In avowed imitation of the plan of the *Journal*, and in a similar form (called 12mo in the catalogues, but more of the size of a 24mo of the present time), the *Nouvelles de la République des Lettres* was commenced as a monthly publication, "to be written as well as printed" at Amsterdam, in March, 1684; and maintained, with breaks, until 1718. During one of these interruptions, a similar publication, with the title *Histoire des Ouvrages des Sçavans*, was started at Rotterdam, in September, 1687, and continued until June, 1709. The last two Journals are quoted sometimes as *Nova* and *Historia Bataviæ*, or *Nouvelles d'Amsterdam* and *de Rotterdam*, respectively.

The *Acta Eruditorum* (Note, p. 11), which was aided by a subvention from the Elector of Saxony, continued to appear in yearly volumes until 1777. The *Acta Helvetica*, quoted p. 15, was founded in 1751, and maintained until 1787. But, before the middle of the eighteenth century, the *Mémoires* of the Academies of Paris (1666) and Berlin (1702) had become the medium for the more elaborate essays of the analysts of the time.

Proofs of Steiner's Theorems relating to Circumscribed and Inscribed Conics. *By* Professor G. B. Mathews.

[*Read Nov. 13th*, 1890.]

The theorems here discussed are enunciated without proof in Steiner's memoir, entitled "Teoremi relativi alle coniche inscritte e circonscritte" (*Werke* II., p. 329, or *Crelle* XXX., p. 97). The most important are those relating to the maximum conics inscribed in a given quadrilateral, and the minimum conics circumscribed to a given quadrangle; the others, in fact, are preliminary to these, but, for the sake of completeness, proofs of them all have been given. It will be observed that the second principal problem admits of three proper solutions, besides nine improper ones, so that a purely geometrical method would necessarily involve the employment of curves other than conics and straight lines.

1. Adopting ordinary trilinear coordinates referred to a triangle ABC, let

$$\phi(x, y, z) \equiv (u, v, w, u', v', w' \between x, y, z)^2 = 0$$

be the equation of any conic, and let

$$\Phi(\xi, \eta, \zeta) \equiv (U, V, W, U', V', W' \between \xi, \eta, \zeta)^2 = 0$$

be the equation of the same conic in line-coordinates, so that

$$U = vw - u'^2, \text{ \&c.}$$

Further, let Δ denote the discriminant

$$\Delta = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

Then, if Γ be written for $\Phi(a, b, c)$, where a, b, c are the sides of ABC, the square of the area of the conic is equal to $M\Delta^2/\Gamma^3$, where M is a constant.

In order to determine M, let the conic be the circumcircle of ABC; then we may put

$$u = v = w = 0, \quad u' = a, \quad v' = b, \quad w' = c,$$

giving

$$\Delta = 2abc,$$

$$\Gamma = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$$

$$= 16S^2,$$

where S is the area of ABC.

Hence

$$\pi^2 R^4 = M \frac{a^2b^2c^2}{4(16)^3 S^6}$$

or, since

$$R = abc/4S,$$

$$M = 4\pi^2 a^2b^2c^2 S^2,$$

and the square of the area of the conic

$$= 4\pi^2 a^2b^2c^2 S^2 \Delta^2/\Gamma^3 \quad \text{.........................(1).}$$

2. Now, let

$$l^2x^2 + m^2y^2 + n^2z^2 - 2mnyz - 2nlzx - 2lmxy = 0$$

be any proper conic inscribed in ABC, and having its centre at (α, β, γ).

Then

$$l : m : n = a\,(b\beta+c\gamma-a\alpha) : b\,(c\gamma+a\alpha-b\beta) : (a\alpha+b\beta-c\gamma).$$

Suppose the sides of ABC are bisected in the points A', B', C'; then, if α', β', γ' are the perpendiculars from (α, β, γ), or P, upon the sides of $A'B'C'$,

$$\alpha' = S/a-\alpha = (b\beta+c\gamma-a\alpha)/2a, \text{ \&c.},$$

so that $$l : m : n = a^2\alpha' : b^2\beta' : c^2\gamma' \quad\text{.....................(2).}$$

For the conic now considered,

$$\Delta = -4l^2m^2n^2,$$

$$\Gamma = 4lmn\,(bcl+cam+abn);$$

therefore, if E is the area of the conic,

$$E^2 = M\,\frac{lmn}{4\,(bcl+cam+abn)^3}$$

$$= M\,\frac{\alpha'\beta'\gamma'}{4abc\,(a\alpha'+b\beta'+c\gamma')^3};$$

whence, substituting for M its value, and observing that

$$a\alpha'+b\beta'+c\gamma' = S,$$

we find $$E^2 = 4\pi^2 R\alpha'\beta'\gamma' \quad\text{..............................(3).}$$

3. In a similar way, if the circumscribed conic

$$2u'yz+2v'zx+2w'xy = 0$$

has its centre at (α, β, γ),

$$u' : v' : w' = a\alpha\alpha' : b\beta\beta' : c\gamma\gamma' \quad\text{.....................(4),}$$

and replacing these proportions by equalities, it will be found, after a few reductions, that

$$\Delta = 2abc\alpha\beta\gamma\alpha'\beta'\gamma',$$

$$\Gamma = 8Sabc\alpha'\beta'\gamma'.$$

Hence, if F be the area of the conic,

$$F^2 = \frac{\pi^2 abc}{4S}\,\frac{\alpha^2\beta^2\gamma^2}{\alpha'\beta'\gamma'}$$

$$= \pi^2 R\alpha^2\beta^2\gamma^2/\alpha'\beta'\gamma' \quad\text{.......................(5).}$$

4. Now, let conics with centre at P be inscribed and circumscribed to the triangle $A'B'C'$, and let $A''B''C''$ be derived from $A'B'C'$ in the same way as $A'B'C'$ from ABC. Then, if E_1, F_1 are the areas of the two new conics

$$E_1^2 = 4\pi^2 R_1 \alpha_1' \beta_1' \gamma_1',$$

$$F_1^2 = \pi^2 R_1 \alpha_1^2 \beta_1^2 \gamma_1^2 / \alpha_1' \beta_1' \gamma_1',$$

where $\quad R_1 = \frac{1}{2} R,$

$$\alpha_1, \beta_1, \gamma_1 = \alpha', \beta', \gamma',$$

$$\alpha_1', \beta_1', \gamma_1' = \tfrac{1}{2}(\alpha - \alpha'),\ \tfrac{1}{2}(\beta - \beta'),\ \tfrac{1}{2}(\gamma - \gamma');$$

so that $\quad E_1^2 = \frac{1}{4}\pi^2 R\,(\alpha - \alpha')(\beta - \beta')(\gamma - \gamma'),$

$$F_1^2 = 4\pi^2 R \alpha'^2 \beta'^2 \gamma'^2 / (\alpha - \alpha')(\beta - \beta')(\gamma - \gamma').$$

Hence, by multiplication,

$$E_1^2 F_1^2 = \pi^4 R^2 \alpha'^2 \beta'^2 \gamma'^2 = \tfrac{1}{16} E^4;$$

and therefore, taking the absolute values of E_1, F_1, that is, disregarding sign,

$$E^2 = 4E_1 F_1 \quad \text{.................................(6)}.$$

The process of derivation may of course be repeated indefinitely.

But it is to be further observed that the conics E, F are such that an infinity of triangles can be drawn circumscribed to E, and inscribed in F. Let XYZ be any one of these, and let $X_0 Y_0 Z_0$ be related to it as ABC is to $A'B'C'$; then, if E_0 be the area of a conic with centre P inscribed in $X_0 Y_0 Z_0$, we have

$$E_0^2 = 4EF,$$

so that the area of all such conics E_0 is constant.

5. Suppose the inscribed conic E is constrained to touch the fixed line

$$\lambda x + \mu y + \nu z = 0;$$

then $\quad l\mu\nu + m\nu\lambda + n\lambda\mu = 0 \quad$ ....(7),

and therefore, by (2), the locus of P is the line

$$a^2 \mu\nu\alpha' + b^2 \nu\lambda\beta' + c^2 \lambda\mu\gamma' = 0 \quad \text{.....................(8)}.$$

Conversely, if the centre of E describes the line

$$f\alpha'+g\beta'+h\gamma'=0,$$

E will always touch the fixed line

$$\frac{a^2}{f}x+\frac{b^2}{g}y+\frac{c^2}{h}z=0.$$

The triangle of reference and the line

$$\lambda x+\mu y+\nu z=0$$

form a quadrilateral in which an infinity of conics, as above, can be inscribed. In order to find the conics of maximum area, we have to make

$$\alpha'\beta'\gamma'$$

a maximum, subject to the conditions

$$a^2\mu\nu\alpha'+b^2\nu\lambda\beta'+c^2\lambda\mu\gamma'=0 \quad\text{..........................}(9),$$

$$a\alpha'+b\beta'+c\gamma'=S \quad\text{......................}(10).$$

The ordinary process leads to the equation

$$\begin{vmatrix} \beta'\gamma', & \gamma'\alpha', & \alpha'\beta' \\ a^2\mu\nu, & b^2\nu\lambda, & c^2\lambda\mu \\ a, & b, & c \end{vmatrix}=0 \quad\text{......................}(11),$$

and the centres of the maximum conics are determined as the intersections of the line (9) with the conic (11). The latter conic goes through A', B', C', through the centroid, G, of ABC (or $A'B'C'$), and through the point for which

$$a^2\alpha' : b^2\beta' : c^2\gamma' = \lambda : \mu : \nu;$$

that is, by (2), through the centre of

$$\lambda^2x^2+\mu^2y^2+\nu^2z^2-2\mu\nu yz-2\nu\lambda zx-2\lambda\mu xy=0.$$

This point can be easily found geometrically. Namely, if

$$\lambda x+\mu y+\nu z=0$$

meet the sides of ABC in D, E, F, and the points D', E', F' be taken so that

$$(BCDD') = (CAEE') = (ABFF') = -1,$$

then the lines joining A, B, C to the middle points of $E'F'$, $F'D'$, $D'E'$, respectively, will meet together at the required point O.

