of the largest term in the series, where θ tends to a finite limit as R tends to infinity. It is therefore less than $\epsilon_R e^{-(\frac{1}{2}\pi - \epsilon)|v|}$ where $\epsilon_R e^{kR}$ tends to zero with $1/R$ and is independent of v, provided $R + s$ is not in the vicinity of one of the zeros of $\Gamma(-R-s)$. Thus when R is large

$$\left| \sum_{t=0}^{\infty} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t \right| < \eta_R' e^{(-\pi/\mu + \epsilon)|v|}$$

where η_R is independent of $|v|$ and $\eta_R' e^{kR}$ can be made as small as we please by taking R sufficiently large.

Hence $$\left| \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - R - s)}{\Gamma(\mu n + c)} \right| < \eta_R e^{(-\pi/\mu+\epsilon)|v|}$$

where η_R is independent of v and $e^{kR}\eta_R$ can be made as small as we please by taking R sufficiently large, provided $R+s$ is not in the vicinity of one of the points n.

34. THEOREM.—*Let $g_\beta(x, 1)$ be the function defined when $|x| < 1$ by the series $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^\beta}$, and $\Phi_m(x)$ the function similarly defined by $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n + c)}{\Gamma(\mu n + \beta + m + c)} x^n$, the functions being made one-valued by cross-cuts along the positive half of the real axis from 1 to $+\infty$. Further, let the coefficients c_m be defined by the expansion*

$$\left\{ \frac{\log(1-y)}{-y} \right\}^{\beta-1} (1-y)^{\mu-c} = \sum_{m=0}^{\infty} c_m (-y)^m,$$

which is valid when y is sufficiently small. Then the coefficients a can be so chosen that the function

$$F(x) = \mu^{-\beta} g_\beta(x^\mu, 1) - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)}{\Gamma(\beta)} \Phi_m(x^\mu) - \sum_{n=0}^{L+N-1} a_n x^n$$

behaves at $x = 0$ like $x^N P(x)$ where $P(x)$ is finite when $x = 0$, and at $x = 1$ like $(1-x)^L Q(1-x)$ where $Q(1-x)$ is finite when $x = 1$, L, M, and N being finite positive integers, however large, such that $R(\beta) + M > L$.

I have previously shewn that* $g_\beta(x, 1)$ admits, near $x = 1$, the convergent expansion

$$\Gamma(1-\beta)(-\log x)^{\beta-1} x^{-1} + \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta, 1),$$

the latter series being convergent near $x = 1$, and $(-\log x)^{\beta-1}$ being made one-valued by a cross-cut from 1 to $+\infty$.

Also it can be shown that the function defined when $|x| < 1$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(\rho+n)} x^n$ admits, near $x = 1$, the convergent expansion

$$\Gamma(1+a-\rho)(1-x)^{\rho-a-1} x^{1-\rho} - \frac{\Gamma(1+a-\rho)}{\Gamma(\rho-1)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(2+a-\rho+n)} (1-x)^n,$$

wherein there is a similar cross-cut, so that $|\arg(1-x)| < \pi$.

---

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 291.

Now

$$\Phi_m(x^\mu) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n+c)}{\Gamma(\mu n+\beta+m+c)} x^{n\mu} = \frac{1}{\mu} \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+c)}{\Gamma(n+\beta+m+c)} (w^r x)^n$$

(where w is a special root of $y^\mu - 1 = 0$)

$$= \frac{1}{\mu} \sum_{r=0}^{\mu-1} \Big\{ \Gamma(1-\beta-m)(1-w^r x)^{\beta+m-1} (w^r x)^{1-\beta-m-c} - \frac{\Gamma(1-\beta-m)}{\Gamma(\beta+m+c-1)} \sum_{n=0}^{\infty} \frac{\Gamma(c+n)}{\Gamma(2-\beta-m+n)} (1-w^r x)^n \Big\},$$

each expansion being valid near the corresponding singularity. Therefore, since near $x = 1$ all the functions in the summation $\sum_r$ except that corresponding to $r = 0$ are analytic, we have, near $x = 1$,

$$\Phi_m(x^\mu) = \frac{1}{\mu} \Gamma(1-\beta-m)(1-x)^{\beta+m-1} x^{1-\beta-m-c} + \sum_{n=0}^{\infty} d_n (1-x)^n,$$

the latter series being convergent near $x = 1$.

Hence the function $F(x)$ behaves in the vicinity of its singularity $x = 1$ like

$$\mu^{-\beta} \Gamma(1-\beta)(-\mu \log x)^{\beta-1} x^{-\mu} - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(1-\beta-m)}{\mu \Gamma(\beta)}$$

$$\times (1-x)^{\beta+m-1} x^{1-\beta-m-c} + \sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n$$

$$= \frac{\Gamma(1-\beta) x^{-c}}{\mu} \left(\frac{1-x}{x}\right)^{\beta-1} \left[\left(-\frac{x \log x}{1-x}\right)^{\beta-1} x^{c-\mu} - \sum_{m=0}^{M} c_m \left(\frac{1-x}{x}\right)^m\right]$$

$$+ \sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n.$$

Now near $x = 1$ we may put $x = 1/(1-y)$, so that $(1-x)/x = -y$; then the expression inside the square brackets becomes

$$\left\{ \frac{-\log(1-y)}{y} \right\}^{\beta-1} (1-y)^{\mu-c} - \sum_{m=0}^{M} c_m (-y)^m$$

$$= (-y)^{M+1} \left\{ \sum_{m=0}^{\infty} c_{M+m+1} (-y)^m \right\}$$

if $| y |$ be small. Thus $F(x)$ near the singularity $x = 1$ behaves like

$$\sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n + (1-x)^{\beta+M} Q'(x),$$

where $Q'(x)$ is finite near $x = 1$.

Suppose now, as is evidently possible, that we choose the first N coefficients a so that

$$F(x) = x^N P(x),$$

where $P(x)$ is finite at $x=0$. Then we may further choose the remaining L coefficients a so that

$$\sum_{n=0}^{\infty} e_n(1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n = (1-x)^L Q(1-x)$$

where $Q(1-x)$ is finite near $x=1$.

If, then, $R(\beta)+M > L$, we have the proposition stated.

35. *We will next shew that, if N be a multiple of μ and if σ be such that*

$$N > R(\sigma) > -L-R(c)+1,$$

$$\frac{1}{\Gamma(c+\sigma)}\int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x)\,dx$$

$$= -\sum_{n=N}^{L+N-1} a_n \frac{\Gamma(n-\sigma)}{\Gamma(n+c)}$$

$$+\sum_{r=N/\mu}^{\infty} \frac{\Gamma(\mu r-\sigma)}{\Gamma(\mu r+c)}\left\{\frac{1}{(\mu r+\mu)^\beta} - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\mu r+\beta+m+c)}\right\}.$$

We assume that all the numbers involved are defined as in the previous paragraph.

