THE DEVELOPMENT OF ALGEBRAIC SYMBOLISM FROM PACIUOLO TO NEWTON.

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Introductory Note: In considering the improvement of our present algebraic symbolism, the possible elimination of certain signs that may have outlived their usefulness, the conservation of the best that have come down to us, the variation observed in passing from the literature of one country to that of another, and the possible unifying of the best of our present symbols, it has been found helpful to consider the early history of the subject. In particular, a study of the formative period from about 1500 to 1650 has been found of greatest value and it is to this period that Miss Benedict has given some attention. Since the subject is both valuable and interesting, and since her information has been secured from original sources available to but few teachers, I have asked Miss Benedict to present in condensed form the historical facts which she has secured. This she has undertaken to do, omitting much information commonly found in the histories of mathematics, and all discussion of the significance of the investigation. The result is a contribution to the history of mathematical education of such interest and value that this department is glad to publish it for the benefit of the profession. David Eugene Smith.

The growth of the symbolism of elementary algebra from its first appearance in print at the end of the fifteenth century to the time of Newton when it was practically complete, roughly divides itself into two periods, each about a century in length. There were in the sixteenth century two general methods of writing algebraic symbols, the one in use in Italy and Spain, and the other in the northern countries. Of these two systems the Italian is the older and therefore should have first attention.

The earliest printed work on algebra is contained in a volume published in Venice in 1494. It was written by a Franciscan monk who, after a youth of travel and adventure, retired to a monastery and there made for himself an honorable name through his works. This man was Luca Paciuolo, commonly known from his birthplace as Fra Luca di Borgo San Sepulcro, and his great work was entitled, *Summa de Arithmetica, Geometria, Proportione et Proportionalita.* The book is merely a compilation made by a man with the interests of a teacher; and what has been said of his methods, that he gave little that was new, is also true of his symbols, many of which are found in the fifteenth century manuscripts.[1] But just as his summary of methods is valuable, so his table of *Caratteri Alghibratici,* the earliest in which they are systematically explained, is of great historic interest. The first character used is a radical sign, simply an abbreviation of the word *radix* or *radice,* signifying root. The symbol is ℞ and it is usually followed by abbreviations of the words *prima, secunda,* etc.; the first two and the last of the thirty roots being written as follows: ℞.*p*^{a}., ℞.*2*^{a}., ℞.30^a. Besides using these forms, the second, third, and fourth roots are frequently written ℞., ℞.*cu.,* and ℞.℞. and there is still another symbol, ℞.*v.,* which Paciuolo calls *Radice Universale* and uses to indicate the root of a polynomial.

Among Italian writers of this period it was very common to indicate the omission of one or more letters by a line drawn above. So we find in the *Summa* the words for *plus* and *minus,* in Italian *piú* and *meno.,* indicated by $\tilde{p}$ and and $\tilde{m}$. In common with many of the early algebraists Paciuolo calls the absolute number *numero* and symbolizes it .*n*°. The unknown quantity was *thing,* in the Italian *cosa,* and from this word were derived the names *Coss* and *Cossic Art,* which at the opening of the Renaissance were synonymous with algebra. The name for the square was *censo.* Cossali, in his *Origine dell' Algebra,*[2] says that the word is probably of Latin derivation and was adopted by Leonardo of Pisa, but that its original meaning is unknown. *Cubo,* for the cube, needs no explanation, and for *relato,* a term signifying a power whose exponent is not a multiple of two or three, there seems to be none. The symbols used are abbreviations of the names: .*co.* for the first power, .*ce.* for the second. .*cu.* for the third, and .$\overset{\circ}{p}$.$\overset{\circ}{r}$., $\overset{\circ}{2}$.$\overset{\circ}{r}$. etc., for the *relati.* All other powers are built up by a multiplicative plan through a combination of .*ce.* and .*cu.* with each other and with the *relati,* but it must be noticed that two *relati* are never written together; for example, the twenty-fifth power is not .$\overset{\circ}{p}$.$\overset{\circ}{r}$.$\overset{\circ}{p}$.$\overset{\circ}{r}$. as we might expect, but .8.$\overset{\circ}{r}$.[3]

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[1] Several of these manuscripts have been examined in the preparation of this report, but no reference is made to them for the reason that they antedate Paciuolo.

[2] Vol. I, p. 12.

[3] For Paciuolo's symbolism, see *Summa,* f. 67, v.