Since five points upon it are known, the conic can be constructed geometrically; the line represented by (9) is, as is well known, the line through the middle points of the diagonals of the quadrilateral $BCEF$. This being drawn, its intersections with the conic $OA'B'C'G$ are the centres of the maximum conics.

6. From a geometrical point of view, the construction above indicated is about as simple and satisfactory as could be expected: it is interesting, however, to verify a remarkable metrical theorem of Steiner's with reference to the centres, K, K' suppose, of the maximum conics. The theorem is that, if the line of centres meet the sides of $A'B'C'$ in P_1, P_2, P_3, and if M bisect KK', then M is the mean centre of P_1, P_2, P_3, and

$$MK^2 = MK'^2 = \tfrac{1}{6}(MP_1^2 + MP_2^2 + MP_3^2);$$

also, M is the centroid of the six vertices of the given circumscribing quadrilateral.

In order to verify these statements, we may solve the equations (9), (10), (11). It will simplify the result if we write p, q, r for the determinants of the matrix

$$\begin{vmatrix} a, & b, & c \\ \lambda, & \mu, & \nu \end{vmatrix},$$

so that

$$p = b\nu - c\mu, \quad q = c\lambda - a\nu, \quad r = a\mu - b\lambda \qquad (12).$$

Further, let us put

$$Z = b^2c^2\lambda^2 + c^2a^2\mu^2 + a^2b^2\nu^2 - abc\,(a\mu\nu + b\nu\lambda + c\lambda\mu)$$

$$= \tfrac{1}{2}(a^2p^2 + b^2q^2 + c^2r^2) \qquad (13),$$

an essentially positive quantity.

Then the coordinates of K, K', referred to the triangle $A'B'C'$, are explicitly given by

$$\left.\begin{aligned} \alpha &= \frac{\lambda S}{3aqr}(cr-bq\pm\sqrt{Z}) \\ \beta &= \frac{\mu S}{3brp}(ap-cr\pm\sqrt{Z}) \\ \gamma &= \frac{\nu S}{3cpq}(bq-ap\pm\sqrt{Z}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

The coordinates of M, referred to the same triangle, are therefore

$$\alpha_0 = \frac{\lambda S}{3aqr}(cr-bq),\quad \beta_0 = \frac{\mu S}{3brp}(ap-cr),\quad \gamma_0 = \frac{\nu S}{3cpq}(bq-ap)\ldots(15),$$

while those of P_1, P_2, P_3 are respectively

$$\left.\begin{aligned} \alpha_1, \beta_1, \gamma_1 &= 0,\quad -c\mu S/bp,\quad b\nu S/cp \\ \alpha_2, \beta_2, \gamma_2 &= c\lambda S/aq,\quad 0,\quad -a\nu S/cq \\ \alpha_3, \beta_3, \gamma_3 &= -b\lambda S/ar,\quad a\mu S/br,\quad 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(16).$$

Hence we find without difficulty

$$\alpha_0 = \tfrac{1}{3}(\alpha_1+\alpha_2+\alpha_3),\ \&c.,$$

that is, M is the mean centre of P_1, P_2, P_3.

Also, $(\alpha_0-\alpha_1)^2+(\alpha_0-\alpha_2)^2+(\alpha_0-\alpha_3)^2$

$$\begin{aligned} &= \alpha_1^2+\alpha_2^2+\alpha_3^2-3\alpha_0^2 \\ &= \lambda^2S^2\left\{\frac{c^2}{a^2q^2}+\frac{b^2}{a^2r^2}-\frac{(cr-bq)^2}{3a^2q^2r^2}\right\} \\ &= \frac{2\lambda^2S^2}{3a^2q^2r^2}(c^2r^2+b^2q^2+bcqr) \\ &= \frac{\lambda^2S^2}{3a^2q^2r^2}\{b^2q^2+c^2r^2+(bq+cr)^2\} \\ &= \frac{\lambda^2S^2}{3a^2q^2r^2}(a^2p^2+b^2q^2+c^2r^2) \\ &= \frac{2\lambda^2S^2Z}{3a^2q^2r^2} = \frac{3}{2}\left(\frac{2\lambda S}{3aqr}\sqrt{Z}\right)^2. \end{aligned}$$

From this, the two similar equations, and (14), it follows that

$$MP_1^2+MP_2^2+MP_3^2 = \tfrac{3}{2}KK'^2 = 6MK^2.$$

7. It is possible that the foregoing method, or something like it, is that by which Steiner obtained his results, and that he refrained from publishing a proof of this kind, because he hoped to obtain one of a more purely geometrical character. If, however, we try to solve in a similar manner the analogous problem of finding conics of minimum area which pass through four given points, we are led to results of great complexity. It will be found that the centres of the conics of stationary area are determined as the intersections of a conic and a sextic, so that apparently there are twelve solutions; it will appear, however, from a less symmetrical, but more manageable method, to be presently explained, that there are only three proper solutions, excluding the line-pairs, and the parabolas of the system; whence we infer that the set of twelve solutions is made up by the three proper solutions, the three line-pairs, and the two parabolas each reckoned three times, the conic which is the locus of centres in fact osculating the sextic where it meets it at infinity.

8. Even the problem already discussed may be more simply treated by the unsymmetrical method.

Taking line coordinates ξ, η, ζ, the conic

$$(1+t)\,\lambda\eta\zeta-\mu\zeta\xi-t\nu\xi\eta = 0 \quad\text{..................(17)},$$

where t is a variable parameter, touches the three sides of the triangle of reference, and the fixed line (λ, μ, ν).

As in Art. 1, it can be shown that the square of the area of the conic is proportional to

$$t\,(1+t)\,/\,(bqt-cr)^3.$$

For a stationary value, we have, by logarithmic differentiation,

$$\frac{1}{t}+\frac{1}{t+1}-\frac{3bq}{bqt-cr}=0 \text{ or } \infty.$$

It is easily seen that

$$t=0, \quad -1, \quad \infty$$

give improper solutions, namely, the point-pairs of the system; while

$$bqt-cr=0,$$

corresponds to the single parabola of the system.

The remaining (proper) solutions are given by

$$bqt^2+2(bq+cr)\,t+cr=0 \quad \text{......................(18)},$$

or

$$bqt^2-2apt+cr=0.$$

Eliminating t between (17) and (18), the equation of the maximum conics is obtained in the form

$$ap\lambda^2\eta^2\zeta^2+bq\mu^2\zeta^2\xi^2+cr\nu^2\xi^2\eta^2+2\xi\eta\zeta\,(ap\mu\nu\xi+bq\nu\lambda\eta+cr\lambda\mu\zeta)=0 \text{...(19)}.$$

9. Actually solving (18), we get, in our previous notation,

$$t=\frac{ap\pm\sqrt{Z}}{bq} \quad \text{..............................(20)},$$

and equation (17) becomes

$$(cr\mp\sqrt{Z})\,\lambda\eta\zeta+bq\mu\zeta\xi+(ap\pm\sqrt{Z})\,\xi\eta=0.$$

Now, with the help of the identity

$$ap+bq+cr=0,$$

it can be verified that

$$\left.\begin{aligned}(cr+\sqrt{Z})(ap-cr+\sqrt{Z})&=ap\,(cr-bq+\sqrt{Z})\\(ap-\sqrt{Z})(ap-cr+\sqrt{Z})&=cr\,(bq-ap+\sqrt{Z})\end{aligned}\right\} \text{.........(21)},$$

so that the equations of the maximum conics are separately obtained in the symmetrical form

$$ap\,(cr-bq\pm\sqrt{Z})\,\lambda\eta\zeta+bq\,(ap-cr\pm\sqrt{Z})\,\mu\zeta\xi$$
$$+cr\,(bq-ap\pm\sqrt{Z})\,\nu\xi\eta=0 \text{.........(22)}.$$

It is now easy to obtain the coordinates of their centres, &c., and thus to verify the results of Art. 6.

10. Proceeding now to the other problem, we have, in point-coordinates,

$$(1+t)\,fyz-gzx-thxy=0 \quad \text{.....................(23)},$$

representing a conic through the vertices of the triangle of reference, and through the fixed point (f, g, h). In order that its area may be a maximum or minimum, the value of

$$\frac{t^2(1+t)^2}{\{(af+ch)^2 t^2+2(a^2f^2-bcgh+cahf+abfg)\,t+(af+bg)^2\}^3}$$

must be stationary.