If N be a multiple of μ, and R be an integer $> N/\mu$, we have

$$\int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x)\,dx$$

$$= \int_{1-\epsilon}^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x)\,dx - \sum_{n=N}^{L+N-1}\int_0^{1-\epsilon} a_n x^n (1-x)^{c+\sigma-1} x^{-\sigma-1}\,dx$$

$$+\sum_{r=N/\mu}^{R-1}\int_0^{1-\epsilon} (1-x)^{c+\sigma-1} x^{-\sigma-1+\mu r}$$

$$\times\left\{\frac{1}{(\mu r+\mu)^\beta} - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\mu r+\beta+m+c)}\right\}dx$$

$$+\int_0^{1-\epsilon} (1-x)^{c+\sigma-1} x^{-\sigma-1} \sum_{r=R}^{\infty} \frac{x^{\mu r} J_M(r)}{r^{\beta+M}}\,dx, \tag{1}$$

where $J_M(r)$ is defined by the equality

$$\frac{1}{(\mu r+\mu)^\beta} = \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\beta+m+\mu r+c)} + \frac{J_M(r)}{r^{\beta+M}}. \tag{2}$$

Now I have shown* that, if $|\arg\phi| < \pi$,

$$\frac{\Gamma(\beta)}{(\theta+\phi)^\beta} = \sum_{m=0}^{M} \frac{(-)^m c'_m \Gamma(\beta+m)\Gamma(1+\phi)}{\Gamma(\beta+m+\phi+1)} + \frac{J_M}{\phi^{\beta+M}}$$

---

* *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 116–140, § 9.

where $|J_M|$ tends to zero as $|\phi|$ tends to infinity, and the coefficients c'_m are given by the expansion

$$\left\{\frac{-\log(1-y)}{y}\right\}^{\beta-1}(1-y)^{\theta-1}=\sum_{m=0}^{\infty}c'_m(-y)^m.$$

Putting $\qquad \phi=\mu r+c-1, \qquad \theta=\mu-c+1,$

we see that in the formula (2) $|J_M(r)| <$ an arbitrarily assigned small quantity η, if R be a sufficiently large positive quantity and $r \geqslant R$, the quantities c_m being defined as in the previous paragraph.

Now, in the formula (1) the modulus of the first integral tends to zero as ϵ tends to zero if $R(c+\sigma+L)>0$.

The modulus of any one of the second system of integrals tends to a finite limit as ϵ tends to zero if $N > R(\sigma) > -R(c)$. Each of the third set of integrals has a modulus which tends to a finite limit if the same conditions hold.

The modulus of the final integral is

$$<\eta\int_0^{1-\epsilon}(1-x)^{R(c+\sigma)-1}\sum_{r=R}^{\infty}\frac{x^{-R(\sigma)-1+\mu r}}{r^{\beta+M}}\,dx.$$

Hence, if $N > R(\sigma) > -R(c)$ and $R(\beta+M)>1$,

$$\int_0^1(1-x)^{c+\sigma-1}x^{-\sigma-1}F(x)\,dx$$
$$=-\sum_{n=N}^{L+N-1}a_n\frac{\Gamma(c+\sigma)\Gamma(n-\sigma)}{\Gamma(c+n)}+\sum_{r=N/\mu}^{R-1}\frac{\Gamma(c+\sigma)}{\Gamma(\mu r+c)}\Gamma(\mu r-\sigma)$$
$$\times\left\{\frac{1}{(\mu r+\mu)^{\beta}}-\sum_{m=0}^{M}\frac{(-)^m c_m\Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\mu r+\beta+m+c)}\right\}+I_R,$$

where $|I_R|$ tends to zero as R tends to infinity.

Thus under the same conditions

$$\frac{1}{\Gamma(c+\sigma)}\int_0^1(1-x)^{c+\sigma-1}x^{-\sigma-1}F(x)\,dx$$
$$=-\sum_{n=N}^{L+N-1}a_n\frac{\Gamma(n-\sigma)}{\Gamma(n+c)}+\sum_{r=N/\mu}^{\infty}\frac{\Gamma(\mu r-\sigma)}{\Gamma(\mu r+c)}$$
$$\times\left\{\frac{1}{(\mu r+\mu)^{\beta}}-\sum_{m=0}^{M}\frac{(-)^m c_m\Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\mu r+\beta+m+c)}\right\}.$$

But, if $R(\beta)+M > L$, the integral on the left-hand side of this equality is an analytic function with no poles if

$$N > R(\sigma) > -L-R(c).$$

Also the series on the right-hand side is absolutely convergent if

$$R(\beta+M+\sigma+c) > 1,$$

and, *a fortiori*, if $$R(\sigma+L+c) > 1.$$

The equality is therefore true under the wider conditions of the enunciation.

36. *Let* $S(\sigma)$ *be the function defined by the series* $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma)}{\Gamma(\mu n+c)(n+1)^{\beta}}$ *when* $R(c+\sigma+\beta) > 1$. *Then, if* $\sigma = u+\iota v$, *and* u *be any real quantity of finite modulus,* $|S(s)| \exp\{(\pi/\mu-\epsilon)|v|\}$ *tends to zero as* $|v|$ *tends to infinity if* $\epsilon > 0$ *and* μ *is an integer* > 1.

Let $\Sigma_m(\sigma)$ be the function defined by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma)}{\Gamma(\mu n+\beta+m+c)}$, if $R(\beta+m+c+\sigma) > 1$.

Then, by the result of the previous paragraph, we have, if

$$N > u > -L-R(c)+1,$$

$$\frac{1}{\Gamma(c+\sigma)}\int_0^1 \left(\frac{1-x}{x}\right)^{c+\sigma-1} x^{c-2} F(x)\,dx$$

$$= \mu^{-\beta} S(\sigma) - \sum_{m=0}^{M} (-)^m c_m \frac{\Gamma(\beta+m)}{\Gamma(\beta)} \Sigma_m(\sigma) - \sum_{n=N}^{L+N-1} a_n \frac{\Gamma(n-\sigma)}{\Gamma(n+c)}$$

$$- \sum_{r=0}^{N/\mu-1} \frac{\Gamma(\mu r-\sigma)}{\Gamma(\mu r+c)} \left\{ \frac{1}{(\mu r+\mu)^{\beta}} - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\mu r+\beta+m+c)} \right\}.$$

Now the modulus of the first integral tends uniformly to zero, as $|v|$ tends to infinity, when multiplied by $\exp\{(\pi/\mu-\epsilon)|v|\}$, as we see by the method of the theorem of § 32. By the properties of the gamma function the same is also true of every term in the two final summations of the previous equality.

Also, if μ be an integer > 1, $|\Sigma_m(\sigma)|$ tends to zero as $|v|$ tends to infinity when multiplied by $\exp\{(\pi/\mu-\epsilon)|v|\}$, as has been proved previously (§ 32).

Hence we have the given theorem.

37. *If* $S(\sigma)$ *be the function defined when* $R(c+\sigma+\beta) > 1$ *by the series* $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma)}{\Gamma(\mu n+c)(n+1)^{\beta}}$, *the function* $\frac{S(\sigma)}{\Gamma(-\sigma)}$ *has for its sole finite*

singularities poles at the points

$$\sigma = 1-\beta-c-n \quad (n = 0, 1, 2, \ldots, \infty).$$

When n is large we know that† $\frac{\Gamma(\mu n-\sigma)}{\Gamma(\mu n+c)(\mu n+\mu)^{\beta}}$ admits the asymptotic expansion

$$\sum_{r=0}^{R} \frac{c_r(\sigma)}{(\mu n)^{\sigma+\beta+c+r}} + \frac{J_R(n)}{(\mu n)^{\sigma+\beta+c+R}},$$

where the coefficients $c_r(\sigma)$ are polynomials in σ and $|J_R(n)|$ can for any finite value of R be made as small as we please by taking n sufficiently large.

Hence the function

$$\frac{S(\sigma)}{\mu^{\beta}} - \sum_{r=0}^{R} \frac{c_r(\sigma)}{\mu^{\sigma+\beta+c+r}} \zeta(\sigma+\beta+c+r)$$

admits, when $R(c+\sigma+\beta) > 1$, the expansion

$$\sum_{n=0}^{\infty}{}'' \frac{J_R(n)}{(\mu n)^{\sigma+\beta+c+R}},$$

the double accent denoting that, when $n = 0$, the term is $\frac{\Gamma(-\sigma)}{\Gamma(c)\,\mu^{\beta}}$. The series is convergent when $R(\sigma+\beta+c) > 1-R$. And hence for this wider range the function

$$\frac{S(\sigma)}{\mu^{\beta}} - \sum_{r=0}^{R} \frac{c_r(\sigma)}{\mu^{\sigma+\beta+r+c}} \zeta(\sigma+\beta+c+r)$$

has no singularities except poles at the points

$$\sigma = n \quad (n = 0, 1, 2, \ldots, \infty).$$

Now $\zeta(s)$ has for its sole finite singularity a pole at the point $s = 1$, at which the residue is unity.

