

*On the Kinematics of non-Euclidean Space.* By Prof. W. BURNSIDE.

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I.

In a note in Vol. XIX of the *Messenger of Mathematics*, "On the Resultant of Two Finite Displacements of a Rigid Body," I have shown that a geometrical construction there given is applicable to non-Euclidean space. The construction, or rather its proof, is materially simplified in a note with a similar title in Vol. XXIII of the same journal, but the phraseology used in this second note is wholly that of ordinary space. It is not, I believe, generally known how simply the kinematics of non-Euclidean space may be treated by the methods of ordinary synthetic geometry; and it is my object in the first part of the present paper, by reproducing in a quite general form the construction above referred to, and by applying it to the deduction of certain kinematical theorems, to bring this out clearly.

The elementary geometry of hyperbolic space has been treated in detail by Herr J. Frischauf (*Elemente der Absoluten Geometrie*, Leipzig, 1876), while the leading theorems in the elementary geometry of elliptic space have been given by Mr. S. Newcomb (*Crelle's Journal*, Vol. LXXXIII). Reference may also be made to an address by Prof. G. Chrystal to the Royal Society of Edinburgh (*Proc. R.S.E.*, 1879-80), in which the leading results of Frischauf and Newcomb are partly summarized and partly treated independently. The more elementary of the results obtained by these authors have been assumed as known in the present paper. The distinction between the single and double elliptic spaces,\* that is, between spaces in which two straight lines in a plane always meet in one or in two points respectively, is of course taken account of where necessary; but most of the results hold equally well for either. There is, however, a fundamental kinematical distinction between the two cases, which may be stated here, as it is given by Prof. Chrystal in the address above referred to. The distinction is that,

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\* In his recent writings Prof. Klein uses the terms "elliptic" and "spherical" space for what are here called "single" and "double" elliptic spaces.

while a translation along a complete straight line in single elliptic space is equivalent to a rotation through two right angles round the line, in double elliptic space it is equivalent to no displacement at all.

In what follows, one of two finite straight lines  $AB$ ,  $A'B'$  is continually spoken of as equal to, greater than, or less than the other. It is to be observed that this does not involve the assumption of any such analytical system of measurement as may be used in ordinary space. (It is, in fact, one of the objects of this paper to show how the appropriate metrical systems of elliptic and hyperbolic space may be deduced from purely synthetical considerations.) The test is one of congruency; namely, the point  $A$  may be made to coincide with the point  $A'$ , and the line  $AB$  to lie along the line  $A'B'$ , and it will then be obvious on inspection whether  $AB$  is equal to, greater than, or less than  $A'B'$ . In the same way, it is clear that a test of congruency can be applied to determine whether two intersecting lines  $AB$ ,  $AC$  are or are not at right angles.

The following definitions are introduced to avoid any possible ambiguity.

A *motion* is a displacement of the points of space such that all congruent figures remain congruent. The word "displacement" without qualification is, however, generally here used for "motion" as just defined.

A *rotation* is a motion in which all the points of one straight line are undisplaced. A rotation through two right angles is for shortness called a *half-turn*.

A *translation* is a motion in which a straight line and the two parts into which it divides, or appears to divide, any plane passing through it are respectively displaced into themselves.

*Lemma I.*—At least one straight line can in general be drawn to meet any two given lines at right angles; and in hyperbolic space there is never more than one such straight line.

If the two straight lines intersect, the line through their point of intersection perpendicular to their plane is a line meeting them both at right angles. Suppose now that the space is hyperbolic, and that  $AB$ ,  $AC$  are the two straight lines. Then, if  $BC$  were a straight line meeting  $AB$  and  $AC$  at right angles, and if  $A$  is a finite point,  $ABC$  would be a rectilinear triangle, the sum of whose angles is greater than two right angles; while, if  $A$  is a point at infinity, then  $ABC$  would be a rectilinear triangle whose area is not infinitesimal, while the sum of its angles is equal to two right angles. Neither of these

results is possible (cf. Chrystal, *loc. cit.*), and therefore no such line as  $BO$  exists. It is to be noticed that, if  $AB$ ,  $AO$  meet at infinity, the line meeting them both at right angles cannot actually be drawn.

