and the equations for the foci are

$$y^2-5 = x^2-8 = -xy\sec\omega = -2xy;$$

therefore $\dfrac{y^2+2xy}{5} = \dfrac{x^2+2xy}{8}$, $5x^2-6xy-8y^2 = 0$, $x = 2y$, or $-\dfrac{4y}{5}$,

$$x = 2y \text{ gives } y^2 = 1, \quad x = -\frac{4y}{5} \text{ gives } y^2 = -\frac{25}{3};$$

thus the real foci are $(2, 1)$, $(-2, -1)$; and the impossible

$$\left(\pm\frac{4}{\sqrt{-3}}, \mp\frac{5}{\sqrt{-3}}\right);$$

while the real directrices are $\dfrac{x}{4}+\dfrac{y}{5} = \pm 1$; and the impossible directrices $\dfrac{x}{2}-y = \pm\sqrt{-3}$.]

The Application of Elliptic Coordinates and Lagrange's Equations of Motion to Euler's Problem of Two Centres of Force. By A. G. Greenhill, M.A.

[*Read April 8th*, 1880.]

Denoting by $2c$ the distance between the centres of force; then, if $c\cos\theta$, $c\sin\theta$ be the $\frac{1}{2}$ axes of the hyperbola $c\cosh\phi$, $c\sinh\phi$ of the ellipse, passing through a point, and having their foci at the centres of force, θ and ϕ may be called the elliptic coordinates of the point; and if the axes of these conics be taken as coordinate axes, then the Cartesian coordinates of the point are $x = c\cos\theta\cosh\phi$ and $y = c\sin\theta\sinh\phi$.

Therefore, for a particle of unit mass, the kinetic energy

$$\begin{aligned} T &= \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) \\ &= \tfrac{1}{2}c^2\{(-\sin\theta\cosh\phi\,\dot\theta+\cos\theta\sinh\phi\,\dot\phi)^2 \\ &\qquad +(\cos\theta\sinh\phi\,\dot\theta+\sin\theta\cosh\phi\,\dot\phi)^2\} \\ &= \tfrac{1}{2}c^2\{(\sin^2\theta\cosh^2\phi+\cos^2\theta\sinh^2\phi)\dot\theta^2 \\ &\qquad +(\cos^2\theta\sinh^2\phi+\sin^2\theta\cosh^2\phi)\dot\phi^2\} \\ &= \tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)(\dot\theta^2+\dot\phi^2). \end{aligned}$$

If r, s denote the distances of the particle from the centres of force, then

$$\begin{aligned} r &= c(\cosh\phi-\cos\theta), \\ s &= c(\cosh\phi+\cos\theta); \end{aligned}$$

and if A, B be the strengths of the centres of force, supposed to attract with intensity inversely proportional to the square of the distance, the gravitation potential

$$U = \frac{A}{r} + \frac{B}{s}$$
$$= \frac{A}{c(\cosh\phi - \cos\theta)} + \frac{B}{c(\cosh\phi + \cos\theta)}.$$

[If we have a third centre of force, midway between the other two, attracting with intensity proportional to the distance, of strength C, we must add to this value of U the term

$$\tfrac{1}{2}C(x^2+y^2)$$
$$= \tfrac{1}{2}Cc^2(\cos^2\theta\cosh^2\phi + \sin^2\theta\sinh^2\phi)$$
$$= \tfrac{1}{4}Cc^2(\cosh 2\phi + \cos 2\theta).]$$

Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot\theta}\right) - \frac{\delta T}{\delta\theta} = \frac{\delta U}{\delta\theta},$$
$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot\phi}\right) - \frac{\delta T}{\delta\phi} = \frac{\delta U}{\delta\phi};$$

δ denoting partial, and d total differentiation.

Therefore $\tfrac{1}{2}c^2(\cosh 2\phi - \cos 2\theta)\,\ddot\theta + \tfrac{1}{2}c^2\sin 2\theta\,(\dot\theta^2 - \dot\phi^2) + \sinh 2\phi\,\dot\theta\dot\phi$

$$= -\frac{A}{c}\,\frac{\sin\theta}{(\cosh\phi - \cos\theta)^2} + \frac{B}{c}\,\frac{\sin\theta}{(\cosh\phi + \cos\theta)^2}$$
$$-\tfrac{1}{2}Cc^2\sin 2\theta \;\ldots\ldots\ldots\; (1),$$

and $\tfrac{1}{2}c^2(\cosh 2\phi - \cos 2\theta)\,\ddot\phi + \sin 2\theta\,\dot\theta\dot\phi - \tfrac{1}{2}c^2\sinh 2\phi\,(\dot\theta^2 - \dot\phi^2)$

$$= -\frac{A}{c}\,\frac{\sinh\phi}{(\cosh\phi - \cos\theta)^2} - \frac{B}{c}\,\frac{\sinh\phi}{(\cosh\phi + \cos\theta)^2}$$
$$+\tfrac{1}{2}Cc^2\sinh 2\phi \;\ldots\ldots\ldots\; (2).$$