Hence the sole finite singularities of $S(\sigma)/\Gamma(-\sigma)$ are poles at the points

$$\sigma = 1-\beta-c-n \quad (n = 0, 1, 2, \ldots, \infty);$$

and at

$$\sigma = 1-\beta-c-n$$

the residue is $\mu^{-1}c_n(1-\beta-c-n)$.

---

† With this theorem the reader may compare the similar proposition established in § 5 of the author's paper "The Use of Factorial Series in an Asymptotic Expansion," *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 116–140.

38. *We proceed now to consider the behaviour of the function* $S(\sigma)$, *defined when* $R(c+\sigma+\beta) > 1$ *by the series*

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma)}{\Gamma(\mu n+c)(n+1)^{\beta}},$$

for values of $\sigma = u+\iota v$ *for which u is large, real, and positive when* $|v|$ *does or does not tend to infinity.*

We will shew that

$$S(\sigma) < \eta_u \exp\{(-\pi/\mu+\epsilon)\,|v|\},$$

where η_u *is independent of* v, *and* $e^{ku}\eta_u$, $k > 0$ *and finite, can be made as small as we please by taking* u *sufficiently large and positive, unless* σ *be in the immediate vicinity of one of the points* n.

Suppose that $\sigma = u+\iota v$, where u is finite. Choose M such that $R(\beta+M) > 0$. Let R be very large, real, and positive, so that

$$R(c+\sigma+R) > 0, \qquad R(c+\sigma+R+\beta) > 1,$$

and

$$R(\beta+m+c+\sigma+R) > 1 \quad (0 \leqslant m \leqslant M).$$

Choose N a multiple of μ so large that $N > u+R-1$. Then, with the previous notation,

$$S(\sigma+R) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma-R)}{\Gamma(\mu n+c)(n+1)^{\beta}}$$

and

$$\Sigma_m(\sigma+R) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\sigma-R)}{\Gamma(\mu n+\beta+m+c)}.$$

Let the a's be so chosen that, near $x = 0$,

$$F(x) \equiv \mu^{-\beta} g_{\beta}(x^{\mu}, 1) - \sum_{m=0}^{M} (-)^m c_m \frac{\Gamma(\beta+m)}{\Gamma(\beta)} \Phi_m(x^{\mu}) - \sum_{n=0}^{N-1} a_n x^n$$

is of the form $x^N P(x)$ where $P(x)$ is finite when $x = 0$. Then, by the result of § 35,

$$\frac{1}{\Gamma(c+\sigma+R)} \int_0^1 \left(\frac{1-x}{x}\right)^{\sigma} (1-x)^{c+R-1} x^{-R-1} F(x)\,dx$$

$$= \mu^{-\beta} S(\sigma+R) - \sum_{m=0}^{M} (-)^m c_m \frac{\Gamma(\beta+m)}{\Gamma(\beta)} \Sigma_m(\sigma+R)$$

$$- \sum_{r=0}^{N/\mu-1} \frac{\Gamma(\mu r-\sigma-R)}{\Gamma(\mu r+c)} \left\{ \frac{1}{(\mu r+\mu)^{\beta}} - \sum_{m=0}^{M} \frac{(-)^m c_m \Gamma(\beta+m)\Gamma(\mu r+c)}{\Gamma(\beta)\Gamma(\mu r+\beta+m+c)} \right\}. \tag{1}$$

Now we have seen in § 33 that, when $|v|$ is large,

$$\Sigma_m(\sigma+R) < {}_m\eta_R \exp\{(-\pi/\mu+\epsilon)\,|v|\},$$

where ${}_m\eta_R$ is independent of v, and $e^{kR}\,{}_m\eta_R$ can be made as small as we please by taking R sufficiently large.

Also, if we employ the transformation of § 32, and take the integral along a suitable circular arc, its modulus is less than

$$\frac{1}{|\Gamma(c+\sigma+R)|}\exp\{-(\tfrac{1}{2}\pi+\pi/\mu-\epsilon)|v|\}\,\mu_1^R K,$$

where μ_1 is the maximum value of $|(1-x)|$ on the arc, and K is finite and independent of v and R. The modulus of the integral is thus less than

$$\frac{\mu_1^R}{\Gamma(R)}K\exp\{(-\pi/\mu+\epsilon)\,|v|\} < \eta_R'\exp\{-\pi/\mu+\epsilon)|v|\},$$

where η_R' is independent of v, and $e^{kR}\eta_R'$ can be made as small as we please by taking R sufficiently large.

The final N/μ terms in the expression (1) have a sum whose modulus is less than N/μ times the modulus of the largest term, *i.e.*,

$$< \eta_R''\exp\{(-\tfrac{1}{2}\pi+\epsilon)\,|v|\}$$

where η_R'' is independent of v, and $e^{kR}\eta_R''$ can be made as small as we please by taking R sufficiently large.

Hence, finally, when $|v|$ is large,

$$|S(\sigma+R)| < \eta_R\exp\{(-\pi/\mu+\epsilon)\,|v|\}$$

where η_R is independent of v, and $e^{kR}\eta^R$ can be made as small as we please by taking R sufficiently large.

This is equivalent to the given theorem.

39. *Let*

$${}_qF_{q-1}\{a_1, \ldots, a_q;\ b_1, \ldots, b_q;\ x\} = \sum_{n=0}^{\infty}\prod_{t=1}^{q}\frac{\Gamma(a_t+n)}{\Gamma(b_t+n)}x^n,$$

when $|x|<1$, *where one of the quantities* $b_1, \ldots, b_q$ *is unity. Further, let the coefficients* $f_r(b, a)$ *be defined by the asymptotic expansion*

$$\prod_{t=1}^{q}\frac{\Gamma(a_t+n)}{\Gamma(b_t+n)} = \sum_{r=0}^{R}\frac{f_r(b, a)}{n^{\Sigma b-\Sigma a+r}}+\frac{J_R(n, b, a)}{n^{\Sigma b-\Sigma a+R}},$$

where, for all finite values of R, $|J_R(n, b, a)|$ *can be made as small as we please by taking* n *sufficiently large. Then, however large the finite* *quantity M may be, if we take* $R(\Sigma b - \Sigma a + R - 1) > M$, *it is always possible to find coefficients d such that, near* $x = 1$,

$$ {}_qF_{q-1}(x) - \sum_{r=0}^{R} f_r(b, a)\,\Gamma(1-\theta-r)(-\log x)^{\theta+r-1} - \sum_{m=0}^{M} d_m (1-x)^m = (1-x)^M Q(x), $$

where $Q(x)$ *vanishes at* $x = 1$, *and* $\theta = \Sigma b - \Sigma a$.

If the double accent denote that the terms $\sum_{r=0}^{R}$ do not exist when $n = 0$, we have

$$ \sum_{n=0}^{\infty}{}'' \left\{ \prod_{t=1}^{q} \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)} - \sum_{r=0}^{R} \frac{f^r(b, a)}{n^{\Sigma b - \Sigma a + r}} \right\} x^n = \sum_{n=0}^{\infty} \frac{J_R(n, b, a)}{n^{\Sigma b - \Sigma a + R}}\, x^n. $$

By the properties of $J_R(n, b, a)$ this series and its k-th derivate with regard to x, where $R(\Sigma b - \Sigma a) + R - k > 1$, is convergent when $x = 1$. But, when $|x| < 1$, the series is equal to

$$ {}_qF_{q-1}(x) - \sum_{r=0}^{R} x f_r(b, a)\, g_{\Sigma b - \Sigma a + r}(x, 1), $$

where, when $|x| < 1$, $\quad g_\beta(x, \theta) = \sum_{n=0}^{\infty} \dfrac{x^n}{(n+\theta)^\beta}$.