Suppose next that the two straight lines  $AB$ ,  $OD$  do not intersect. From every point  $P$  of  $AB$  draw lines  $Pp$  in every possible direction perpendicular to  $AB$ , and such that each of them can be brought to congruence with a given finite straight line.

The locus of the extremities of these lines will be called an equidistant surface of  $AB$ , and  $Pp$ , ... will be called its radii. The equidistant surface remains congruent with itself for all translations along and rotations round its axis  $AB$ , and it must therefore be a continuous surface with a definite tangent plane at every point, while the radius through any point is perpendicular to the tangent plane at it. If now the radius to the equidistant round  $AB$  is sufficiently small, then, since, by supposition,  $AB$  and  $CD$  are non-intersecting lines,  $CD$  must lie entirely outside (*i.e.*, on the opposite side to  $AB$ ) of the equidistant. Hence, when the radius is taken larger and larger, there will be some definite finite value of the radius for which  $CD$  first meets the equidistant. If the point in which  $CD$  first meets an equidistant is a finite point, it necessarily touches it at this point, and the radius to the equidistant through the point is a straight line meeting  $AB$  and  $CD$  at right angles. If, now, a second equidistant touch  $CD$ , then  $CD$  is necessarily a line returning into itself, and the space must therefore be elliptic. Hence, again, in this case, not more than one line can be drawn in hyperbolic space to meet two given lines at right angles.

Suppose, secondly, if possible, that  $CD$  first meets an equidistant at infinity, so that the space is necessarily hyperbolic. If  $AB$ ,  $OD$  are not in the same plane, this is clearly impossible. For, if from points taken further and further along  $OD$  perpendiculars be let fall on the plane  $ABO$ , these perpendiculars increase without limit, and, *a fortiori*, the same must be true of the perpendiculars let fall on  $AB$ . If, on the other hand,  $AB$ ,  $OD$  are in the same plane, and if through any point  $C$  of  $OD$  the lines  $CD'$ ,  $CD''$  be drawn to meet  $AB$  at infinity, then  $CD$  must make with one or the other of these lines an angle less than any finite angle, or, in other words,  $CD$  must coincide with either  $CD'$  or  $CD''$ . Hence this second possibility reduces to the previously considered case in which the two lines meet at infinity.

*Lemma II.*—If  $AB$  be any straight line, and  $Aa$ ,  $Bb$  two straight

lines in the same plane, both of which are at right angles to  $AB$ , then successive half-turns round  $Aa$  and  $Bb$  are equivalent to a translation  $2AB$  along  $AB$ . Let  $P$  be any point in  $BA$  produced, and take  $P'$  and  $Q$  in this line, so that  $PA$  and  $AP'$ , and also  $P'B$  and  $BQ$ , can be respectively brought to congruence. Then  $2AB$  is congruent with  $PQ$ . Now the half-turn round  $Aa$  brings  $P$  to  $P'$ , and the half-turn round  $Bb$  brings  $P'$  to  $Q$ , so that the two half-turns displace every point of  $AB$  through a distance congruent with  $2AB$  along  $AB$ , while the two halves of the plane  $aABb$  on either side of  $AB$  are displaced each into itself. The resultant displacement is therefore a translation  $2AB$  along  $AB$ .

*Lemma III.*—If  $Oa$ ,  $Ob$  are any two intersecting lines, and if  $cOc'$  is perpendicular to both of them, successive half-turns round  $Oa$  and  $Ob$  are equivalent to a rotation  $2aOb$  round  $cOc'$ .

Let  $OP$  be any line through  $O$  in the plane  $aOb$ , and take  $OP'$ ,  $OQ$  such that the angles  $POa$ ,  $aOP'$  and the angles  $P'Ob$ ,  $bOQ$  are respectively equal. The half-turn round  $Oa$  changes  $OP$  to  $OP'$ , and  $Oc$  to  $Oc'$ , and the half-turn round  $Ob$  changes  $OP'$  to  $OQ$  and  $Oc'$  to  $Oc$ . The successive half-turns therefore keep  $Oc$  undisplaced and change  $OP$  into  $OQ$ , and it is evident that the angle  $POQ$  is equal to  $2aOb$ .

*Lemma IV.*—If  $Aa$ ,  $Bb$  are any two lines, and  $AB$  is a line meeting them both at right angles, successive half-turns round  $Aa$  and  $Bb$  are equivalent to a translation  $2AB$  along  $AB$ , and a rotation round  $AB$  through twice the angle between the planes  $aAB$  and  $ABb$ .