Another great name, in connection with the Italian symbolism, is that of Tartaglia, to whom we owe the first solution of the cubic equation. In his *General Trattato* (Venice, 1556) he uses most of Paciuolo's symbols, but, since the first power and the first root are the same, he discards *.co.* and uses ℞. *prima* in its place. In this work, then, we find the powers of the unknown written as follows: ℞. *prima, ce, cu, ce.ce, rel, ce.cu,* $\frac{0}{2}$ *rel., etc.*[4] The roots are indicated by ℞. followed by *ce* or *cu*, as the case may be, and the universal radical appears as ℞.V. For *plus* Tartaglia writes the word *piu,* or more frequently the letter p. the upper part of which is nearly surrounded by a curved line, and his *minus* may be *men, me,* or *m̃.* But the most interesting bit of symbolism in the *Trattato* is the trace of the parenthesis in the expression *22 men* (*22 men* ℞ 6.) which is given as the product of

$$\sqrt{22} + \sqrt{22 - \sqrt{6}} \text{ and } \sqrt{22} - \sqrt{22 - \sqrt{6}}.^{5}$$

The arithmetic of Antich Rocha (Barcelona, 1565), in which we find one of the earliest chapters on algebra written in the Spanish language, also uses the Italian symbols, although they are modified even more than in the *Trattato.* The radical sign does not appear, but the symbols for the powers of the unknown quantities, including the zero power or the absolute number, are as follows: *N, Co, Ce, Cu, Cce, R. CeCu, RR, Ccce, Ccu, Rce,* etc.[6] For the plus sign Rocha uses the word *Mas* or its abbreviation *ma,* and for the minus sign *me,* an abbreviation of *ménos.*

These three writers are by no means the only ones to use the Italian symbols. Many others, among them Cardan,[7] use all or part of them, but the books which have been mentioned may well be taken as types of the others. This symbolism was, however, restricted almost wholly to the south, and in the meantime there was growing up in the north a system which became far more popular. This is described in Stifel's edition of Rudolff's *Coss* (Nürnberg, 1553) and in Robert Recorde's *Whetstone of Witte* (London, 1557).

In Stifel's work the absolute number is called *dragma,* indicated by a modified *d,* and the unknown is *radix* whose symbol has never been explained.[8] There are, however, in Paciuolo's

[4] *Trattato*, Vol. II, f. 39, r.

[5] *Trattato*, Vol. II, f. 169, r.

[6] *Arithmetica*, f. 253.

[7] *Practica Aritmetica*, Milan, 1537.

[8] For Stifel's symbolism, see the table at the end of this report.

Summa several cases in which the radix is written with a small *r*[9]. Reasoning from this fact and remembering that radix often means the unknown quantity as well as the root, it seems very possible that manuscript writers might easily have changed that symbol to the one in question. *Census* of the Italians is *zensus* with the Germans and the symbol ʒ is merely the initial letter. For the cube we find a sign which again is evidently developed from the Italian symbol, for in manuscripts the letters of *cu* are frequently joined at the top, from which the evolution of the German form is an easy matter. Instead of *relato*, the Germans used *sursolidus*, from whose symbol, together with those for *zensus* and *cubus*, all the other powers are formed by a multiplicative plan. The + and — are seen here as signs of operation, and indeed Stifel is generally considered to have been the first to use them as such. He is not entitled to this credit, however, for they appeared "in a work by Grammateus in 1514,[10] and again in 1537" in a remarkable little book published in Antwerp by one Gillis vander Hoecke.[11] Stifel's radical sign is nearly like our own, appearing as $\sqrt{}$ followed by the symbol to show the index, and, in the case of the universal radical, separated from the quantity to which it refers by a large dot. But these were not the earliest forms, as appears from Stifel's commentary on Rudolff, who he says used a sign with three angles for the cube root and two for the fourth.[12] Recorde's *Whetstone of Witte,* which was the first algebra printed in English, and Masterson's *Arithmeticke* which followed it,[13] represent the powers of the unknown by symbols which differ from Stifel's only in that they are printed with much bolder type, and that *sursolidus* has a slightly different form. Masterson's radicals are also the same as Stifel's, but Recorde follows the plan of Rudolff. There was, nevertheless, one important contribution made by Recorde, namely, the equality sign, which he says is "a paire of parallels or Gemowe lines of one length, thus ═ becaufe noe. 2. thynges can be moare equalle."[14]

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[9]*Summa*, f. 128, v.

[10]The page from Grammateus is reproduced in Professor Smith's *Rara Arithmetica*, Boston, 1908, p. 125.

[11]The page from vander Hoecke is also reproduced in the same work, p. 185. The student may thus consult the fac similes of the original sources.