But the author has shown that, near $x = 1$,

$$ g_\beta(x, \theta) - \Gamma(1-\beta)(-\log x)^{\beta-1} x^{-\theta} $$

is one-valued and equal to

$$ \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}}\, \bar{\zeta}_{n+1}(\beta, \theta). $$

Therefore, near $x = 1$, the series is equal to

$$ {}_qF_{q-1}(x) - \sum_{r=0}^{R} f_r(b, a) \left\{ \Gamma(1-\Sigma b + \Sigma a - r)(-\log x)^{\Sigma b - \Sigma a + r - 1} + \sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n \bar{\zeta}_{n+1}(\Sigma b - \Sigma a + r, 1) \right\}. $$

This expression and all derivates up to the k-th converge to a definite value as x tends to unity, the various terms being made one-valued by a cross-cut from 1 to ∞ along the positive half of the real axis.

Hence we can always find coefficients d so as to satisfy the proposition.

40. A suggestive deduction from the previous theorem may be noticed.

Since one of the constants b is unity, the function ${}_qF_{q-1}(x)$ is a hypergeometric series in x of order q, and therefore by the researches of Thomae and Pochhammer* we know that, near $x = 1$,

$$ {}_qF_{q-1}(x) = \sum_{m=1}^{q-1} A_m P_m(1-x) + A_q(1-x)^{\theta-1} P_q(1-x) $$

where the coefficients of the constants A are solutions of the differential equation for ${}_qF_{q-1}(x)$ near $x = 1$. Moreover the functions $P_m(1-x)$ $(m = 1, 2, \ldots, q)$ are one-valued power series of $(1-x)$ convergent near $x = 1$.

We must therefore have, near $x = 1$, if $R(\theta+R-1) > M$,

$$ \sum_{r=0}^{R} f_r(b, a)\, \Gamma(1-\theta-r)(-\log x)^{\theta+r-1} - A_q(1-x)^{\theta-1} P_q(1-x) = (1-x)^M Q(x), $$

where $Q(x)$ vanishes at $x = 1$.

We may therefore anticipate that

$$ \sum_{r=0}^{\infty} f_r(b, a)\, \Gamma(1-\theta-r)(-\log x)^{\theta+r-1} $$

is either an asymptotic or a convergent expansion near $x = 1$ of that solution of the differential equation for ${}_qF_{q-1}(x)$ which is multiform near $x = 1$.

41. *Let* ${}_qF_{q-1}\{x\}$ *denote the function represented by the series*

$$ \sum_{n=0}^{\infty} \prod_{t=1}^{q} \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)} x^n $$

when $|x| < 1$, *with a cross-cut from* 1 *to* ∞ *along the positive half of the real axis. As in* § 29, *let* $U_r(s)$ *be the function defined when*

$$ R(\mu s+\mu+\Sigma b-\Sigma a) > 1 $$

by the series

$$ \sum_{n=0}^{\infty} \frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)} \prod_{t=1}^{q} \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)}. $$

Let R, M, *and* N *be integers such that, if* $\theta = \Sigma b - \Sigma a$,

$$ R(\theta)+R-1 > M, \qquad R(\mu-\mu\rho_r+\mu N) > R(\mu s) > -M-\mu. \qquad (1) $$

---

* Thomae, *Mathematische Annalen*, Bd. II., p. 433; Pochhammer, *Crelle*, T. CII., pp. 97, &c.

Then, if the c's be suitably chosen, we shall have

$$\frac{1}{\Gamma(\mu s+\mu)}\int_0^1\left(\frac{1-x}{x}\right)^{\mu s+\mu-1}x^{2\mu-2-\mu\rho_r}$$
$$\times\left\{{}_qF_{q-1}\{x^\mu\}-\sum_{\tau=0}^{R}x^\mu f_\tau(b,a)\,g_{\theta+\tau}(x^\mu,1)-\sum_{m=0}^{M+N}c_m x^{\mu m}\right\}dx$$
$$=U_r(s)-\sum_{\tau=0}^{R}f_\tau(b,a)\sum_{n=1}^{\infty}\frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)\,n^{\theta+\tau}}$$
$$-\sum_{m=0}^{M+N}c_m\frac{\Gamma(\mu m+\mu-\mu s-\mu\rho_r)}{\Gamma(\mu m+2\mu-\mu\rho_r)}. \qquad (2)$$

By § 39, if the quantities $d_0, d_1, \ldots, d_M$ be suitably chosen,

$${}_qF_{q-1}\{x^\mu\}-\sum_{\tau=0}^{R}x^\mu f_\tau(b,a)\,g_{\theta+\tau}(x^\mu,1)-\sum_{m=0}^{M}d_m(1-x^\mu)^m=(1-x^\mu)^M Q(x^\mu)$$

where $Q(x^\mu)$ vanishes where $x=1$, if $R(\theta)+R-1>M$.

We may now choose $d_{M+1}, d_{M+2}, \ldots, d_{M+N}$ so that the previous expression *minus* $\sum_{m=M+1}^{M+N} d_m(1-x^\mu)^m$ is near $x=0$ equal to $x^{\mu N}P(x^\mu)$ where $P(x^\mu)$ is finite at $x=0$.

If, now, the coefficients c are given in terms of the quantities d thus chosen by the relation

$$\sum_{m=0}^{M+N}d_m(1-x)^m=\sum_{m=0}^{M+N}c_m x^m,$$

we see that the integral in the equality (2) is convergent at $x=1$ if

$$R(\mu s+\mu+M)>0,$$

and is convergent at $x=0$ if

$$R(\mu-\mu\rho_r+\mu N-\mu s)>0.$$

It is therefore convergent at both limits, provided the conditions (1) hold good.

Let I denote the integral in the equality (2) and suppose that L is a large positive integer $>M+N$, and let $c_n=0$ when $n>M+N$. Then

$$I=\sum_{n=0}^{L}{}''\left\{\prod_{t=1}^{q}\frac{\Gamma(a_t+n)}{\Gamma(b_t+n)}-\sum_{\tau=0}^{R}\frac{f_\tau(b,a)}{n^{\theta+\tau}}-c_n\right\}\frac{\Gamma(\mu n+\mu-\mu\rho_r-\mu s)}{\Gamma(\mu n+2\mu-\mu\rho_r)}+J_L$$

where $$J_L=\frac{1}{\Gamma(\mu s+\mu)}\int_0^1\left(\frac{1-x}{x}\right)^{\mu s+\mu-1}x^{2\mu-2-\mu\rho_r}\left\{\sum_{n=L}^{\infty}\frac{J_R(b,a,n)}{n^{\theta+R}}x^{\mu n}\right\}dx,$$

and the double accent denotes that the summation $\sum_{r=0}^{R}$ does not exist when $x = 0$.

Now we may take L so large that $|J_R(b, a, n)| < \epsilon$ when $n \geqslant L$. The final integral J_L is then such that

$$|J_L| < \frac{\epsilon}{|\Gamma(\mu s+\mu)|} \int_0^1 (1-x)^{R(\mu s)+\mu-1} x^{R(\mu-1-\mu\rho_r-\mu s+\mu L)} \sum_{n=0}^{\infty} \frac{x^{\mu n}}{(L+n)^{M+1}}\, dx.$$

Thus $|J_L|$ can be made as small as we please by taking L sufficiently large, provided

$$R(\mu-\mu\rho_r+\mu L) > R(\mu s) > -\mu.$$

If, then, $R(\mu-\mu\rho_r+\mu N) > R(\mu s) > -\mu$, we have the equality

$$I = \sum_{n=0}^{\infty}{}'' \left\{ \prod_{t=1}^{q} \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)} - \sum_{r=0}^{R} \frac{f_r(b, a)}{n^{\theta+r}} - c_n \right\} \frac{\Gamma(\mu n+\mu-\mu\rho_r-\mu s)}{\Gamma(\mu n+2\mu-\mu\rho_r)}.$$

But under the wider conditions (1) of the enunciation I is an analytic function, and so also is the series just written.