If we multiply (1) by $\dot\theta$, and (2) by $\dot\phi$, add and integrate, we obtain

$$\tfrac{1}{4}c^2(\cosh 2\phi - \cos 2\theta)(\dot\theta^2 + \dot\phi^2)$$
$$= \frac{A}{c}\,\frac{1}{\cosh\phi - \cos\theta} + \frac{B}{c}\,\frac{1}{\cosh\phi + \cos\theta}$$
$$+\tfrac{1}{4}Cc^2(\cosh 2\phi + \cos 2\theta) - H \;\ldots\ldots\ldots\; (3),$$

the equation of energy.

To obtain the second integral of these equations of motion, multiply

(1) by $\cosh 2\phi\,\dot\theta$, and (2) by $\cos 2\theta\,\dot\phi$, and add; then

$$\tfrac{1}{4}c^2\frac{d}{dt}(\cosh 2\phi-\cos 2\theta)(\cosh 2\phi\,\dot\theta^2+\cos 2\theta\,\dot\phi^2)$$
$$=-\frac{A}{c}\frac{\sin\theta\cosh 2\phi\,\dot\theta+\cos 2\theta\sinh\phi\,\dot\phi}{(\cosh\phi-\cos\theta)^2}$$
$$+\frac{B}{c}\frac{\sin\theta\cosh\phi\,\dot\theta-\cos 2\theta\sinh\phi\,\dot\phi}{(\cosh\phi+\cos\theta)^2}$$
$$-\tfrac{1}{2}Cc^2(\sin 2\theta\cosh 2\phi\,\dot\theta-\cos 2\theta\sinh 2\phi\,\dot\phi);$$

and, integrating,

$$\tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)(\cosh 2\phi\,\dot\theta^2+\cos 2\theta\,\dot\phi^2)$$
$$=\frac{A}{c}\frac{2\cos\theta\cosh\phi-1}{\cosh\phi-\cos\theta}-\frac{B}{c}\frac{2\cos\theta\cosh\phi+1}{\cosh\phi+\cos\theta}$$
$$+\tfrac{1}{4}Cc^2\cos 2\theta\cosh 2\phi-D \quad\text{.........}\ (4),$$

equivalent to Euler's second integral, D and H being arbitrary constants, determined by the initial circumstances of the motion.

From (3) and (4),

$$\tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)^2\,\dot\theta^2$$
$$=\frac{A}{c}\frac{2\cos\theta\cosh\phi-1-\cos 2\theta}{\cosh\phi-\cos\theta}-\frac{B}{c}\frac{2\cos\theta\cosh\phi+1+\cos 2\theta}{\cosh\phi+\cos\theta}$$
$$-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta$$
$$=2\frac{A-B}{c}\cos\theta-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta \quad\text{..................}\ (5),$$

$$\tfrac{1}{4}Cc^2(\cosh 2\phi-\cos 2\theta)^2\,\dot\phi^2$$
$$=\frac{A}{c}\frac{\cosh 2\phi+1-2\cos\theta\cosh\phi}{\cosh\phi-\cos\theta}+\frac{B}{c}\frac{\cosh 2\phi+1+2\cos\theta\cosh\phi}{\cosh\phi+\cos\theta}$$
$$+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi$$
$$=2\frac{A+B}{c}\cosh\phi+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi \quad\text{..............}\ (6).$$

Therefore

$$\left(\frac{d\theta}{d\phi}\right)^2=\frac{2\dfrac{A-B}{c}\cos\theta-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta}{2\dfrac{A+B}{c}\cosh\phi+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi},$$

the differential equation of the orbit, a differential equation in which the variables θ and ϕ are separated.

Euler employs new variables u and v, such that $u=\tan\tfrac{1}{2}\theta$, $v=\tanh\tfrac{1}{2}\phi$; his p and q being respectively $c\cos\theta$ and $c\cosh\phi$.