Through  $B$  draw  $Bb'$  in the plane  $aAB$  perpendicular to  $AB$ . Then successive half-turns round  $Aa$  and  $Bb$  are equivalent to successive half-turns round  $Aa$  and  $Bb'$  followed by successive half-turns round  $Bb'$  and  $Bb$ .

The first displacement is, by Lemma II., the same as a translation  $2AB$  along  $AB$ , and the second, by Lemma III., is the same as a rotation through  $2b'Bb$ , i.e., through twice the angle between the planes  $aAB$  and  $ABb$ , round  $AB$ .

It is obvious that the resultant of these last two displacements is independent of their order.

These lemmas lead to a very simple construction for the resultant of any given finite displacements. Suppose first that two displacements each consisting of a translation along and a rotation round a given line are to be compounded; and let  $a'Aa$ ,  $b'Bb$  be the axes of

the given displacements. Take  $AB^*$  a line meeting both axes at right angles, and in  $a'Aa$  take  $a'$  such that half the translation along  $a'Aa$  will bring  $a'$  to  $A$ ; then through  $a'$  draw  $a'a$  perpendicular to  $a'Aa$ , such that half the rotation round  $a'Aa$  brings the plane  $aa'A$  into the position  $a'AB$ . The first displacement is then equivalent to half-turns round  $a'a$  and  $AB$ , by Lemma IV.; and in the same way  $b\beta$  may be constructed meeting  $b'Bb$  at right angles, such that the second displacement is equivalent to successive half-turns round  $AB$  and  $b\beta$ . The resultant displacement is therefore equivalent to successive half-turns about  $a'a$  and  $b\beta$ ; and, if  $a\beta$  be a line meeting these two at right angles, the resultant displacement is the same as a translation  $2a\beta$  along  $a\beta$  and a rotation round  $a\beta$  through twice the angle between the planes  $a'a\beta$  and  $a\beta b$ . Any number of successive displacements may now be compounded in this way, and the axis, translation, and rotation of the resultant screw-motion so determined.

As was pointed out in the introduction, these constructions hold equally well for non-Euclidean as for Euclidean space; but the nature of the displacement arising by compounding two given displacements depends obviously upon the geometrical relations of the lines denoted by  $aa'$ ,  $a'A$ ,  $AB$ ,  $Bb$ ,  $b\beta$ , above.

When the two displacements are translations and the space Euclidean, the resultant displacement is again a translation. Suppose now that the space is hyperbolic, that is, that the two points at infinity on every straight line are real and distinct, and that the axes of the two translations do not lie in the same plane. If the resulting displacement were a translation, it would be necessary that  $a'a$  and  $b\beta$  should lie in the same plane, but it may be easily shown that this is impossible.

Thus, if the planes through  $A$  and  $B$  perpendicular to  $AB$  be spoken of for a moment as the planes  $P$  and  $Q$ ,  $a'a$ , and therefore every plane passing through it, is at right angles to the plane  $P$ , while  $b\beta$  is at right angles to the plane  $Q$ . Now, if  $a'a$ ,  $b\beta$  lie in a plane, then  $b\beta$ , being the line of intersection of the planes  $a'ab\beta$  and  $ABb\beta$ , both of which are perpendicular to the plane  $P$ , is itself perpendicular to the plane  $P$ ; and there are therefore two common perpendiculars to the lines  $P$  and  $Q$ . But this is in contradiction to the fact that in hyperbolic space one line only can be drawn to meet two given lines at right angles.

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\* The exceptional case in which the points  $A$  and  $B$  lie at infinity is dealt with at the beginning of Section III.

Hence in hyperbolic space the resultant of two translations along axes that do not lie in the same plane is never a translation.

If the axes of the two translations lie in a plane and meet, their resultant is equivalent to two half-turns about axes perpendicular to the plane, and is thus always a translation whose axis is in the same plane as the given axes.

If the axes lie in a plane and do not meet, the resultant displacement is equivalent to two half-turns about axes lying in the plane, and will thus be a rotation or a translation according as the axes of these half-turns do or do not meet.