If, then, $\sum_{n=1}^{\infty} \frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)\, n^{\theta+r}}$ denotes the function $T_r(\mu s)$, which is expressible in the form of the series when $R(\theta+r+\mu s+\mu) > 0$, the equality (2) will hold within the assigned range (1).

We thus have the theorem.

42. *We can now shew that, if $R(s)$ be finite and if $s = u+\iota v$,*

$$|U_r(s)| \exp\{(\pi-\epsilon)\,|v|\}$$

tends to zero as $|v|$ tends to infinity if $\epsilon > 0$ and μ is an integer > 1.

Take the equality (2) established in the previous paragraph. If we employ the transformation of § 32 and take the integral along a suitable circular arc, we can shew that its modulus is less than

$$\frac{1}{|\Gamma(\mu s+\mu)|} \exp\{-(\tfrac{1}{2}\pi+\pi/\mu-\epsilon)\,|\mu v|\}\, K$$

when $|v|$ is large, where K is finite and independent of v, and

$$R(\mu-\mu\rho_r+\mu N) > \mu u > -M-\mu.$$

The modulus of the integral is therefore less than

$$\exp\{(-\pi+\epsilon)\,|v|\}$$

if $\epsilon > 0$ and $|v|$ is sufficiently large.

Again, by the theorem already proved in § 36, when u is finite

$$|T_r(\mu s)| \exp\{(\pi-\epsilon)|v|\},$$

where $\epsilon > 0$, can be made as small as we please by taking $|v|$ sufficiently large.

Again, each term of the final series in the equality (2) of the previous paragraph has a modulus less than

$$\exp\{(-\mu\pi+\epsilon)|v|\}$$

where $|v|$ is sufficiently large.

Hence, since μ is an integer > 1, $|U_r(s)| \exp\{(\pi-\epsilon)|v|\}$ can be made as small as we please by taking $|v|$ sufficiently large, if $\epsilon > 0$.

We thus have the given theorem, since M and N may have any finite integral values as large as we please.

43. *We can now shew that, if ${}_pS_q(s)$ be the function defined when*

$$R(s) > R\{\tfrac{1}{2}(\mu-1)+\Sigma a-\Sigma\rho\}/\mu$$

by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^{q} \Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^{\mu} \Gamma\left(\frac{t+r}{\mu}\right) \prod_{r=1}^{p} \Gamma(1-a_r+t/\mu-s)},$$

then, for all finite values of u, $|{}_pS_q(s)| \exp[\{(\mu+1)\pi-\epsilon\}|v|]$ can be made as small as we please by taking $|v|$ sufficiently large.

For we have seen in § 29 that $\sin \pi\mu s\, {}_pS_q(s)$ may be written

$$\sum_{r=1}^{q} A_r \mu^{\mu s} U_r(s)$$

where A_r is a numerical coefficient independent of s. Also, by the previous paragraph, $|U_r(s)| \exp\{(\pi-\epsilon)|v|\}$ can be made as small as we please, if u be finite, by taking $|v|$ sufficiently large. The same is therefore true of

$$|\sin \pi\mu s\, {}_pS_q(s)| \exp\{(\pi-\epsilon)|v|\}.$$

We thus have the given theorem.

44. It is now evident from the previous investigations that ${}_pS_q(s)$ *is a one-valued function which can be continued over the whole of the finite portion of the s-plane.*

We will now prove that it has simple poles at the points

$$s = [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu - 1)]/\mu - n/\mu \quad (n = 0, 1, 2, \ldots, \infty)$$

and no other finite singularities except those of $\Gamma(-\mu s)$ *and*

$$\Gamma(\mu - \mu\rho_r - \mu s) \quad (r = 1, 2, \ldots, q).$$

We have seen that, if $R(\mu s + \mu + \Sigma b - \Sigma a) > 1$,

$$U_r(s) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu\rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu\rho_r + 2\mu)} \prod_{t=1}^{q} \frac{\Gamma({}_r a_t + n)}{\Gamma({}_r b_t + n)}$$

$$= \sum_{n=0}^{\infty} {}_r V_n(s) \quad \text{(say)}.$$

Also (§ 29) the constants ${}_r a_t$ and ${}_r b_t$ are such that

$${}_r a_t = 2 + t/\mu - \rho_r \quad (t = 1, 2, \mu - 1),$$

$$= 1 + a_t - \rho_r \quad (t = \mu, \mu + 1, \ldots, q),$$

$${}_r b_t = 1 + \rho_t - \rho_r \quad (t = 1, 2, \ldots, q).$$

And therefore the condition under which $U_r(s)$ can be represented by the series is equivalent to

$$R[\mu s + \Sigma\rho - \Sigma a - \tfrac{1}{2}(\mu - 3)] > 1.$$

Now, when n is large, we know that we have an asymptotic expansion of the type

$${}_r V_n(s) = \sum_{t=0}^{T} \frac{c_t(s)}{n^{\mu s + \mu + \Sigma b - \Sigma a + t}} + \frac{J_T(n)}{n^{\mu s + \mu + \Sigma b - \Sigma a + T}},$$

where, for all finite values of T, $|J_T(n)|$ can be made as small as we please, provided n is greater than an assignable number N. The coefficients $c_t(s)$ are polynomials in s, and can be calculated from the known asymptotic expansion for $\dfrac{\Gamma(a+n)}{\Gamma(b+n)}$ by sufficient labour.

We see therefore, as in § 37, that the function

$$U_r(s) - \sum_{t=0}^{T} c_t(s)\, \zeta(\mu s + \mu + \Sigma b - \Sigma a + t)$$

has no finite singularities except those of $\Gamma(\mu n + \mu - \mu\rho_r - \mu s)$ when

$$R(\mu s + \mu + \Sigma b - \Sigma a) > 1 - T.$$

Hence the sole finite singularities of $U_r(s)/\Gamma(\mu-\mu\rho_r-\mu s)$ are simple poles at the points

$$s = (1-\mu+\Sigma a-\Sigma b)/\mu - n/\mu \quad (n = 0, 1, 2, \ldots, \infty),$$

i.e.,
$$s = [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu-1)]/\mu - n/\mu.$$

Now we have seen that

$$_pS_q(s) \sin \pi\mu s = \sum_{r=1}^{q} A_r \mu^{\mu s} U_r(s),$$

where the quantities A are independent of s.

Hence all possible finite singularities of $_pS_q(s)$ are included in

(1) simple poles at $s = n/\mu \qquad (n = 0, 1, 2, \ldots, \infty)$,

(2) ,, $s = n/\mu - \rho_r + 1 \quad (n = 0, 1, 2, \ldots, \infty;\ r = 1, 2, \ldots, q)$,

(3) ,, $s = -n/\mu + [\Sigma a - \Sigma\rho + \frac{1}{2}(\mu - 1)]/\mu$

$(n = 0, 1, 2, \ldots, \infty)$.

We thus have the given theorem, and we see that the residues at the last system of poles can be calculated with sufficient labour by use of the asymptotic expansion which gives $\dfrac{\Gamma(a+n)}{\Gamma(b+n)}$, when n is large.

45. *We proceed now to show that, if* $s = u+\iota v$, *and* $|v|$ *be finite or infinite,*

$$|{}_pS_q(s)| < \eta_u \exp\{[-(\mu+1)\pi+\epsilon]\,|v|\},$$

if $\epsilon > 0$, *where* $\eta_u e^{ku}$ ($k > 0$ *but finite*), *can be made as small as we please by taking* u *sufficiently large and positive if* s *be not in the immediate vicinity of one of the poles of* $_pS_q(s)$.