Then $\dfrac{d\theta}{du}=\dfrac{2}{1+u^2},\quad \cos\theta=\dfrac{1-u^2}{1+u^2},\quad \cos 2\theta=2\left(\dfrac{1-u^2}{1+u^2}\right)^2-1,$

$$\frac{d\phi}{dv}=\frac{2}{1-v^2},\quad \cosh\phi=\frac{1+v^2}{1-v^2},\quad \cosh 2\phi=2\left(\frac{1+v^2}{1-v^2}\right)^2-1;$$

and if C be put $=0$, the differential equation becomes

$$\frac{du^2}{2\dfrac{A-B}{c}(1-u^4)-D(1+u^2)^2+H(1-u^2)^2}$$

$$=\frac{dv^2}{2\dfrac{A+B}{c}(1-v^4)+D(1-v^2)^2-H(1+v^2)^2}=d\lambda^2,\text{ suppose;}$$

and therefore u and v are elliptic functions of λ, and, by the elimination of λ, we obtain the equation of the orbit in terms of u and v, or θ and ϕ.

The integral may also be written

$$\int\frac{du}{\sqrt{\left\{H-D+2\dfrac{A-B}{c}-2(H+D)u^2+\left(H-D-2\dfrac{A-B}{c}\right)u^4\right\}}}$$

$$\pm\int\frac{dv}{\sqrt{\left\{D-H+2\dfrac{A+B}{c}-2(D+H)v^2+\left(D-H-2\dfrac{A+B}{c}\right)v^4\right\}}}=E,$$

a constant;

and, using the notation

$$\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\arg\operatorname{sn}(x,k),$$

this equation may be written

$$\frac{1}{\sqrt{\left[H+D+2\sqrt{\left\{HD+\left(\dfrac{A-B}{c}\right)^2\right\}}\right]}}$$

$$\arg\operatorname{sn}\left\{\frac{H-D-2\dfrac{A-B}{c}}{H+D-2\sqrt{\left\{HD+\left(\dfrac{A-B}{c}\right)^2\right\}}}u,\ k_1\right\}$$

$$\pm\frac{1}{\sqrt{\left[D+H+2\sqrt{\left\{DH+\left(\dfrac{A+B}{c}\right)^2\right\}}\right]}}$$

$$\arg\operatorname{sn}\left\{\frac{D-H-2\dfrac{A+B}{c}}{D+H-2\sqrt{\left\{DH+\left(\dfrac{A+B}{c}\right)^2\right\}}}v,\ k_2\right\}=E,$$

when
$$k_1 = \frac{H+D-2\sqrt{\left\{HD+\left(\frac{A-B}{c}\right)^2\right\}}}{H+D+2\sqrt{\left\{HD+\left(\frac{A-B}{c}\right)^2\right\}}},$$
$$k_2 = \frac{H+D-2\sqrt{\left\{HD+\left(\frac{A+B}{c}\right)^2\right\}}}{H+D+2\sqrt{\left\{HD+\left(\frac{A+B}{c}\right)^2\right\}}}.$$

A discussion of the different cases that arise from giving different values to D and H is given in Legendre's "Traité des Functions Elliptiques," tome i.

Theorems in the Calculus of Operations. *By* J. J. WALKER.

[*Read April 8th,* 1880.]

I. The subjects of operation are functions of a single variable x, or if other variables enter these are supposed independent of x, so that x alone is considered to vary. Let u, φ be any such functions. The kind of operation considered is that of multiplying u by some integer power of ϕ, and then taking the differential coefficient of the product of an order differing from the index of ϕ by a given number, which may be 0, 1, 2 ... Thus the symbols for a completed set of such operations may be $Du\phi$, $D^2u\phi^2 \dots D^ru\phi^r$, where D stands for $\frac{d}{dx}$; or $Du\phi^2$, $D^2u\phi^3 \dots D^{r-1}u\phi^r \dots$. A series of such terms, for shortness, may be called a *progressive series*, in the sense that the subjects of successive differentiations are not, as in Taylor's Series or Leibnitz's Theorem, one and the same function, but form a geometric progression, the common ratio being the function ϕ.

The first theorem establishes the development of $D^nu\phi^{n+1}$ in a progressive series, the terms of which are of the form $D^ru\phi^r$; viz., writing ϕ' for $D\phi$, it is proved that

$$D^nu\phi^{n+1} = \phi D^nu\phi^n + n\phi'\phi D^{n-1}u\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}D\phi'\phi^2\,.\,D^{n-2}u\phi^{n-2}$$
$$+ \dots + nD^{n-2}\phi'\phi^{n-1}\,.\,Du\phi + D^{n-1}\phi'\phi^n\,.\,u \quad \dots\dots\dots\dots\dots (a).$$

The second theorem similarly establishes the development of $D^nu\phi^{n-1}$ in another progressive series, in which the terms of the same type, $D^ru\phi^r$, are multiplied by different functional coefficients; viz., now, for convenience, writing $\psi = \phi^{-1}$, $\psi' = D\psi$,