Representing the denominator by u, for the moment, we have

$$\frac{2}{t}+\frac{2}{t+1}-\frac{3}{u}\cdot\frac{du}{dt}=0 \text{ or } \infty.$$

As before, $t=0, -1, \infty$ give the three line-pairs of the system, while $u=0$ corresponds to the two parabolas of the system. Rejecting these solutions as irrelevant, we obtain a cubic equation in t, which, written out in full, is

$$(af+ch)^2 t^3+(a^2f^2+2c^2h^2+bcgh+3cahf-abfg)\,t^2$$
$$-(a^2f^2+2b^2g^2+bcgh-cahf+3abfg)\,t-(af+bg)^2=0 \ldots (24).$$

Eliminating t between (23) and (24), we obtain the equation of the three critical conics in the following form, where, for simplicity, X, Y, Z are written for fyz, gzx, hxy respectively :—

$$\begin{aligned}
&(bg+ch)^2X^3+(ch+af)^2Y^3+(af+bg)^2Z^3\\
&\quad-(2b^2g^2+c^2h^2+3bcgh-cahf+abfg)\,X^2Y\\
&\quad-(b^2g^2+2c^2h^2+3bcgh+cahf-abfg)\,X^2Z\\
&\quad-(2c^2h^2+a^2f^2+3cahf-abfg+bcgh)\,Y^2Z\\
&\quad-(c^2h^2+2a^2f^2+3cahf+abfg-bcgh)\,Y^2X\\
&\quad-(2a^2f^2+b^2g^2+3abfg-bcgh+cahf)\,Z^2X\\
&\quad-(a^2f^2+2b^2g^2+3abfg+bcgh-cahf)\,Z^2Y\\
&\quad+4(a^2f^2+b^2g^2+c^2h^2+bcgh+cahf+abfg)\,XYZ=0 \ldots (25).
\end{aligned}$$

The separate equations of the conics are to be obtained, either by solving (24) and substituting in (23), or by applying the theory of ternary cubics to resolve the left-hand side of (25) into three factors, each linear in X, Y, Z.

On an Algebraic Integral of Two Differential Equations.

By R. A. Roberts.

[*Read Nov.* 13*th*, 1890.]

1. If u, v denote two quadratic expressions in a variable x, I propose to show that the differential equations

$$\frac{dx_1}{\sqrt[3]{(u_1 v_1^2)}} + \frac{dx_2}{\sqrt[3]{(u_2 v_2^2)}} + \frac{dx_3}{\sqrt[3]{(u_3 v_3^2)}} = 0,$$

$$\frac{dx_1}{\sqrt[3]{(u_1^2 v_1)}} + \frac{dx_2}{\sqrt[3]{(u_2^2 v_2)}} + \frac{dx_3}{\sqrt[3]{(u_3^2 v_3)}} = 0,$$

where x_1, x_2, x_3 are three values of x, and u_1, v_1; u_2, v_2; u_3, v_3 the corresponding values of u, v, necessarily involve an algebraic relation between the variables x_1, x_2, x_3 containing two arbitrary constants.

2. Let

$$u = lx^2 + mx + n, \quad v = l'x^2 + m'x + n';$$

then, if

$$f(x) = (\alpha + \beta x)^3 u - (\gamma + \delta x)^3 v \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

there are evidently two relations, independent of α, β, γ, δ, connecting the roots of the quintic

$$f(x) = 0.$$

These may be easily found in the form of systems of determinants involving the cube roots of u, v; for, if x_1, x_2, x_3, x_4 are four roots of

$$f(x) = 0,$$

we have

$$(\alpha + \beta x_1)\sqrt[3]{u_1} - (\gamma + \delta x)\sqrt[3]{v_1} = 0,$$

and three other similar equations, from which we get, eliminating α, β, γ, δ,

$$\begin{vmatrix} \sqrt[3]{u_1}, & x_1\sqrt[3]{u_1}, & \sqrt[3]{v_1}, & x_1\sqrt[3]{v_1} \\ \sqrt[3]{u_2}, & x_2\sqrt[3]{u_2}, & \sqrt[3]{v_2}, & x_2\sqrt[3]{v_2} \\ \sqrt[3]{u_3}, & x_3\sqrt[3]{u_3}, & \sqrt[3]{v_3}, & x_3\sqrt[3]{v_3} \\ \sqrt[3]{u_4}, & x_4\sqrt[3]{u_4}, & \sqrt[3]{v_4}, & x_4\sqrt[3]{v_4} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (2);$$

and, in the same way, we have four other determinants involving x_1, x_2, x_3, x_4 and the fifth root x_5.

3. We may also find these relations in other forms, for, substituting a, b, the roots of $u = 0$, and c, d, the roots of $v = 0$, in the identity (1) successively, we get

$$\left.\begin{array}{ll} \sqrt[3]{f(a)} \propto \gamma+\delta a, & \sqrt[3]{f(b)} \propto \gamma+\delta b \\ \sqrt[3]{f(c)} \propto \alpha+\beta c, & \sqrt[3]{f(d)} \propto \gamma+\delta d \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3);$$

from which and the relations

$$(\alpha+\beta x_1)\sqrt[3]{u_1}-(\gamma+\delta x_1)\sqrt[3]{v_1} = 0,$$

&c., we can eliminate α, β, γ, δ linearly, and so obtain a number of other determinants. A particular form of the two relations in this case, when two of the roots of

$$f(x) = 0$$

are supposed to be given, is worth noticing. From

$$(\alpha+\beta x_4)\sqrt[3]{u_1}-(\gamma+\delta x_4)\sqrt[3]{u_4} = 0,$$

$$(\alpha+\beta x_5)\sqrt[3]{u_5}-(\gamma+\delta x_5)\sqrt[3]{u_5} = 0,$$

where x_4, x_5 are the given roots, we have two linear relations connecting α, β, γ, δ, so that from (3) we obtain

$$\left.\begin{array}{l} \lambda\sqrt[3]{f(a)}+\mu\sqrt[3]{f(b)}+\nu\sqrt[3]{f(c)} = 0 \\ \lambda'\sqrt[3]{f(a)}+\mu'\sqrt[3]{f(b)}+\nu'\sqrt[3]{f(d)} = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4),$$

where λ, μ, ν, λ', μ', ν' are known quantities, and $f(a)$, $f(b)$, $f(c)$, $f(d)$ are respectively proportional to

$$(a-x_1)(a-x_2)(a-x_3), \quad (b-x_1)(b-x_2)(b-x_3),$$

$$(c-x_1)(c-x_2)(c-x_3), \quad (d-x_1)(d-x_2)(d-x_3).$$

4. I now proceed to show that the relations between the five roots of

$$f(x) = 0$$

can be written in a form involving differentials.

Suppose α, β, γ, δ to be functions of a variable t; then, if

$$f(x) = 0$$

is a relation connecting the variables x and t, we obtain, by differentiation,

$$f'(x)\frac{dx}{dt}+3(\alpha+\beta x)^2(\alpha'+\beta' x)u-3(\gamma+\delta x)^2(\gamma'+\delta' x)v=0 \quad\text{......(5)},$$

where α', β', γ', δ' are the differential coefficients of α, β, γ, δ with respect to t. Dividing out now by $\sqrt[3]{(uv^2)}f'(x)$, we get

$$\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\frac{3(\alpha+\beta x)^2(\alpha'+\beta' x)}{f'(x)}\sqrt[3]{\left(\frac{u^2}{v^2}\right)}$$

$$-\frac{3(\gamma+\delta x)^2(\gamma'+\delta' x)}{f'(x)}\sqrt[3]{\left(\frac{v}{u}\right)}=0,$$

which, since

$$(\alpha+\beta x)^3 u=(\gamma+\delta x)^3 v,$$

becomes

$$\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\frac{3(\gamma+\delta x)^2(\alpha'+\beta' x)-3(\gamma+\delta x)(\alpha+\beta x)(\gamma'+\delta' x)}{f'(x)}=0 \quad\text{.................(6)}.$$

Now, let x take successively the values of the five roots of the equation

$$f(x)=0;$$

then, summing with regard to these five quantities, we obtain

$$\Sigma\frac{1}{\sqrt[3]{(uv^2)}}\frac{dx}{dt}+\Sigma\frac{\phi(x)}{f'(x)}=0,$$

where $\phi(x)$ is an expression of the third degree in x; but the expression on the right-hand side, namely

$$\Sigma\frac{\phi(x)}{f'(x)},$$

vanishes, in accordance with a well-known theorem. Hence we have

$$\Sigma\frac{dx}{\sqrt[3]{(uv^2)}}=0 \quad\text{............................(7)},$$

omitting the variable t.

In precisely the same way, by dividing (5) by $\sqrt[3]{(u^2v)}f'(x)$ and summing, we get

$$\Sigma\frac{dx}{\sqrt[3]{(u^2v)}}=0 \quad\text{............................(8)}.$$

5. If we suppose now that two of the roots x_4, x_5 of $f(x)$ are constants, these quantities will be eliminated in the differential equations,

as there result

$$\frac{dx_1}{\sqrt[3]{(u_1v_1^2)}}+\frac{dx_2}{\sqrt[3]{(u_2v_2^2)}}+\frac{dx_3}{\sqrt[3]{(u_3v_3^2)}}=0,$$

$$\frac{dx_1}{\sqrt[3]{(u_1^2v_1)}}+\frac{dx_2}{\sqrt[3]{(u_2^2v_2)}}+\frac{dx_3}{\sqrt[3]{(u_3^2v_3)}}=0,$$

which are the equations that we proposed to show were equivalent to two algebraic conditions connecting x_1, x_3. These latter may be most simply written in the forms (4), namely,

$$\lambda^3 f(a)+\mu^3 f(b)+\nu\sqrt[3]{f}(c)=0,$$

$$\lambda'\sqrt[3]{f}(a)+\mu'\sqrt[3]{(b)}+\nu\sqrt[3]{f}(d)=0.$$

6. The foregoing results are a special application of the general theorem of Abel concerning the comparison of transcendents; but seem worth noticing on account of their simplicity. The algebraical relations are thus consistent with the transcendental equations, in consequence of the integrals belonging to the class called Abelian. It seems, however, worth giving some special attention to the integrals involved.