We can always choose the coefficients c so that the function

$$\Psi(x) \equiv {}_qF_{q-1}\{x^\mu\} - \sum_{\tau=0}^{K} x^\mu f_\tau(b, a)\, g_{\theta+\tau}(x^\mu, 1) - \sum_{m=0}^{N-1} c_m x^{\mu m} = x^{\mu N} P(x^\mu),$$

where $P(x^\mu)$ is finite at $x = 0$. Choose N so large that

$$R(\mu-\mu\rho_r-\mu s-\mu R+\mu N) > 0$$

and let $T_\tau(\mu s)$ be the function defined when

$$R(\theta+\tau+\mu s+\mu) > 0,$$

by the series $$\sum_{n=1}^{\infty} \frac{\Gamma(\mu n-\mu\rho_r+\mu-\mu s)}{\Gamma(\mu n-\mu\rho_r+2\mu)\, n^{\theta+\tau}}.$$

Then, as in § 41, we have, if R be very large and positive,

$$\frac{1}{\Gamma(\mu s+\mu R+\mu)} \int_0^1 \left(\frac{1-x}{x}\right)^{\mu s+\mu-1} (1-x)^{\mu R}\, x^{2\mu-2-\mu\rho_r-\mu R}\, \Psi(x)\, dx$$
$$= U_r(s+R) - \sum_{\tau=0}^{K} f_\tau(b, a)\, T_\tau(\mu s+\mu R) - \sum_{m=0}^{N-1} c_m \frac{\Gamma(\mu m+\mu-\mu s-\mu R-\mu\rho_r)}{\Gamma(\mu m+2\mu-\mu\rho_r)}.$$

When R is very large, and whether $|v|$ be large or not, the modulus of the integral is less than

$$\frac{1}{|\Gamma(\mu s+\mu R+\mu)|} \exp\left\{-(\tfrac{1}{2}\pi+\pi/\mu-\epsilon)\, |\mu v|\right\} K'\mu_1^{\mu R}$$

where K' is finite and independent of R and $|v|$, and μ_1 is the maximum value of $|1-x|$ on the circular arc which, as in § 32, we take to be the modified form of the contour of integration.

Thus, if $s+R$ be not in the immediate vicinity of one of the poles of $U_r(s)$, we have

$$|U_r(s+R)| < {}_1\eta_R \exp\{(-\pi+\epsilon)\, |v|\},$$

where ${}_1\eta_R e^{kR}$ can be made as small as we please by taking the positive quantity R sufficiently large.

Now we have seen that

$$\sin \pi\mu s\; {}_pS_q(s) = \sum_{r=1}^{q} A_r \mu^{\mu s}\, U_r(s),$$

where the quantities A are independent of s.

Hence a similar inequality is true of $|{}_pS_q(s+R)|$, and we have the given theorem.

46. *Let Φ denote the linear combination of hypergeometric integral functions*

$$\frac{\prod_{r=1}^{q} \Gamma(1-\rho_r)}{\prod_{r=1}^{p} \Gamma(1-a_r)}\; {}_pF_q\{a_1, \ldots, a_p;\, \rho_1, \ldots, \rho_q;\, (-)^\mu x\}$$
$$+ \sum_{r=1}^{q} x^{1-\rho_r}\, \Gamma(\rho_r-1) \frac{\prod'_{t=1}^{q} \Gamma(\rho_r-\rho_t)}{\prod_{t=1}^{p} \Gamma(\rho_r-a_t)}\; {}_pF_q\{1+a_1-\rho_r, \ldots, 1+a_p-\rho_r;\; 2-\rho_r, \ldots, \rho_q-\rho_r+1,\, (-)^\mu x\}$$

wherein $\mu = q+1-p$ and $q > p$.

We proceed to shew that

$$\Phi = \exp\{-\mu x^{1/\mu}\}(2\pi)^{\frac{1}{2}(\mu-1)}\mu^{-\frac{1}{2}} I$$

where
$$I = -\frac{1}{2\pi\iota}\int_Q {}_pS_q(s)\, x^s\, ds,$$

the integral being taken round a contour Q which embraces the positive half of the real axis and encloses the poles

$$n/\mu, \quad n/\mu - \rho_r + 1 \quad (n = 0, 1, 2, \ldots, \infty;\ r = 1, 2, \ldots, q),$$

but not the poles
$$-n/\mu + [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu-1)]/\mu$$

of the subject of intergration.

The integral is convergent by the result of § 45 for all values of $|x|$. Let T be the greatest integer such that

$$T/\mu + n \leqq N$$

and let T_r be the greatest integer such that

$$T_r/\mu + 1 - \rho_r + n \leqq N.$$

Then, by Cauchy's theory of residues,

$$I = \sum_{n=0}^{N} \frac{(-)^n}{n!} \sum_{t=0}^{T} \frac{\prod\limits_{r=1}^{q} \Gamma(1-\rho_r-n)\, x^{t/\mu+n}}{\prod\limits_{r=1}^{p} \Gamma(1-a_r-n) \prod\limits_{r=1}^{\mu} \Gamma\left(\frac{t+r}{\mu}\right)}$$

$$+ \sum_{r=1}^{q} \sum_{n=0}^{N} \frac{(-)^n}{n!} \sum_{t=0}^{T_r} \frac{\Gamma(\rho_r - 1 - n) \prod\limits_{m=1}^{q}{}' \Gamma(\rho_r - \rho_m - n)}{\prod\limits_{m=1}^{\mu} \Gamma\left(\frac{t+m}{\mu}\right) \prod\limits_{m=1}^{p} \Gamma(\rho_r - a_m - n)} x^{t/\mu+1-\rho_r+n} + I_N.$$

I_N denotes the integral $-\frac{1}{2\pi\iota}\int {}_pS_q(s)\, x^s\, ds$ taken round a contour on which $R(s) > N$ and which embraces the real axis and encloses poles of ${}_pS_q(s)$ whose residues have not been included in the previous summations.

By the theorem of the previous paragraph it is evident that I_N can be made as small as we please by taking N sufficiently large.

Now, by the multiplication formula for the gamma function,

$$\prod_{q=1}^{\mu} \Gamma(t/\mu + q/\mu) = (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-t-\frac{1}{2}} \Gamma(t+1).$$

Hence, on making N infinite, we have

$$I = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\prod_{r=1}^{q} \Gamma(1-\rho_r-n)}{\prod_{r=1}^{p} \Gamma(1-a_r-n)} \frac{\mu^{\frac{1}{2}} x^n}{(2\pi)^{\frac{1}{2}(\mu-1)}} \sum_{t=0}^{\infty} \frac{x^{t/\mu}\mu^t}{\Gamma(t+1)}$$

$$+ \sum_{r=1}^{q} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\Gamma(\rho_r-1-n) \prod_{m=1}^{q}{}' \Gamma(\rho_r-\rho_m-n)}{\prod_{m=1}^{p} \Gamma(\rho_r-a_m-n)} \frac{\mu^{\frac{1}{2}} x^{1-\rho_r+n}}{(2\pi)^{\frac{1}{2}(\mu-1)}} \sum_{t=0}^{\infty} \frac{x^{t/\mu}\mu^t}{\Gamma(t+1)}.$$

Thus

$$I \exp\{-\mu x^{1/\mu}\} \mu^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}(\mu-1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\prod_{r=1}^{q} \Gamma(1-\rho_r-n)}{\prod_{r=1}^{p} \Gamma(1-a_r-n)} x^n$$

$$+ \sum_{r=1}^{q} x^{1-\rho_r} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\Gamma(\rho_r-1-n) \prod_{m=1}^{q}{}' \Gamma(\rho_r-\rho_m-n)}{\prod_{m=1}^{p} \Gamma(\rho_r-a_m-n)} x^n = \Phi.$$

We thus have the given theorem.