[12]Stifel's Rudolff's *Coss*, f. 82, v.

[13]London, 1592-94.

[14]*Whetstone of Witte*, f. Ff, j, v.

The first French work on algebra, that of Jacques Peletier (Paris, 1560), combines the Italian and the German symbolism. Here we find the German symbols for the cube, the fifth power, and the radical sign, but for the first power ℞ is used, while the square is q, and the plus and minus signs appear $\tilde{p}$ as and $\tilde{m}$.

One of the most important writers to use the German symbols was the Jesuit Christopher Clavius, whom we ordinarily think of as belonging to the Italian school, since he lived and taught in Rome. He was a native of Bamberg, however, and his German origin is very evident in his *Algebra* (Rome, 1608). Clavius uses Stifel's system in its entirety except that N stands for the absolute number, and the parenthesis is used to show the universal radical. Such an expression as $\sqrt{3}(15+\sqrt{3}144)$[15] occurs very often, a form which was probably suggested to him by Bombelli's *Algebra.*

Although during the sixteenth century most of the writers were using one or the other of these systems, or a combination of both, there were a few by whom its limitations were felt, and whose efforts at improvement are noteworthy. The first of these is Cardan, a man whose powerful intellect was felt in every phase of life. In his earlier work he adopted the Italian symbols, but in his masterpiece, the *Ars Magna* (Nürnberg, 1545), they are must less used. ℞ appears as the sign for the root, and *plus* and *minus* are still indicated by p and m although without the line above, and always separated from the term to which they refer by a colon. *Primo relato* is also used for the fifth power, but the symbol $R^{\circ}p^{\circ}$ differs from the one in common use. Here the similarity to the old method disappears, for the other powers of the unknown are indicated by the Latin words *res* or *positiones, quadratus,* and *cubus* which Cardan writes in full or abbreviates to suit the context, using the proper terminations to designate the Latin case. Thus, for example, the first power is sometimes pot^{is} and sometimes reb^{9}, and the square may be $\tilde{q}d^{i}$, qd°, or any one of a variety of other forms. Cardan used a multiplicative plan of combination, and his symbols are even more unsatisfactory than those he discards; but nevertheless the change is an evidence of his dissatisfaction with those in existence.

In this connection, there are two men whose work is most

[15] *Algebra,* p. 90.

important, namely, Raphael Bombelli of Bologna, and the Hollander Simon Stevin, both of whom used a method approaching the exponential system. Bombelli's *Algebra* was published in two editions, in 1572 and in 1579, which is evidence of its worth and appreciation. Here the powers of the unknown are expressed by the figures 1, 2, 3, etc., written above small arcs[16], and it is evident that Bombelli had some idea of the significance of the exponent, for he explains at considerable length that the product of $3\,\overset{1}{\smile}$ and $10\,\overset{1}{\smile}$ is $30\,\overset{2}{\smile}$ because "the whole sum of 1 and 1 is 2[17]." These symbols, when used in the text, follow the coefficients, but in adding or subtracting polynomials they are often written above, thus furnishing an early example of detached coefficients. Bombelli's radical sign is simply an *R*, followed by *q* or *c*, to show the index, but in connection with this sign there is one very interesting point to be noticed, namely, the use of the adjective *legato* and its initial letter *L*. The word was not original with him for we find it in Cardan's arithmetic[18] where it signifies that the radical sign is to apply only to the term immediately following; for example *L* ℞ 3 *p*: ℞ 8 would mean $\sqrt{3} + \sqrt{8}$. Bombelli adopts the name and the symbol, but in his work, *legato* is synonymous with *universale*. Here we find the *L* written after the *R* and the expression is completed by placing an inverted *L* at the end[19] as in *R.q L* 128. p. 8^2 ⌋. The Italian word *legato* means bound, or fastened together, and it is quite possible that our signs of aggregation may have had their origin in the initial letter *L*. In that case the idea of the bracket would not be due to Girard (1629), as the historians assert, but to Bombelli in 1572.

Almost contemporary with Bombelli, there lived in Holland another mathematician, Simon Stevin, who was working along the same line. Stevin was one of the best thinkers of his time and is well known as being among the first to use the decimal fraction. His symbols for the powers of the unknown are the same that he uses to indicate units, tenths, hundredths, and so on, in decimal fractions, namely small figures placed in circles[20]. If Stevin finds that several unknowns are necessary, he forms them by prefixing *sec* or *ter* to the ap-

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16For Bombelli's symbolism, see the table at the end of this report.