Let J denote the Jacobian quadratic of u, v; then we have

$$udv-vdu=Jdx;$$

therefore $$\int\frac{dx}{\sqrt[3]{(uv^2)}}=\int\frac{udv-vdu}{J\sqrt[3]{(uv^2)}};$$

but by a known relation, we have

$$J^2=au^2+\gamma uv+\beta v^2,$$

where a, β, γ are functions of the coefficients of u, v, and, in fact, a, β are proportional to the discriminants of v, u, respectively.

Hence

$$\int\frac{dx}{\sqrt[3]{(uv^2)}}=\int\frac{udv-vdu}{\sqrt[3]{(uv^2)}\sqrt{(au^2+\gamma uv+\beta v^2)}},$$

which becomes

$$3\int\frac{dz}{\sqrt{(a+\gamma z^3+\beta z^6)}}\quad\text{.......................... (9)},$$

by putting $$v=uz^3.$$

In the same way

$$\int\frac{dx}{\sqrt[3]{(u^2v)}}=3\int\frac{zdz}{\sqrt{(a+\beta z^6+\gamma z^3)}}\quad\text{....................(10)}.$$

Now, these integrals in z are apparently of the first class of hyper-elliptic integrals, but can be reduced to elliptic integrals, which are, however, not necessarily real, as I proceed to prove.

7. First, suppose that a and β have the same sign, that is, that the discriminants of u, v have the same sign, in which case u, v have their factors both real or both imaginary; then, writing

$$a = \beta k^6,$$

the foregoing integrals become

$$\int \frac{dz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^3\}}}, \quad \int \frac{zdz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^3\}}}.$$

Putting now
$$z+\frac{k^2}{z}=y,$$

we have
$$z^3+\frac{k^6}{z^3}=y^3-3k^2y,$$

and
$$2\sqrt{z}=\sqrt{(y+2k)}\pm\sqrt{(y-2k)},$$

$$\frac{2k}{\sqrt{z}}=\sqrt{(y+2k)}\mp\sqrt{(y-2k)},$$

so that
$$\frac{2\,dz}{\sqrt{z}}=\frac{dy}{\sqrt{(y-2k)}}\pm\frac{dy}{\sqrt{(y+2k)}},$$

$$\frac{2k\,dz}{\sqrt{z^3}}=\frac{dy}{\sqrt{(y+2k)}}\mp\frac{dy}{\sqrt{(y-2k)}}.$$

Hence

$$\int\frac{dz}{\sqrt{\{\beta(z^6+k^6)+\gamma z^3\}}}=\int\frac{dz}{\sqrt{z^3}}\frac{1}{\sqrt{\left\{\beta\left(z^3+\frac{k^6}{z^3}\right)+\gamma\right\}}}$$

$$=\frac{1}{2k}\int\left\{\frac{dy}{\sqrt{(y-2k)}}\mp\frac{dy}{\sqrt{(y+2k)}}\right\}\frac{1}{\sqrt{\{\beta(y^3-3k^2y)+\gamma\}}},$$

and
$$\int\frac{zdz}{\sqrt{\{a(z^6+k^6)+\gamma z^3\}}}=\int\frac{dz}{\sqrt{z}\sqrt{\left\{\beta\left(z^3+\frac{k^6}{z^3}\right)+\gamma\right\}}}$$

$$=\frac{1}{2}\int\left\{\frac{dy}{\sqrt{(y+2k)}}\pm\frac{dy}{\sqrt{(y-2k)}}\right\}\frac{1}{\sqrt{\{\beta(y^3-3k^2y)+\gamma\}}}.$$

We see thus that (9) and (10) are made to depend upon the elliptic integrals

$$\int \frac{dy}{\sqrt{\{(y+2k)(\beta y^3-3k^2\beta y+\gamma)\}}},$$

$$\int \frac{dy}{\sqrt{\{(y-2k)(\beta y^3-3k^2\beta y+\gamma)\}}},$$

which are real if, as we have supposed, α, β have the same signs; but if the latter have not, that is, if u, say, has real factors, and v imaginary ones, then k is imaginary, and the elliptic integrals are also, and cannot be resolved into their real and imaginary parts, except by means of the more general hyper-elliptic integrals from which they were derived.

8. Some integrals which come under the preceding forms may be noticed. If

$$u = x, \quad v = ax^2+bx+c,$$

and we then put $x = z^3$, we have

$$\int \frac{dx}{\sqrt[3]{(u^2v)}} = 3\int \frac{dz}{(az^6+bz^3+c)^{\frac{1}{3}}},$$

$$\int \frac{dx}{\sqrt[3]{(uv^2)}} = 3\int \frac{zdz}{(az^6+bz^3+c)^{\frac{2}{3}}}.$$

Thus, from what we have proved, the integrals on the right-hand side can be expressed by means of elliptic integrals.

9. An application of the preceding results may be made, so as to obtain the differential equations of a certain system of lines in space satisfying two conditions.

Let x, y, z, u be quadriplanar coordinates of a point; then, if we have the system of cubics

$$\frac{x^3}{a-\lambda}+\frac{y^3}{b-\lambda}+\frac{z^3}{c-\lambda}+\frac{u^3}{d-\lambda} = 0,$$

the coordinates of any point in space can be expressed in terms of the parameters λ_1, λ_2, λ_3 of the three cubics of the system which pass through the point, as follows:

$$x^3 = \frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(a-b)(a-c)(a-d)}, \quad y^3 = \frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{(b-a)(b-c)(b-d)},$$

$$z^3 = \frac{(c-\lambda_1)(c-\lambda_2)(c-\lambda_3)}{(c-a)(c-b)(c-d)}, \quad u^3 = \frac{(d-\lambda_1)(d-\lambda_2)(d-\lambda_3)}{(d-a)(d-b)(d-c)};$$

so that, if

$$\frac{d\lambda_1}{\sqrt[3]{(u_1 v_1^2)}} + \frac{d\lambda_2}{\sqrt[3]{(u_2 v_2^2)}} + \frac{d\lambda_3}{\sqrt[3]{(u_3 v_3^2)}} = 0,$$

$$\frac{d\lambda_1}{\sqrt[3]{(u_1^2 v_1)}} + \frac{d\lambda_2}{\sqrt[3]{(u_2^2 v_2)}} + \frac{d\lambda_3}{\sqrt[3]{(u_3^2 v_3)}} = 0,$$

where $u = (a-\lambda)(b-\lambda), \quad v = (c-\lambda)(d-\lambda),$

the integrals (4) will give

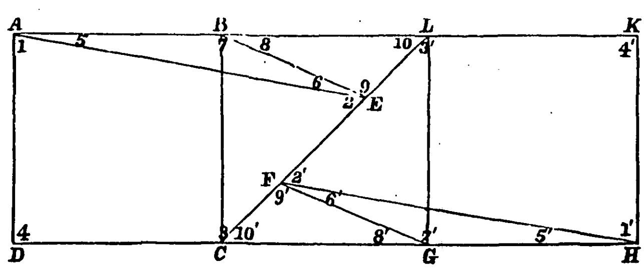
$$lx + my + nz = 0, \quad l'x + m'y + n'u = 0,$$

where l, m, n, l', m', n' are quantities involving the two constants introduced by integration; that is, the differential equations represent a system of lines in space.

Some Theorems in Elementary Geometry. *By* Mr. Oscher Ber.

[*Read Nov. 13th*, 1890.]

I. *To describe a square which is equal to three given equal squares.* Place the squares side by side as in the figure.



Take E on OL so that $OE = OB$,

" F " OL " $LF = OB$.

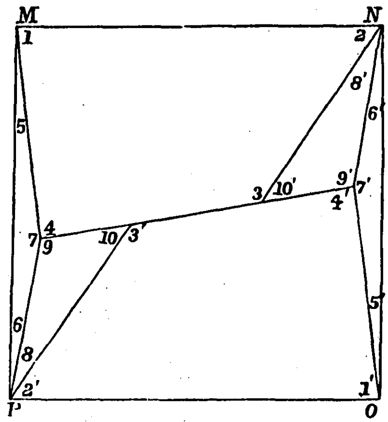
Then cut out the figures

$$AEOD,\ HKLF,\ AEB,\ HFG,\ BEL,\ OFG.$$

The angles have all been numbered.

The angles 1, 1′ are equal; 2, 2′ are equal, and so on.

The parts can now be arranged thus:—



The angles 3, 10′ make up two right angles, and so do 3′, 10.

The angles 4, 7, 9,—*i.e.*, ADC, ABE, BEL,—together equal

$$ADC+ABC+CBE+CBE+ECB.$$

Now $$CB = CE;$$

therefore $$CBE = CEB;$$

therefore 4, 7, 9 together $= ADC+ABC+CBE+CEB+ECB$

$= 4$ right angles.

The side 5 7 = side 1 4.

The side 6 7 = side 8 9.