In elliptic space, in which all straight lines are of finite length, and every two straight lines in a plane meet, the distinction between a translation and a rotation is lost, for the following reason. The lines drawn in a plane, perpendicular to a given line, all meet in either one common or two common points, according as the space is single or double elliptic space; and the locus of these points when the perpendiculars are drawn in all the different planes through the line is a second line, every point of which is at the same distance from the first line. The relation between the two lines\* is reciprocal, and it is immediately evident from the above that a rotation about one of them is equivalent to a translation along the other. If, now, in elliptic space, the two translations to be compounded are along axes not lying in one plane, the lines  $a'a$  and  $b\beta$  will both meet  $AB$ .

Hence  $a'a$  and  $b\beta$  will only lie in a plane if  $ABa'b$  is a plane; and this is contrary to the supposition that the axes of the two translations are non-intersecting lines. Hence the resultant of two translations along non-intersecting axes in elliptic space is never a translation (or rotation). If, on the other hand, the axes lie in one plane, the resultant displacement can be represented indifferently as a translation along some line in that plane or a rotation round the conjugate line.

## II.

In a general displacement in Euclidean or hyperbolic space one line only remains unchanged, while in elliptic space two (conjugate) lines remain fixed. This statement, which is true of the general displacement, is therefore also true of the general infinitesimal displacement and of the set of displacements which result from repeating an infinitesimal displacement any (finite or infinite)

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\* Two such lines will be called conjugate lines.

number of times. There are, however, in Euclidean space certain infinitesimal displacements, namely translations, which keep unchanged each of a doubly-infinite set of straight lines; and the question therefore arises whether there are, in non-Euclidean space, any sets of displacements, arising from the repetition of an infinitesimal displacement, for which more than one (or two) lines remain unchanged.

It is known from considerations of analysis that in hyperbolic space there is no such set of displacements; but that in elliptic space, when a line is given, there are two distinct sets of displacements, each of which keeps a distinct doubly-infinite system of straight lines, of which the given line forms one, unchanged. The latter result is proved by Clifford in his paper on "Biquaternions" (*Proc. Lond. Math. Soc.*, Vol. iv., p. 390); and reference may also be made to a memoir by Sir R. Ball, "On the Theory of the Content" (*Trans. R.I.A.*, 1889).

The preceding lemmas and construction may be applied to obtain and amplify these results by elementary geometrical considerations, which are in part at least distinct from Clifford's.

If round the axis of the displacement one of its equidistant surfaces be described, no line which cuts this surface can remain unaltered by an infinitesimal displacement. For, if  $P, Q$  be the points where the line meets the surface, then, since both the surface and the line are changed into themselves by the displacement, the points  $P, Q$  must be either unaltered or interchanged; and both these suppositions are clearly impossible. If then a line remains unaltered by a displacement, it must lie on one of the equidistant surfaces of the axis of the displacement. Now in hyperbolic space the tangent plane at any point of an equidistant surface must lie wholly outside it, since the common perpendicular to two lines is also necessarily their only shortest distance. The equidistant is therefore, in this case, not a ruled surface, and no such displacement as that considered is possible.

That, in elliptic space, the equidistant is a ruled surface, may be seen directly as follows.

Let  $A$  and  $B$  be any two points on a line and its conjugate respectively, and take points  $A_1, A_2, \dots$  on the line, and  $B_1, B_2, \dots$  on the conjugate such that the finite lines  $AA_1, A_1A_2, \dots, BB_1, B_1B_2, \dots$  are all equal. Join  $AB, A_1B_1, A_2B_2$  by lines which will be all of equal length, and all at right angles both to  $AA_1A_2 \dots$  and  $BB_1B_2 \dots$ . Finally, take  $C, C_1, C_2, \dots$  on  $AB, A_1B_1, \dots$  such that  $AC, A_1C_1, \dots$

are all equal. Then  $C, C_1, C_2, \dots$  all lie on an equidistant of  $AA_1$ , which is evidently at the same time an equidistant of  $BB_1$ . Join  $CC_1, C_1C_2, C_2C_3, \dots$  by straight lines.

If, now, the figure be rotated through two right angles about  $A_1C_1B_1$ , the points  $A, B$  are brought into the positions  $A_2, B_2$ , while the lines  $AA_1$  and  $BB_1$  are changed into themselves. The points  $C$  and  $C_2$  are therefore interchanged, and hence  $CC_1C_2$  is a straight line. If the points  $A$  be kept fixed, and the points  $B$ , retaining their relative positions, be displaced continuously along the conjugate line, a complete set of generators of one system of the equidistant is obtained; and the other set will be obtained by taking the points  $B, B_1, \dots$  in the opposite direction along the conjugate line.