17*Algebra*, p. 205.

18*Arithmetic*, f. A. 7, v.

19Bombelli's *Algebra*, p. 356.

20For Stevin's symbolism, see the table at the end of this report.

propriate symbol and often combines these in the same term. In this combination we often find an M used to denote multiplication, and there is a corresponding use of D for division which is common but by no means universal with Stevin. The radical sign appears in the German form, $\surd$ representing the square root, and the fourth and eighth roots requiring only the addition of a second and third angle. The symbol for cube root is the sign $\surd$ followed by the 3 in a circle,[21] which is modified by another angle to form the ninth root. These signs when placed before a monomial affect all of its factors unless they are separated from one another by a double arc forming a symbol which resembles an X, in which case the radical affects only the part preceding the sign. To indicate a root of a polynomial, Stevin writes the word *bino* or *trino* after the radical sign and we will frequently find such expressions as $\surd$*bino* 2 + $\surd$3 or $\surd$*trino* $\surd 3 + \surd 2 - \surd 5$ for $\sqrt{2+\sqrt{3}}$ and $\sqrt{\sqrt{3}+\sqrt{2}-\sqrt{5}}$.

None of the early attempts to form a symbolism was really successful, and the first noteworthy improvement was made toward the close of the sixteenth century. This was accomplished by François Vieta, the greatest French mathematician of his generation, in whose mind was conceived the idea of a general symbolism by which known as well as unknown numbers might be represented. He wrote at different times during the last quarter of the century and much of his work has been lost; but from what was recovered and edited by Franciscus Van Schooten in 1646, a fair idea of his symbolism can be formed. In the theoretical part of his work the unknowns are expressed by the capital vowels and the known quantities by the consonants. To raise these to any power, the words *quadratus, cubus,* etc., or their abbrevations, are written after, as *A quad* 4, or *A cub* in *E plano* 2, for $4x^2$ and $2x^3y$. One of the treatises saved by Van Schooten is called *De Aequationum recognitione et emendatione,* and was first published after Vieta's death by his friend Alexander Anderson. In this paper appears still another symbolism which may or may not have been due to Vieta.[22] Here a fair beginning is made in the theory of equations and the statements are made in a symbolism which is generally known as Vieta's. But below each theoretical statement

[21] *Œuvres,* Girard edition, p. 19.

[22] For Cantor's discussion of this point see *Vorlesungen uber Geschichte der Mathematik,* Vol. II, p. 582.

there appears in smaller type a numerical equation as an illustration, with a symbolism altogether different. The unknown number is represented by N, the square by Q, and the cube by C, and these are combined by an additive plan to form the higher powers. For example Vieta says, "*If Acubus—B—D—G in Aquad+B in D+B in G+Din Gin A* equals *B in D in G,* then *N* is either *B, D or G,*" and below it, by way of explanation, "If 1*C*—6*Q*+11*N* equals 6, then *N* is either 1, 2, or 3."[23] Another noteworthy feature is the vinculum, which was first used as a sign of aggregation in this work, and the symbol ═, by which Vieta indicates the absolute difference.

These symbols of Vieta were used for many years although they underwent several modifications, notably in the work of Harriot.[24] In this work the *A* was changed to *a* and the powers were formed by a repetition of the letter, as in *aaa* for our x^3. Here also the equality sign is more commonly used than in any earlier algebra, and the signs $<$ and $>$ first appear to express inequality.[25] Harriott also uses a double bar in writing complex fractions, as in the case of $\dfrac{\dfrac{aaa}{\text{b}}}{\text{d}} = \dfrac{aaa}{\text{bd}}$ a symbolism not accepted by his successors.

Harriot's symbolism was not a bad one and might have remained in use until the present time had there not appeared, in the person of Descartes, a man whose authority was universally felt. In writing *La Géométrie* (Paris, 1637), Descartes felt it better to discard most of the symbols then in use, and to adopt for knowns and unknowns, respectively, the first and last letters of the alphabet as we use them to-day. The exponents which in this work are always positive and integral, are represented by small figures placed above and to the right, but in case of the square the letter is always repeated instead of being affected by the exponent 2. In dealing with the equation, Descartes usually transposes all the terms to the left side, marking by an asterisk the places of any powers that may be wanting, and representing the equality by the sign ∞, as in x^5 * * * * $-a^4b \infty o$.

[23] *Opera Mathematica*, p. 158.

[24] *Artis analytica*, London, 1631.