The angles 6, 8, 2′ together $= BEA+EBL+LFH$

$= BEA+EBL+AEC$

$= BEC+EBL$

$= CBE+EBL$

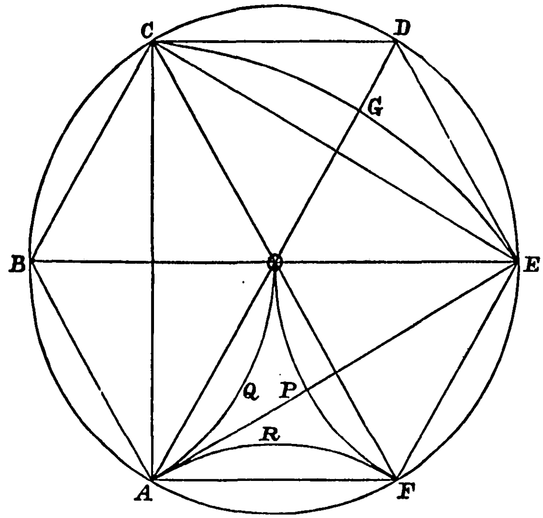
$= CBL =$ a right angle.

The angles 1, 5 together = 1′, 5′ together = a right angle.

The sides 1 2 and 5 6 are equal.

Hence the figure $MNOP$ is a square equal to three given equal squares.

2. *On an area equal to a given semicircle.*



Inscribe a regular hexagon $ABCDEF$ in the circle $ABCDEF$.

With A as centre, and AC as radius, describe a circular arc CE.

Then the figure bounded by the straight lines AC, AE, and the arc CGE is equal to half the whole circle.

For the triangle AEC = half the hexagon.

It remains, therefore, to show that the segment $CGEC$ is half the sum of the six segments of the circle outside the hexagon.

Consider the segments $CGEC$ and the segment of the circle cut off by DE.

The arc of the segment $CGEC$ subtends at the centre A of its circle an angle equal to the angle of an equilateral triangle.

The arc of the segment DE subtends at centre O of its circle the same angle.

Hence the segments are similar.

Hence area of segment $CEGC$: area of segment cut off by DE

$$:: CE^2 : DE^2 :: 3 : 1;$$

therefore area of segment $CEGC$

$$= 3 \text{ (area of segment cut off by } DE)$$

$$= \tfrac{1}{2} \text{ sum of 6 segments outside hexagon;}$$

but triangle $AEC = \frac{1}{2}$ of hexagon;

therefore figure bounded by straight lines AE, AC, and arc CGE

$= \frac{1}{2}$ of whole circle.

3. If with E as centre and EO as radius the arc OPF be drawn,

if with B „ „ BO „ „ OQA „

and if an equal arc be described on AF:

Then these three circular arcs touch where they meet, and form a triangular figure $OQARFPO$.

Now, each of the segments $OQAO$, $ARFA$, $OPFO$ is equal to each of the segments outside the hexagon.

Therefore their sum is equal to the segment $CGEC$.

Also triangle AOF = triangle CED.

Hence the figure $OQARFPO$ (bounded by three arcs and triangular) = figure $CDEGC$ (bounded by two straight lines and one arc).

On the Analytical Representation of Heptagrams.

By L. J. Rogers.

[*Read November 13th,* 1890.]

Contents.

1. In the standard works on the Theory of Substitutions—I allude to Jordan's *Traité des Substitutions*, Netto's *Substitutionentheorie*, and to those sections in Serret's *Algebra* referring to the same—there is an extract made from a paper of M. Hermite's, which appeared in the

Comptes Rendus, Vol. LVII., on the method of analytically expressing a substitution of a prime number of letters by means of a congruence-quantic of order two unities less than the prime in question. Since the subject is dealt with in greatest detail by Serret, it will be best to make all references to his work in preference to the other works mentioned above.

To explain this more fully, let us take p symbols, which we may adequately represent by the numbers $0, 1, 2, 3 \ldots (p-1)$, and suppose that p is prime. Now, suppose these symbols subjected to a substitution so that the arrangement becomes shifted into $\alpha, \beta, \gamma \ldots$, which are the same symbols in a different order. Then it is proved in Serret's *Algebra*, § 474, that an algebraic function ϕx can always be found, such that

$$\phi 0 \equiv \alpha, \text{ mod. } p, \quad \phi 1 \equiv \beta, \quad \phi 2 \equiv \gamma, \text{ \&c.,}$$

for the whole set of symbols, and this function can by Fermat's theorem always be reduced to degree $(p-2)$, at the highest.

For instance, if $p = 7$, and the substitution be that of rearranging 0, 1, 2, 3, 4, 5, 6 into 2, 4, 1, 5, 6, 3, 0, we shall find that

$$\phi x \equiv 3x^5 + 4x^3 + 2x + 2.$$

However, the converse is not always true, that every quantic of order not greater than $(p-2)$ should represent a substitution. For, since we must have the final set of symbols identical with the first, though not in the same order, it is obvious that the only conditions we must and need but have, are that $\phi 0$, $\phi 1$, $\phi 2$, &c. should be congruent non-respectively with 0, 1, 2, 3 ... &c., or, in other words, all different. Thus, for modulus 7, the function x^3+x^2 does not represent a substitution, for its values got by giving x the values 0, 1, 2... are 0, 2, 5, 1, 3, 3, 0, which are not all different.

Now, in general, it is obvious from this consideration, that if ϕx represents a substitution, so also will $\phi x + k$, where k is independent of x. Hence, in testing these forms, it is sufficient to test those in which

$$\phi 0 \equiv 0.$$

The necessary and sufficient conditions have been discovered by M. Hermite, and put into the simple and elegant form, that the $(p-3)$ coefficients of x^{p-1}, obtained in calculating the values of the powers of ϕx, viz.:

$$(\phi x)^2, \; (\phi x)^3 \ldots (\phi x)^{p-2},$$

and reducing by Fermat's theorem all powers greater than the $(p-1)^{th}$, should be severally congruent with zero (see § 476).

As I intend to deal in detail with quantics representing substitutions of seven letters, I shall now leave the general case, and take all congruences according to the modulus 7.

It is also my object to refer especially to the geometrical significance of these quantics, and to show how a substitution of seven letters may be adequately represented by a seven-point polygon or heptagram, understood in its most general form; either as a complete heptagon, or as a triangle and a quadrangle, or as a pentagon and a line, &c. This may be done by arranging seven dots, exactly or approximately at the vertices of a perfectly regular heptagon, and numbering them 0, 1, 2, 3, 4, 5, 6, in order. We will agree, moreover, to place the zero-vertex vertically highest, so to speak, and arrange the others symmetrically about the *vertical* line through the zero-vertex, ascending in value in the direction of the hands of a clock.

If then
$$\phi\alpha \equiv \beta$$
for any particular values α, β, we shall join the vertex α to the vertex β, and when all vertices are thus joined, we shall get a complete seven-point figure. With this convention, it will not always be necessary to number the vertices.

For instance, if
$$\phi x \equiv 3x^5 + 4x^3 + 2x + 2,$$
we shall easily trace the figure from the values of $\phi 0$, $\phi 1$, &c., given above, namely, 2, 4, 1, 5, 6, 3, 0, to be that given in Fig. 1, while the arrow-head marks the direction in which we draw the lines. This is obviously necessary for each sub-polygon, except in the case of a single line or dot. Here we need only place an arrow-head on one side of the pentagon.

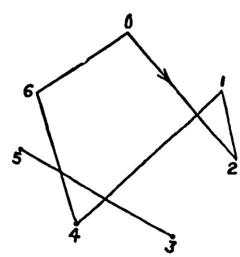


Fig. 1.

M. Hermite has found that all substitutions of 7 letters may be represented by the functions
$$\alpha z + \beta, \text{ or } \alpha\theta(z+\beta)+\gamma,$$
where
$$\theta z \equiv z^4 \pm 3z \quad \text{.............................(1)},$$
or
$$z^5 \pm 2z^2 \quad \text{.............................(2)},$$

or $$z^5+az^3+3a^2z, \text{ where } a^7-a \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or $$z^5+az^3 \pm z^2+3a^2z, \text{ where } a^3 \equiv -1 \quad \ldots\ldots\ldots\ldots\ldots (4);$$

giving in all 5040 forms, as is shown in Serret's *Algebra.* These simplified forms he refers to as *formes réduites*, or reducts.

2. I propose, however, to rearrange these in seven classes, on the principle that, if θz be any form, then every reduct of the same form may be congruent with $\lambda\theta\mu z$, where λ, μ are constant. This will necessitate dividing $z^5+az^3+3a^2z$ into three new classes, according as

$$a \equiv 0, \quad a^3 \equiv 1, \text{ or } a^3 \equiv -1.$$

If we give θz the following values—

$$z^4+3m^3z, \quad z^5+2m^3z^2, \quad z^5, \quad z^5+m^2z^3+3m^4z, \quad z^5-m^2z^3+3m^4z,$$
$$z^5+3m^2z^3+m^3z^2+6m^4z,$$

we shall obtain, by generalizing, the same 5040 forms as obtained by M. Hermite.

Now, these reducts may be replaced by others, equally general, and of such forms that M. Hermite's conditions may be immediately verified.

Thus, it can be easily shown that

$$z^5+m^2z^3+3m^4z \equiv 5m^4z\,(z^2+m^2)^3,$$

and that $$z^5-m^2z^3+3m^4z \equiv 4z^5\,(z^4+\lambda^4)^3,$$

where $$\lambda^2 \equiv 4m^2.$$

Similarly, $$z^4+3m^3z \equiv m^3z\,(z^3+4m^3)^2,$$

and $$z^5+2m^3z^2 \equiv 5z^5\,(z^3-4m^3)^2.$$

M. Hermite's last reduct can be brought to a simpler form. For

$$z^5+3m^2z^3+m^3z^2+6m^4z \equiv 6m^4\left\{(z+2m)^5+4m^5\right\}^5+m^5,$$

which, without loss of generalization, can be replaced by the reduct

$$(z^5+m^5)^5.$$

I shall therefore substitute the following seven reducts in place of those given by M. Hermite:—

$$x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$x^5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$x(x^2+m^2)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$m^2x^5(x^4+m^4)^3 \quad \ldots\ldots\ldots\ldots\ldots \quad (4),$$

$$x(x^3+4m^3)^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$x^5(x^3+4m^3)^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$(x^5+m^5)^5 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

If these be generalized in the manner indicated by M. Hermite, we shall obtain the same 5040 substitution-functions as can be derived from his reducts.