47. *We can now shew that, if μ be an integer > 1 and if*

$$|\arg x| < (\mu+1)\pi,$$

the linear combination of hypergeometric integral functions which has been denoted by Φ admits the asymptotic expansion

$$\exp\{-\mu x^{1/\mu}\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} x^{[\Sigma a - \Sigma\rho + \frac{1}{2}(\mu-1)]/\mu} \left\{ \sum_{n=0}^{N} \frac{\lambda_n}{x^{n/\mu}} + \frac{J_N}{x^{N/\mu}} \right\}$$

where the quantities λ are definite functions of the parameters a and ρ and $|J_N|$ can be made as small as we please by taking $|x|$ sufficiently large.

We have seen that, for all finite values of $|\mu|$, Φ is equal to

$$\exp\{-\mu x^{1/\mu}\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} I$$

where I is the integral $-\dfrac{1}{2\pi\iota}\displaystyle\int_Q {}_pS_q(s)\, x^s\, ds$ considered in the previous paragraph.

Also, it has been established in §§ 43 and 45 that, if $|\arg x| < (\mu+1)\pi$, $|{}_pS_q(s)x^s|$ will tend exponentially to zero as $|s|$ tends to infinity if $R(s)$ is greater than a finite negative quantity.

Let L be a contour parallel to the real axis which passes between the points

$$[\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)]/\mu - N/\mu \quad \text{and} \quad [\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)]/\mu - (N+1)/\mu.$$

Then I is equal to $-\frac{1}{2\pi\iota}\int_L {}_pS_q(s)\,x^s\,ds$ together with the sum of the residues of ${}_pS_q(s)\,x^s$ at the points

$$[\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)]/\mu - n/\mu \quad (n = 0, 1, 2, \ldots, N).$$

Let λ_n be the typical residue of ${}_pS_q(s)$ at such a pole: we have seen that λ_n is a function of the parameters a and ρ which can be determined with sufficient labour.

Further,

$$-\frac{1}{2\pi\iota}\int_L {}_pS_q(s)\,x^s\,ds = x^{[\Sigma a - \Sigma\rho + \frac{1}{2}(\mu-1)]/\mu - N/\mu}\,J_N,$$

and it is evident that $|J_N|$ can be made as small as we please by taking $|x|$ sufficiently large.

We thus have the given theorem.

48. *We can now shew that the previous theorem is equivalent to* $(\mu+1)$ *different results.*

For the asymptotic equality is valid if $|\arg x| < (\mu+1)\,\pi$. Thus it is valid for the $(\mu+1)$ ranges

$$(\mu-1)\,\pi < \arg x < (\mu+1)\,\pi,$$

$$(\mu-1-2)\,\pi < \arg x < (\mu+1-2)\,\pi,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$(\mu-1-2\mu)\,\pi < \arg x < (\mu+1-2\mu)\,\pi.$$

We therefore have, if $m = 0, 1, 2, \ldots, \mu$, and $|\arg x| < \pi$, the asymptotic equalities

$$\frac{\prod_{r=1}^{q}\Gamma(1-\rho_r)}{\prod_{r=1}^{p}\Gamma(1-a_r)}\,{}_pF_q\{a\,;\ \rho\,;\ (-)^{\mu}x\}$$

$$+\sum_{r=1}^{q} e^{(\mu-2m)(1-\rho_r)\pi\iota}\,x^{1-\rho_r}\,\frac{\Gamma(\rho_r-1)\prod_{t=1}^{q}{}'\,\Gamma(\rho_r-\rho_t)}{\prod_{t=1}^{p}\Gamma(\rho_r-a_t)}$$

$$\times\,{}_pF_q\{1+a-\rho_r\,;\ 2-\rho_r,\ \ldots,\ \rho_q-\rho_r+1\,;\ (-)^{\mu}x\}$$

$$= \exp\{-\mu\, e^{(\mu-2m)\pi\iota/\mu}\,x^{1/\mu}\}\,(2\pi)^{\frac{1}{2}(\mu-1)}\,\mu^{-\frac{1}{2}}\,x^{\theta/\mu}\,e^{(\mu-2m)\theta\pi\iota/\mu}\sum_{n=0}^{\infty}\frac{\lambda_n\,e^{n(2m-\mu)\pi\iota\,\mu}}{x^{n\,\mu}},$$

where

$$\theta = \Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1).$$

Only μ of the $(\mu+1)$ results here given are independent: those corresponding to $m=0$ and $m=\mu$ can be deduced from one another by means of other asymptotic expansions of the hypergeometric functions.

For brevity write, as in § 1,

$$Q_0(x) = \frac{\prod_{r=1}^{q} \Gamma(1-\rho_r)}{\prod_{r=1}^{p} \Gamma(1-a_r)} \, {}_pF_q\left\{a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_q;\ (-)^\mu x\right\},$$

$$Q_r(x) = x^{1-\rho_r} \frac{\Gamma(\rho_r - 1) \prod_{t=1}^{q}{}' \Gamma(\rho_r - \rho_t)}{\prod_{t=1}^{p} \Gamma(\rho_r - a_t)}$$

$$\times {}_pF_q\left\{1+a_1-\rho_r, \ldots, 1+a_p-\rho_r;\ 2-\rho_r, \ldots;\ \rho_q-\rho_r+1;\ (-)^\mu x\right\}.$$

Then we have, if $|\arg x| < \pi$,

$$Q_0(x) + \sum_{r=1}^{q} e^{(\mu-2m)(1-\rho_r)\pi i} Q_r(x)$$

$$= \exp\left\{-\mu e^{(\mu-2m)\pi i/\mu} x^{1/\mu}\right\} \frac{(2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\frac{1}{2}}} x^{\theta/\mu} e^{(\mu-2m)\theta\pi i/\mu} \sum_{n=0}^{\infty} \frac{\lambda_n e^{n(2m-\mu)\pi i/\mu}}{x^{n/\mu}}, \quad \text{(A)}$$

when $m = 0, 1, \ldots, \mu.$

Also, by the result of § 7, if $|\arg(-x)| < \frac{1}{2}\mu\pi + \pi$,

$$\frac{1}{\sin \pi a_m} Q_0(x) + \sum_{r=1}^{q} \frac{1}{\sin \pi(\rho_r - a_m)} e^{\pi i(1-\rho_r)} Q_r(x)$$

$$= \frac{\Gamma(a_m) \prod_{r=1}^{q} \Gamma(a_m - \rho_r + 1)}{\pi \prod_{r=1}^{p}{}' \Gamma(1 + a_m - a_r)} x^{-a_m} e^{-\pi i a_m}$$

$$\times {}_{q+1}F_{p-1}\left\{a_m, a_m+1-\rho_1, \ldots, a_m+1-\rho_q;\ a_m-a_1+1, \ldots, a_m-a_p+1;\ \frac{1}{x}\right\}, \quad \text{(B)}$$

when $m = 1, 2, \ldots, p.$

Now, in the relations (A) take $m = 0$ and $m = \mu$. We have

$$Q_0(x) + \sum_{r=1}^{q} e^{-\mu(\rho_r-1)\pi i} Q_r(x) = \exp\left[\mu x^{1/\mu}\right] x^{\theta/\mu} e^{\theta\pi i}\,\Theta,$$

$$Q_0(x) + \sum_{r=1}^{q} e^{\mu(\rho_r-1)\pi i} Q_r(x) = \exp\left[\mu x^{1/\mu}\right] x^{\theta/\mu} e^{-\theta\pi i}\,\Theta,$$

where Θ denotes the asymptotic series

$$\frac{(2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\lambda_n(-)^n}{x^{n/\mu}}.$$

From these we deduce

$$Q_0(x)\sin\pi\theta + \sum_{r=1}^{q} Q_r(x)\sin\pi(\theta+\mu\rho_r-\mu) = 0.$$

This equality means that when $|\arg x| < \pi$ the particular combination of functions Q just written down admits an asymptotic expansion whose dominant term is of order less than that of $\exp\{\mu x^{1/\mu}\}$ when divided by any algebraic power of x. Thus it must be possible to form the particular combination from the q other relations contained in (A) and (B).*

Hence in (A) and (B) there are $(q+1)$ relations which are independent.