A displacement  $AA_1$  along the line  $AA_1$  and a displacement  $BB_1$  along the conjugate now clearly displace  $CC_1C_2$  along itself, whatever the length  $AC$  may be, and wherever  $B$  is taken on the conjugate line. Hence a translation along a line and an equal translation along its conjugate leave undisplaced all the generators of one system of all the equidistants of the two lines. If the second translation is reversed in direction, the doubly-infinite set of generators of the other system are undisplaced. To these two displacements, or rather to the velocity-systems connected with them, Clifford has given the names right- and left-vectors. The same words may be used here to denote the corresponding finite displacements, while the two sets of lines which remain undisplaced by a right- or a left-vector may be called, with Clifford, a set of right- or left-parallels. The above reasoning shows that any two of a set of parallels, either right or left, are everywhere at the same distance apart. Moreover, if in the above construction  $CC_1C_2$  is a right-parallel of  $AA_1A_2$ , then the lines  $ACB, A_1C_1B_1, \dots$  are left-parallels, and conversely. A right-vector is therefore equivalent to successive half-turns about two left-parallels.

Suppose, now, with the previous notation, that the two displacements, of which  $a'Aa, b'Bb$  are axes, are both right-vectors. Then  $a'a$  and  $AB$  are left-parallels, as also are  $AB$  and  $b\beta$ . The resultant displacement, consisting of successive half-turns about  $a'a$  and  $b\beta$ , which are left-parallels, is therefore a right-vector. Right-vectors, therefore, form a group of displacements, in the sense that the resultant of any two right-vectors is again a right-vector; and the same is, of course, true of left-vectors. The groups of displacements thus formed are not, however, like the group of translations in Euclidean space, composed of permutable operations; viz., the re-

sultant of two right- (or left-) vectors depends upon the order in which they are performed.

Finally, it may be shown that a right-vector and a left-vector are always permutable. Thus, let  $a'Aa$  be the axis of a right-vector, and  $b'Bb$  that of a left-vector,  $AB$  being a common perpendicular to these two lines. Take  $a'a'$  and  $aa$  perpendicular to  $a'Aa$  and such that the right-vector is equivalent to successive half-turns round  $a'a'$  and  $AB$ , and also to successive half-turns round  $AB$  and  $aa$ ; and construct  $b'\beta'$  and  $b\beta$  similarly for the left-vector. Then  $a'a'$  and  $aa$  are opposite generators of the same system of an equidistant of  $AB$ , so that any common perpendicular to them meets  $AB$  (necessarily at right angles). So also  $b'\beta'$  and  $b\beta$  are opposite generators of the other system on another equidistant of  $AB$ . The five lines  $a'a'$ ,  $b'\beta'$ ,  $AB$ ,  $b\beta$ ,  $aa$  therefore have a common perpendicular  $\alpha'\beta'O\beta a$ , and from the construction of the equidistants it follows that  $\alpha'\beta$  is equal to  $\beta'a$ , and the angle between the planes  $\alpha'a'\beta$  and  $\alpha'\beta b$  is equal to that between  $b'\beta'a$  and  $\beta'aa$ . Hence successive half-turns round  $a'a'$  and  $b\beta$  are equivalent to successive half-turns round  $b'\beta'$  and  $aa$ ; or, in other words, the displacement resulting from the right-vector followed by the left-vector is identical with that resulting from the left-vector followed by the right-vector.

Any displacement in elliptic space is the resultant of a right-vector and a left-vector. For it has been seen that any displacement is equivalent to a rotation  $\Theta$  round some line, and a rotation  $\Theta'$  round its conjugate, and, since these two displacements are permutable, they are equivalent to rotations  $\frac{1}{2}(\Theta + \Theta')$  round the line and its conjugate, followed by rotations  $\frac{1}{2}(\Theta - \Theta')$  round the line and  $-\frac{1}{2}(\Theta - \Theta')$  round its conjugate, that is, to a right-vector  $\frac{1}{2}(\Theta + \Theta')$  and a left-vector  $\frac{1}{2}(\Theta - \Theta')$  with the line for their common axis.\*

Now, it has been seen that right-vectors form a group of motions in the sense that the resultant of any two right-vectors is again a right-vector, and that the same is true of left-vectors, while every displacement of the one group is permutable with every displacement of the other. Hence, to determine completely the nature of the general group of motions in elliptic space, it is only necessary to consider the laws according to which right- and left-vectors separately combine.