It is interesting to notice that all the reducts except (7) can be written in the form

$$x^r (fx^s)^{(p-1)/s},$$

where r is prime to and less than $p-1$ (*i.e.*, 1 or 5), and fx^s is a rational integral function of x^s which can never become zero, and is not a perfect power of any function of x^s, and where s is a factor of $p-1$.

It is, moreover, easy to see that such an expression always satisfies the required conditions for a substitution-quantic. For, if the reduct be raised to a power other than the s^{th}, we shall have a set of terms whose indices are of the form $ns+r$; and, since r is prime to s and $p-1$, this can never be equal to a multiple of $p-1$. Again, if it be raised to the s^{th} power, we get, by Fermat's theorem, x^{rs}, which cannot $\equiv x^{p-1}$, since r is prime to $p-1$.

The seventh reduct is obviously a substitution-function, for it is the result of the operation of θ on $\theta z + m^5$, where

$$\theta z \equiv z^5.$$

3. *Isomorphism and Vertex-shifting.*

It is proved in all works on Substitutions that ϕx and any function of the form $f^{-1}\phi fx$ represent similar or isomorphous substitutions, if fx also represents a substitution. For instance, if ϕx represent Fig. 1, § 1, then $f^{-1}\phi fx$ will represent some heptagram consisting of a pentagon and a line, and moreover $f(x)$ can be chosen to make it represent any such heptagram.

The geometrical meaning of $f^{-1}\phi fx$ can be very simply demonstrated.

Let ϕx be represented by Fig. 1. Then the congruence

$$y \equiv \phi x,$$

shows to what point any vertex x must be drawn.

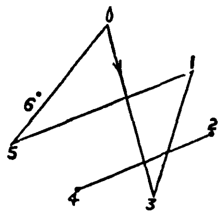


Fig. 1.

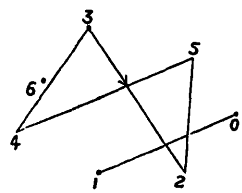


Fig. 2.

Now, suppose $f0$, $f1$, $f2$ ... have the values 3, 5, 0, 2, 1, 4, 6, and let us re-number the vertices according to this substitution, so that any vertex formerly marked a is now called fa. Now, draw the polygram of Fig. 1 exactly as before, as in Fig. 2. The method of tracing it is now given by the congruence

$$fy \equiv \phi fx,$$

for the same relation now holds between fy and fx as before held between y and x.

Now rearrange the vertices in their proper order, and we get Fig. 3, which is represented by

$$fy \equiv \phi fx,$$

i.e., $$y \equiv f^{-1}\phi fx.$$

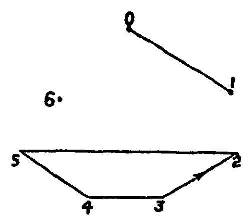


Fig. 3.

The geometric method of deriving $f^{-1}\phi fx$ from ϕx shows that the two figures are isomorphous or like-membered.

With this section, *cf.* Serret, § 413.

If $$fx \equiv x+n,$$

we have to shift each vertex back n places without moving the figure, which, on restoring 0 to its proper place, means that we shift on the figure n places.

Thus $\phi(x+n)-n$ denotes the same figure as ϕx, rotated back n places.

4. *Reciprocal Polygrams.*

The most important investigation in the theory of polygrams is that of finding the reciprocal of any quantic, by which is meant that quantic whose figure is the same as that of the given quantic, with the arrow-head turned in the opposite direction. Or, analytically speaking, if

$$y \equiv \phi x,$$

where ϕ is known in form, the problem is to find the form of $\phi^{-1} x$.

Let $$y \equiv Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex,$$

so that $$\phi 0 \equiv 0,$$

and suppose $$x \equiv \alpha y^5 + \beta y^4 + \gamma y^3 + \delta y^2 + \eta y.$$

Then $$xy \equiv A + Bx^5 + \ldots$$

$$\equiv \alpha + \beta y^5 + \ldots.$$

Now, by Hermite's conditions for a valid quantic, we know that, if $x \equiv 0$, then y, y^2, y^3, y^4, y^5 all $\equiv 0$, so that we get

$$\alpha \equiv A \quad \text{.................................(1).}$$

Again, $$xy^2 \equiv \alpha y + \beta + \gamma y^5 + \ldots$$

$$\equiv x\,(Ax^5 + Bx^4 + \ldots)^2.$$

Putting $x \equiv 0$ after reducing the last expression by Fermat, we see that

$$\left.\begin{aligned} \beta &\equiv \text{coefficient of } x^5 \text{ in } (\phi x)^2; \\ \text{similarly,}\quad \gamma &\equiv \quad\text{,,}\quad\text{,,}\quad\text{,,}\quad (\phi x)^3 \\ \delta &\equiv \quad\text{,,}\quad\text{,,}\quad\text{,,}\quad (\phi x)^4 \\ \eta &\equiv \quad\text{,,}\quad\text{,,}\quad\text{,,}\quad (\phi x)^5 \end{aligned}\right\} \quad \text{....................(2).}$$

Thus, if $$\phi 0 \equiv 0,$$

the reciprocal of ϕx may be directly calculated, though in the general case the process is laborious.

If $$y \equiv \phi x + b,$$

it is obvious that $$x \equiv \phi^{-1}(y-b) \quad \text{..........................(3)},$$

so that the reciprocal of every substitution-quantic may be determined. This, by rotation through b points, can be brought into the form $\phi^{-1}y - b$.

I shall now proceed to the calculation of the reciprocals of the several standard forms given in § 2, but it will be advisable for the sake of greater generality to multiply each of these reducts by a constant, say a.

In the first place, the forms (1), (2), (7) present no difficulty,

since, if $$y \equiv ax,$$

then $$x \equiv a^5 y;$$

if $$y \equiv ax^5,$$

then $$x \equiv ay^5;$$

and if $$y \equiv a(x^5+m^5)^5,$$

then $$y^5 \equiv a^5(x^5+m^5),$$

so that $$x \equiv a^5(y^5 - a^5m^5)^5.$$

The reciprocal forms corresponding to the other reducts have, however, to be obtained according to the method indicated above.

Thus, if $$y \equiv ax(x^2+m^2)^3,$$

then $$y^2 \equiv a^2x^2,$$

so that $$\beta \equiv \delta \equiv 0;$$

$$y^3 \equiv a^3x^3(x^2+m^2)^3,$$

so that $$\gamma \equiv a^3 . 3m^2;$$

$$y^5 \equiv a^5x^5(x^2+m^2)^3,$$

so that $$\eta \equiv a^5(1+m^6);$$

while the coefficient of x^6 in xy is $a . 3m^4$.

These results give $$x \equiv a^5y(y^2+a^2m^2)^3,$$

as may be easily verified.

Similarly, if $$y \equiv am^2x^5(x^4+m^4)^3,$$
then $$x \equiv am^2y^5(y^4+a^4m^4)^3.$$

In form (5), we have $y \equiv ax(x^3+4m^3)^2$,

so that $$y^2 \equiv a^2x^2(3+x^3m^3)^2 \equiv a^2x^2(x^3-4m^3)^2,$$
and $$y^3 \equiv a^3x^3;$$
and finally, applying the method explained above,

$$x \equiv a^5y(y^3-4a^3m^3)^2.$$

Similarly, from form (6), where

$$y \equiv ax^5(x^3+4m^3)^2,$$
we get $$x \equiv ay^5(y^3+4a^3m^3)^2,$$
as may be obtained from the last by changing x into x^5.

Collecting the above results, we get the following seven pairs of reciprocal equations:—

$y \equiv ax$,	$x \equiv a^5y$,	$\mu^2 \equiv a^4$,
$y \equiv ax^5$,	$x \equiv ay^5$,	$\mu^6 \equiv 1$,
$y \equiv ax.(x^2+m^2)^3$,	$x \equiv a^5y(y^2+a^2m^2)^3$,	$\mu^2 \equiv a^4$,
$y \equiv am^2x^5(x^4+m^4)^3$,	$x \equiv am^2y^5(y^4+a^4m^4)^3$,	$\mu^2 \equiv a^2$,
$y \equiv ax(x^3+4m^3)^2$,	$x \equiv a^5y(y^3-4a^3m^3)^2$,	$\mu \equiv -a^5$,
$y \equiv ax^5(x^3+4m^3)^2$,	$x \equiv ay^5(y^3+4a^3m^3)^2$,	$\mu^3 \equiv a^3$,
$y \equiv a(x^5+m^5)^2$,	$x \equiv a^5(y^5-a^5m^5)^5$,	$\mu \equiv -a^5$.