There are $(q+1)$ functions Q which, when x is replaced by $(-)^\mu x$, are independent solutions of the equation

$$\left[(\vartheta+\alpha_1)\dots(\vartheta+\alpha_p) - \frac{d}{dx}(\vartheta+\rho_1-1)\dots(\vartheta+\rho_q-1)\right]y = 0 \tag{1}$$

valid over the whole of the finite part of the plane.

There are similarly $(q+1)$ asymptotic solutions valid near infinity. These are, with the same transformation, given by the expressions on the right-hand sides of equations (A) ($m = 1, 2, \dots, \mu$) and (B). The relations between these solutions are given by the equations (A) and (B).

49. But the equations (A) and (B) of the previous paragraph only express the $(q+1)$ principal asymptotic solutions near $|x| = \infty$ of the differential equation (1), transformed by writing $(-)^\mu x$ for x, in terms of linear combinations of the $(q+1)$ hypergeometric integral functions $Q_r(x)$ ($r = 0, 1, 2, \dots, q$), which are principal solutions in the finite part of the plane.

We have conversely to express each of the functions $Q_r(x)$ in terms of

* The same phenomenon occurs in the simple case $p = 0$, $q = 1$, $\mu = 2$, and was discussed in detail in the author's earlier paper. *Loc. cit.*, § 1, Paper (η), Part II.

linear combinations of the asymptotic expansions. In § 11 it has been shewn that it is possible to express a suitable linear combination of μ, and not less than μ, functions $Q_r(x)$ in terms of the p asymptotic solutions whose dominant terms are algebraic. The expression of a linear combination of any number less than μ of the functions $Q_r(x)$ will involve asymptotic expansions whose dominant term is exponentially infinite, and such series will entirely overshadow the other series which were obtained in Part I.

Evidently *as typical of the general converse problem we may find the asymptotic expansion of* ${}_pF_q\{a; \rho; x\}$ *when* $|\arg x| < \pi$.

For this purpose we consider the integral

$$-\frac{1}{2\pi i}\int U(s)\, x^s\, ds, \tag{1}$$

where $U(s)$ is the function of s defined, when

$$R(s) > R\{\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)\}/\mu,$$

by the convergent series of gamma functions

$$\sum_{n=0}^{\infty} \frac{\prod\limits_{r=1}^{p}\{\Gamma(a_r+n)\}\,\Gamma(\mu n-\mu s)}{\prod\limits_{r=1}^{q}\{\Gamma(\rho_r+n)\}\,\Gamma(n+1)}\,\mu^{\mu s-\mu n+1}.$$

By the multiplication formula for the gamma function

$$\Gamma(\mu n-\mu s) = \frac{\mu^{\mu n-\mu s-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}}\prod_{r=0}^{\mu-1}\Gamma(n-s+r/\mu).$$

Hence the above series for $U(s)$ may be written

$$U(s) = \frac{\mu^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}}\sum_{n=0}^{\infty}\frac{\prod\limits_{r=1}^{p}[\Gamma(a_r+n)]\prod\limits_{r=0}^{\mu-1}\Gamma(n-s+r/\mu)}{\Gamma(n+1)\prod\limits_{r=1}^{q}(\rho_r+n)}.$$

Evidently $U(s)$ is a function of the same type as the function $U_r(s)$ introduced in § 29: it possesses the same properties.

It has, besides the poles of the functions $\Gamma(\mu n-\mu s)$, as its only singularities poles at the points

$$s = [\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)]/\mu - r/\mu, \tag{2}$$

and the residue at the general pole we may take to be $\mu^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-\mu)}\, l_r$. The quantity l_r can be calculated with sufficient labour from the asymptotic expansion of the general term of the series for $U(s)$ when

$$R(s) > R\,\{\Sigma a - \Sigma\rho + \tfrac{1}{2}(\mu-1)\}/\mu.$$

In particular we readily see that $l_0 = 1$.

Let now the contour C of the integral $-\frac{1}{2\pi\iota}\int_C U(s)\, x^s\, ds$ embrace the real axis, enclose all the poles

$$s = n + t/\mu \quad (n = 0, 1, 2, \ldots, \infty;\ t = 0, 1, 2, \ldots, \infty),$$

and enclose none of the poles (2).

Then the integral is convergent for all finite values of $|x|$. Also

$$-\frac{1}{2\pi\iota}\int_C \Gamma(\mu n - \mu s)\, \mu^{\mu s - \mu n + 1}\, x^s ds = \sum_{t=0}^{\infty} \frac{(-)^t \mu^t}{t!}\, x^{n+t/\mu} = x^n \exp\{-\mu x^{1/\mu}\}.$$

Hence the integral is equal to

$$\exp\{-\mu x^{1/\mu}\}\, \frac{\prod_{r=1}^{p}\Gamma(a_r)}{\prod_{r=1}^{q}\Gamma(\rho_r)}\, {}_pF_q\{x\}.$$

Again, by the results of §§ 42 and 45 the integral may be swung back as in § 8, provided $|\arg x| < \pi$.

We thus get for its asymptotic value

$$x^{[\Sigma a - \Sigma\rho + \frac{1}{2}(\mu-1)]/\mu}\, \mu^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-\mu)} \left\{1 + \sum_{r=1}^{R} \frac{l_r}{x^{r/\mu}} + \frac{J_R}{x^{R/\mu}}\right\},$$

where, for any finite value of R, $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity, and the quantities l_r can be determined with sufficient labour.

We thus obtain the asymptotic equality

$$\frac{\prod_{r=1}^{p}\Gamma(a_r)}{\prod_{r=1}^{q}\Gamma(\rho_r)}\, {}_pF_q\{a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_q;\ x\}$$
$$= \exp\{\mu x^{1/\mu}\}\, x^{[\Sigma a - \Sigma\rho + \frac{1}{2}(\mu-1)]/\mu}\, \mu^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-\mu)} \left\{1 + \sum_{r=1}^{R} \frac{l_r}{x^{r/\mu}} + \frac{J_R}{x^{R/\mu}}\right\},$$

valid provided $|\arg x| < \pi$.

Evidently this includes the result of Stokes stated in § 3.

When $p = 0$, $q = 1$, $\mu = 2$ we have the particular case previously discussed by the author. In this case the coefficients l_r take a simple form, and we have* the asymptotic equality

$$ {}_0F_1(\rho\,;\ x) = \frac{\Gamma(\rho)}{2\pi^{\frac{1}{2}}}\, x^{\frac{1}{4}-\frac{1}{2}\rho}\, e^{2x^{\frac{1}{2}}}\, {}_2F_0 \left\{ \rho-\tfrac{1}{2},\ \tfrac{3}{2}-\rho\,;\ 1/(4x^{\frac{1}{2}}) \right\}. $$

50. In a similar manner we can obtain the asymptotic expansions of any of the hypergeometric integral functions which are the principal solutions of the differential equation (1) of § 48.

The theory is evidently complete.

CORRIGENDA in above paper.—

On p. 62, line 18, *for* "$\rho_r - 1$" *read* "$1-\rho_r$."

,, line 19, *for* "$e^{(\mu-2m)\theta\pi i}$" *read* "$e^{(\mu-2m)\theta\pi i/\mu}$."

,, line 20, *read* "$\theta = [\Sigma\alpha - \Sigma\rho + \tfrac{1}{2}(\mu-1)]$."

On p. 80, line 8 from foot *for* "$\prod_{m=1}^{p} \Gamma(\rho_r - \rho_m)$" *read* "$\prod_{m=1}^{p}{}' \Gamma(\rho_r - \rho_m)$."

* *Vide* § 23 of the memoir previously cited.