[25] "Comparitionis figna in feqentibus furpanda
Æqualitatis $=$ *vt* $a = b$. fignificet *a* æqualem ipfi *b*
Maioritatis $<$ *vt* $a > b$. fignificet *a* maiorem quam *b*
Minoritatis $<$ *vt* $a < b$. fignificet *a* minorem quam *b*."—p. 6.

With Descartes, therefore, our modern elementary symbolism became fairly perfected, and it needed only the authority of a Newton to make it permanent: Newton's *Arithmetica Universalis,* containing his lectures on algebra at the University of Cambridge, appeared in London in 1707. In the first few pages of this work the symbolism is explained. The symbols were not original with Newton but were essentially those of Descartes, with a few due to Harriot, Girard, and Recorde. The fact that he judged them worthy, however, and that he made use of them in one of his best known works, was sufficient to give them such standing that they are still the recognized language of algebra.

It would be easy to select from the works consulted numerous other symbols of interest which have been suggested, and which have served their purpose either for the time, or even until our own day. In Harriot's work, for example, some useful symbols of aggregation are introduced, and in Recorde's the equations are particularly interesting, but the space allowed for this report permits of reference to only a few of the more important symbols not generally studied at first hand even by certain writers on the history of the subject. It is hoped that the information given, fragmentary as it is, may be helpful to teachers of mathematics.

In conclusion, there is appended a table showing some of the symbols as they appear in the original works. The copy of Vieta consulted is in the library of Columbia University, but for an opportunity to examine the writings of the other mathematicians I am indebted to Professor David Eugene Smith, through whose kindness the books in his own library as well as many of those belonging to George A. Plimpton, Esq., of New York, have been placed at my disposal.

Present Symbols		$\sqrt{\ }$	$\sqrt[3]{\ }$	$\sqrt[4]{\ }$	$\sqrt{\overline{\ }}$	1	x	x^2	x^3	x^4	x^5	+	−	=
Paciuolo	℞.p^a.	℞.2^a.	℞.3^a.	℞.4^a.	℞.v.	.n^o.	.co.	.ce	.cu.	.ce.ce.	.p̊.r̊.	.p̃.	.m̃.	
Tartaglia		℞	℞cu	℞℞	℞V		℞prima	ce	cu	ce.ce	rel	p̃	m̃	
Rocha						N	Co	Ce	Cu	Cce	R	ma	me	
Stifel		√	√𝔠𝔢	√ʒʒ	√.	𝔖	𝔯	ʒ	𝔠𝔢	ʒʒ	ß	+	−	
Recorde		√	w	w		𝔖	𝔯	ʒ	𝔠𝔢	ʒʒ	ß	+	−	=
Masterson		√ʒ	√𝔠𝔢	√ʒʒ	√ʒ.		𝔯	ʒ	𝔠𝔢	ʒʒ	ß	+	−	
Peletier							℞	q	𝔠𝔢	qq	ß	p	m	
de la Roche		℞					ρ	𝔠𝔢	□	c𝔠𝔢		p̃	m̃	
Clavius		√ʒ	√𝔠𝔢	√ʒʒ	√( )	N	𝔯	ʒ	𝔠𝔢	ʒʒ	ß	+	−	
Halcke						N	𝔯	ʒ	𝔠𝔢	ʒʒ	ß	∔	−, ÷ }	
Ghaligai		℞□	℞▭	℞□di□		n^o	c^o	□	▭	□di□	▯	piu	m^o	
Cardan		℞	cu℞		℞V:	\	res	q̃d	cub^9	q̃d q̃d	R^o p^o	p:	m:	
Bombelli		R.q	R.c	RR.q	RL⌋		$\overset{1}{\smile}$	$\overset{2}{\smile}$	$\overset{3}{\smile}$	$\overset{4}{\smile}$	$\overset{5}{\smile}$	p	m	
Vander Hoecke	h	℞^{2a}	℞^{3a}	℞^{4a}		n	Pri	2^a	3^a	4^a	5^a	+	−	
Stevin		√	√③	w	√bino	⓪	①	②	③	④	⑤	+	−	
Vieta		√	√C.	w	√‾		A	Aquad	Acubus	Aquad-quad	Aquad-cub	+	−, = }	
Harriot		√	√3)				a	aa	aaa	aaaa	aaaaa	+	−	=
Descartes		√	√C.		√‾		x	xx	x^3	x^4	x^5	+	−	∞
Newton		√	$\sqrt[3]{\ }$	$\sqrt[4]{\ }$	√‾		x	xx	x^3	x^4	x^5	+	−	=