One remarkable fact is to be noticed concerning these reciprocal quantics. If

$$y \equiv \phi x,$$

where ϕx is a reduct, then in every case $\phi^{-1}x$ can be reduced to the form $\mu\phi\mu x$, where μ is a constant. In other words, if

$$y \equiv \phi x,$$
then $$x \equiv \mu\phi\mu y;$$
that is, if $$z \equiv \phi\mu y,$$

then $$x \equiv \mu z,$$

so that $$y \equiv \phi\mu z;$$

which shows that $\phi\mu x$ is a self-reciprocal quantic, a species which will be treated of further on. The congruences giving the values of μ for each form are to be found in the third column above, each placed on the same line as the form to which it refers.

More generally, if $y \equiv \psi x \equiv \phi(x-a)+b$,

where ϕx is a reduct, then evidently

$$\psi^{-1}x \equiv \phi^{-1}(x-b)+a.$$

Now $$\phi^{-1}x \equiv \mu\phi\mu x;$$

therefore $$\phi^{-1}(x-b) \equiv \mu\phi\mu(x-b).$$

But $$\phi x+b \equiv \psi(x+a), \text{ by hypothesis};$$

therefore $$\phi(\mu x-\mu b)+b \equiv \psi(\mu x-\mu b+a).$$

Hence $$\psi^{-1}x \equiv \mu\psi(\mu x-\mu b+a)-\mu b+a,$$

a congruence giving a general connexion between a quantic and its reciprocal.

If $$\mu b-a \equiv \mu c,$$

then $$\psi^{-1}x \equiv \mu\psi\mu(x-c)-\mu c.$$

Let $$\psi\mu(x-c) \equiv z,$$

so that the above congruence gives us

$$\psi^{-1}x \equiv \mu(z-c).$$

Then $$x \equiv \psi\mu(z-c),$$

so that x, z are connected by a self-reciprocal relation.

Hence, if ψx represent any heptagram, two constants μ, c can always be found, such that $\psi\mu(x-c)$ is a self-reciprocal quantic.

5. *Skew-symmetry and Self-reflexion.*

A polygon is called skew-symmetric when its geometric form is its own reflexion in some line, but the direction-arrows are reversed. If the number of vertices is odd, the axis of symmetry must pass through one of the vertices. The polygon is then said to be symmetric about that vertex.

A polygon is called self-reflective when both its geometric form and the direction-arrows are reflected in some line.

The condition for self-reflexion about the zero-vertex is obviously that

$$\phi x \equiv -\phi(-x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

that is, that ϕx should only contain odd powers of x, as is the case in the first four reducts found in § 2.

The test for skew-symmetry depends upon the form of the reciprocal quantic, the general law being that the reciprocal figure should be the exact reflexion of the original figure in some line through a vertex. If the axis pass through the zero-vertex, we get

$$\phi^{-1}x \equiv -\phi(-x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The reciprocating constant, therefore (§ 4), is congruent with -1. We can, moreover, deduce the very important fact that, if the reduct ϕx gives a skew-symmetric figure, so also will $\phi x+b$. For the reciprocal of the latter is $\phi^{-1}(x-b)$, which, rotated back through b vertices, becomes $\phi^{-1}x-b$. But this is $-\phi(-x)-b$, which is the reflexion of $\phi x+b$, the original figure.

Now the vertex of symmetry for the reduct ϕx is zero, while, for $\phi x+b$, we must shift back the reciprocal b places before we get the reflexion of the original. A little consideration will show that the vertex of symmetry is that whose number is congruent with $\frac{1}{2}b$, *i.e.*, $4b$.

The skew-symmetric reducts are the following :—

$$\pm x, \quad ax^5, \quad \pm x(x^2+m^2)^3, \quad \pm m^2x^5(x^4+m^4)^3,$$

$$+x(x^3+4m^3)^2, \quad ax^5(x^3+4m^3)^2 \quad \text{where} \quad a^3 \equiv -1,$$

and

$$(x^5+m^5)^5.$$

Except for $-x$, we get seven different figures from each of these forms by adding a constant b, so that the totality of skew-symmetric heptagrams* is

$$1+7+42+42+42+14+42+42 = 232.$$

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* It is interesting to note that all pentagrams can be represented by the two congruences

$$y \equiv ax+b, \quad y \equiv ax^3+b, \text{ mod. } 5$$

6. *Self-reciprocal or Line-and-Dot Polygrams.*

If a polygon is its own reciprocal, it is evidently made up of lines and dots.

Now, if
$$\phi^{-1}x \equiv \phi x,$$
the reciprocating constant μ is $\equiv 1$, and the self-reciprocal forms are as follows:—

$$\pm x, \quad ax^5, \quad \pm x(x^2+m^2)^2, \quad \pm m^2x(x^4+m^4)^3, \quad -x(x^3+4m^3)^2,$$
$$ax^5(x^3+4m^3)^2 \text{ where } a^3 \equiv 1, \text{ and } -(x^5+m^5)^5.$$

Their totality is
$$2+6+6+6+2+6+6 = 34.$$

It may be noticed, as is geometrically obvious, that, if ϕx be skew-symmetrical about the zero-axis, then $-\phi x$ is self-reciprocal. Moreover, if ϕx be both self-reflective and skew-symmetric, it must be also self-reciprocal.

7. *Metrical Properties, Sets.*

The seven figures corresponding to the quantic $\phi x+b$, where ϕx is a reduct, and b has the seven values $0, 1, 2 \ldots$, I shall call a *set* of figures, or the figures belonging to the same set. It will be found in general that figures of the same set apparently differ very much in their geometric properties. There are, however, properties possessed by each member of a set, which it is my object now to point out, and which will give us a method of detecting, by mere inspection of the geometric figure, the reduct to which the figure belongs. It is, of course, always possible to find the quantic of the given figure, and reduce it by so rotating that its second term vanishes; but this method is tedious, and furnishes no clue as to the connexion between geometric and analytic similarity.

(see SERRET's *Algebra*, § 486). Moreover, if a be not $\equiv 1$,
$$y \equiv ax+b$$
may be rotated into $y \equiv ax$,
which is self-reflective; also $y \equiv x+b$
is regular, while the reciprocal of
$$y \equiv ax^3+b \quad \text{is} \quad x \equiv a(y-b)^3,$$
which can be rotated into $x \equiv ay^3-b$,
which shows that all figures belonging to this congruence are skew-symmetrical. Combining these results, we see that—*All pentagrams formed by joining the angular points of a regular pentagon are symmetrical.*

The most important property possessed by all members (or none) of a set is skew-symmetry, as we saw in § 5.

However, since each kind of reduct was found to include skew-symmetric forms, this property will not serve to distinguish different kinds, though it will differentiate sub-species, such as are marked by the coefficient of the leading term being a residue or a non-residue.

We may, in fact, subdivide the species already found into twenty-four in all, and we may geometrically distinguish these (1) by observing the number of equal sides, and (2) by the number of parallel sides in any figure. Now, the analytical condition for equality of two or more sides is that $\phi x - x$ should have the same value for two or more values of x, while the condition for parallelism is that $\phi x + x$ should have the same value for two or more values of x. The first fact is easy to see, and the second is obvious when we consider that $\frac{1}{2}(\phi x + x)$ is the number of the vertex halfway between x and ϕx; for these two sides have the same midway vertex, that is, are parallel.

We have to notice one or two special cases :—

(1) Isolated dots must be looked upon as sides of equal length.

(2) An isolated line must be treated as a pair of parallel lines.

(3) An isolated dot midway between the extremities of a side must be considered as a side parallel to that side.

It is very easy to see that every member of a set has the same equality of sides and parallelism of sides. For, if $\phi x_1 \pm x_1$ and $\phi x_2 \pm x_2$ have equal values, so also have $\phi x_1 + c \pm x_1$ and $\phi x_2 + c \pm x_2$. This may be expressed by saying that the isoscelism or parallelism of a set is $q, r, s \ldots$ if we wish to state that the figures have each a group of q equal or parallel sides, and another group of r equal or parallel sides, not equal or parallel to the last, &c.

It may be worth noticing that there is a kind of reciprocal relation connecting equality of sides and parallelism in the figures represented by ϕx and $-\phi x$, or, as we may say, in any figure and its *negative*. For groups of parallel sides depend upon the groups of congruent values in $\phi x + x$, that is, upon groups of congruent values in $-\phi x - x$, which shows that they are equivalent to the groups of equal sides in $-\phi x$. For instance, $x^5 + 2x^3 + 5x + b$ has five equal sides for all values of b, and, consequently, $6(x^5 + 2x^3 + 5x + b)$ has five sides parallel.

8. *Descriptive Properties of sets of Polygrams.*

Besides symmetry, we have hitherto only considered the metrical properties possessed by every member of a set, *i.e.*, those properties, such as isoscelism and parallelism, which depend on geometric lengths and directions. We have now to consider a property which refers simply to the number of lesser polygons into which a polygon may split up, and which we may call a descriptive property.

The fundamental theorem is as follows:—

If ϕx consist of m members, and fx consist of n members, then $f\phi x$ or ϕfx will consist of r members, where

$$r \equiv m+n-1, \text{ mod. } 2.$$

As this is a theorem known to those who have studied the properties of substitutions, I need only refer to Mr. Asquith's paper in the *Quarterly Journal* for October, 1889, p. 114, in which the proposition stands in the form: A cycle is always added to or subtracted from substitution by a transformation. Now, every substitution can be made up of a certain number of transformations, which are represented by figures consisting of one line and $p-2$ dots, where p is the number of sides (prime) of the polygram. It therefore consists of an even number of sub-figures or members, so that we shall call it an even polygram.

Now, it is easy to infer that an odd polygram will be made up of an even number of transpositions, and *vice versâ*.

Hence the theorem, as re-worded above, follows easily.

For instance, if

$$fx \equiv x+\lambda,$$

then fx is a complete or one-membered polygram, so that ϕx and $f\phi x$—*i.e.*, $\phi x+\lambda$—are both even or both odd, or, as we may say, of *like character*. Hence all members of a set are of like character.

This may be readily verified, *e.g.*, in the case of, say, $3x^5$, which has four members, and is therefore even. It will be found that $3x^5+1$ has two, $3x^5+2$ has four, $3x^5+3$ has six, &c.

We may further notice that ϕx and $-\phi x$ have unlike characters, if

$$p = 7,$$

since, in this case, $$f(x) \equiv -x,$$

which consists of a dot and three lines, and is therefore even.

It is in many cases easy from analytical considerations to discover the form and character of the reducts given in § 2.

Let us first draw heptagrams corresponding to $x+\lambda$, $-x+\lambda$, $ax+\lambda$ (where a is a primitive root), and $-ax+\lambda$.

It will be found that $x+\lambda$ gives a complete heptagon; $-x+\lambda$ gives three lines and a dot; $ax+\lambda$ gives a hexagon and a dot; $-ax+\lambda$ gives two triangles and a dot.

Now, let us consider the seventh form

$$a\,(x^5+m^5)^5,$$

or, as we may write it,

$$\{a^5\,(x^5+m^5)\}^5.$$

If

$$fx \equiv x^5,$$

we may write this

$$f^{-1}\,(a^5fx+a^5m^5),$$

which, by § 3, is isomorphous with

$$a^5\,(x+m^5).$$

It is easy to see then that

$$a\,(x^5+m^5)^5$$

will represent a complete heptagon, three lines and a dot, a dot and a hexagon, or two triangles and a dot, according as $a\equiv 1$, $\equiv -1$, $\equiv$ a primitive root, or $\equiv 2$ or 4.

Again, let $$\phi x \equiv ax\,(x^2+m^2)^3;$$

then $$\phi^2 x \equiv a^2x\,(x^2+m^2)^3\,(a^2x^2+m^2)^3 \equiv x, \text{ if } a^2\equiv 1,$$

so that $$\pm x\,(x^2+m^2)^3,$$

gives self-reciprocal heptagrams. Moreover

$$\phi^3 x \equiv a^3x\,(x^2+m^2)^3\,(a^2x^2+m^2)^3\,(a^4x^2+m^2)^3$$
$$\equiv -x, \text{ if } a^3\equiv -1.$$

Now, ϕ^3x consists of three lines and a dot, so that it is easy to see that ϕx must either consist of a dot and a hexagon, or must be self-reciprocal. The latter cannot be true unless $a^2\equiv 1$; therefore, if a be a primitive root, ϕx must consist of a dot and a hexagon, and is therefore even in character.

In similar ways we may establish the fact that if $a^3\equiv -1$ in any reduct, then the character of the corresponding heptagram is even; but it is scarcely necessary to prove the statement in the case of every reduct. It is interesting to notice that if

$$\phi x \equiv am^2x^5\,(x^4+m^4)^3,$$

then ϕ^4x in all cases $\equiv x$, so that ϕx must, if not self-reciprocal, consist partly of a quadrilateral figure.

The following Table represents twenty-four subdivisions in the different species of heptagrams, arranged according to their analytical forms, and with the geometric properties appended which are common to every member of the corresponding sets. The descriptive properties of symmetry and character have been, or can be, found directly by analysis. The metrical properties of isoscelism and parallelism have been found by inspection.

It has been found convenient to use the symbol $\sqrt[3]{1}$ for a quadratic residue, and $-\sqrt[3]{1}$ for a non-residue; a for a primitive root.

		Metrical Properties.		Descriptive Properties.	
		Isoscelism.	Parallelism.	Character.	Symmetry.
Form 1	x	7	0	odd	symmetrical
	$-x$	0	7	even	sym.
	$-ax$	0	0	odd	(self-reflective)
	ax	0	0	even	(self-reflective)
Form 2	$\sqrt[3]{1} \,.\, x^5$	3 . 2 . 2	2 . 2	odd	sym.
	$-\sqrt[3]{1} \,.\, x^5$	2 . 2	3 . 2 . 2	even	sym.
Form 3	$x\,(x^2+m^2)^3$	3	5	odd	sym.
	$-x\,(x^2+m^2)^3$	5	3	even	sym.
	$-ax\,(x^2+m^2)^3$	2 . 2	2 . 2	odd	unsym.
	$ax\,(x^2+m^2)^3$	2 . 2	2 . 2	even	unsym.
Form 4	$m^2x^5\,(x^4+m^4)^3$	. 2 . 2	3 . 3	odd	sym.
	$-m^2x^5\,(x^4+m^4)^3$	3 . 3	3 . 2 . 2	even	sym.
	$-am^2x^5\,(x^4+m^4)^3$	2 . 2	3	odd	unsym.
	$am^2x^5\,(x^4+m^4)^3$	3	2 . 2	even	unsym.
Form 5	$x\,(x^3+4m^3)^2$	2 . 2 . 2	0	odd	sym.
	$-x\,(x^3+4m^3)^2$	0	2 . 2 . 2	even	unsym.
	$-ax\,(x^3+4m^3)^2$	4	2 . 2 . 2	odd	unsym.
	$ax\,(x^3+4m^3)^2$	2 . 2 . 2	4	even	unsym.
Form 6	$\sqrt[3]{1} \,.\, x^5\,(x^3+4m^3)^2$	3	3 . 2	odd	unsym.
	$-\sqrt[3]{1} \,.\, x^5\,(x^3+4m^3)^2$	3 . 2	3	even	sym.
Form 7	$(x^5+m^5)^5$	4 . 2	2 . 2	odd	sym.
	$-(x^5+m^5)^5$	2 . 2	4 . 2	even	unsym.
	$-a\,(x^5+m^5)^5$	2 . 2	3 . 2	odd	unsym.
	$a\,(x^5+m^5)^5$	3 . 2	2 . 2	even	unsym.

December 11th, 1890.

Prof. GREENHILL, F.R.S., President, in the Chair.

The following gentlemen were elected members:—F. S. Carey, M.A., late Fellow of Trinity College, Cambridge, Professor of Mathematics, University College, Liverpool; M. W. J. Fry, M.A., Fellow of Trinity College, Dublin; H. S. Romer, M.A., late Scholar of Trinity Hall, Cambridge; and Hari Dâs Sastri, M.A., Director of Public Instruction, Jaypur State, Rajputana.

The Auditor made his Report. Upon the motion of Sir J. Cockle, seconded by Mr. S. Roberts, the Treasurer's Report was then adopted. A vote of thanks was unanimously accorded to Mr. Heppel for the trouble he had taken in auditing the accounts.

The following communications were made:—

On the Stability of a Plane Plate under Thrusts in its own Plane, with applications to the "Buckling" of the Sides of a Ship: Mr. G. H. Bryan (communicated by Mr. Love).

On the Extension to Matrices of any Order of the Quaternion Symbols S and V: Dr. Taber.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables: Mr. E. B. Elliott.

Newton's Classification of Cubic Curves: Mr. W. W. R. Ball.

Steiner's Poristic Systems of Spheres: Prof. G. B. Mathews.

On the q-series derived from the Elliptic and Zeta Functions of $\frac{1}{3}k$ and $\frac{1}{4}k$: Dr. Glaisher.

The following presents were received:—

"Educational Times," for December.

"Proceedings of the Physical Society of London," Vol. x., Part iv.; Nov., 1890.

"Proceedings of the Cambridge Philosophical Society," Vol. vii., Part ii.

"Nautical Almanack," for 1894.

"Bulletin des Sciences Mathématiques," Tome xiv., Nov., 1890.

"Nieuw Archief voor Wiskunde," Deel xvii., Stuk 1 and 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vi., Fasc. 4, 6, and 7; Roma, 1890.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 116, 117, and 118.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xx.-xl.

"Acta Mathematica," xiii., 1 and 2.

"Memorias de la Sociedad Científica—Antonio Alzate," Tomo III., Nos. 11 and 12.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 6.

Pamphlets by A. Voss:—"Ueber die mit einer bilinearen Form vertauschbaren bilinearen Formen," 8vo; "Ueber die conjugirte Transformation einer bilinearen Form in sich selbst," 8vo; "Ueber einen Satz aus der Theorie der Determinanten," 8vo; "Ueber die cogredienten Transformationen einer bilinearen Form in sich selbst," 4to.

On the Stability of a Plane Plate under Thrusts in its own Plane, with Applications to the "Buckling" of the Sides of a Ship.

By G. H. BRYAN.

[*Read Dec. 11th*, 1890.]

Introduction.

1. The problems discussed in this paper are the analogues for a plane rectangular or circular plate of the well-known investigations of the stability of a thin wire or shaft, due in the first place to Euler, and since developed by Greenhill. I have employed the energy criterion of stability, the use of which I have already illustrated in this connexion in two papers published in the *Proceedings of the Cambridge Philosophical Society.**

The case of a plate supported on equidistant parallel ribs will be considered more fully, on account of the practical use of such structures in the construction of ships.

Suppose a plane elastic plate is submitted to edge tractions in its own plane which produce compression of its middle surface, and let every point of that surface receive a displacement normal to the plane, such displacements being chosen in accordance with the prescribed boundary conditions. If this displacement be everywhere of the first order of small quantities, the surface of the plate will thereby become extended by small quantities of the second order,

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* *Camb. Phil. Proc.*, Vol. VI., pp. 199, 286.