Through any point of space one, and only one, of a set of right-parallel lines will pass. Hence, when two right-vectors are given whose

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\* Cf. Clifford on "Biquaternions" (*Proc. Lond. Math. Soc.*, Vol. iv., p. 390).

resultant is required, intersecting lines  $OA$  and  $OB$  may be taken as their axes; these being the two lines drawn through any chosen point  $O$  which belong respectively to the two sets of right-parallels that are displaced into themselves by the two right-vectors.

Through  $OA$  draw a plane  $AOO$  such that the right-vector whose axis is  $OA$  displaces it to  $AOB$ ; and through  $OB$  draw a plane  $BOB$  such that the plane  $AOB$  is displaced into it by the second right-vector. The angles between the pairs of planes  $AOO$ ,  $AOB$  and  $BOA$ ,  $BOO$  will then measure the amplitudes of the two right-vectors. Now every plane contains one, and only one, of a set of right- (or left-) parallels, and therefore the plane  $AOO$  must contain a line which is transformed into itself by the resultant displacement; but the plane  $AOO$  is changed into  $BOO$  by the resultant displacement, and therefore  $OC$  must be that axis of the resultant right-vector which passes through  $O$ . The amplitude of the resultant is the angle between the planes  $AOO$  and  $BOO$ , since it displaces one of these planes into the other. The axis and amplitude of the resultant of any two right- (or left-) vectors is thus completely determined. The result may be stated as follows:—

Right-vectors combine according to the same law as finite rotations round a point, the amplitudes of the rotations being twice those of the corresponding right-vectors. It is also clear that exactly the same statement holds concerning left-vectors.

The group of right-vectors (or left-vectors) is therefore isomorphous with the group of rotations round a point; and the structure of the general group\* of real motions in elliptic space is thus deduced from

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\* It has been suggested by one of the referees, to both of whom I owe my best thanks for the trouble they have taken with this paper, that since there is at present no English treatise on the subject of continuous groups, it would be advisable to give such definitions and explanations of some of the terms used in the present paper as will suffice to make their meaning definite to the reader.

I have attempted in the following note to carry out this suggestion; purposely abstaining from any reference to the analytical form in which Herr Sophus Lie, to whom the theory of continuous groups owes its origin, has presented it.

A set of operations  $1, S, S', S'', \dots$ ,

which contains every possible combination of the individual operations, taken either directly or inversely, is said to form a group. When the individual operations depend upon a finite number,  $n$ , of quantities, each of which is capable of continuous variation through a range which is not infinitely small, the group is spoken of as a "finite continuous group." Since each of the  $n$  quantities on which the determination of a particular operation depends is capable of an infinite number of values, the group contains in a quite definite sense  $\infty^n$  different operations. To denote such a group Lie uses the phrase "*n-gliedrige kontinuierliche Gruppe*."

Such a group necessarily contains infinitesimal operations, *i.e.*, operations which produce an infinitesimal change in any possible operand. If  $S$  and  $T$  are two infinitesimal operations, the difference of the changes produced by  $ST$  and  $TS$  in any

purely synthetical considerations. It is, in fact, now seen to arise from the combination of two permutable and isomorphous groups of known type. The structure of this group renders it very simple to enumerate all types of sub-group contained in it; that is, all those sets of motions in elliptic space which have the group property, and at the same time do not include all possible motions. Thus any sub-group must be formed by the combination of the group of right-vectors or one of its sub-groups with a sub-group of the group of left-vectors, or *vice versa*; and the combination may either be general or may be such that, an isomorphous correspondence having been established between the two sub-groups, corresponding operations are combined together.

Now the group of rotations round a point contains no real sub-group with a doubly-infinite number of operations; its only sub-groups, in fact, being rotations round a fixed axis, which form a singly-infinite set.

Hence the general group of motions in elliptic space which con-

operand is necessarily infinitesimal in comparison with the changes produced by  $S$  or  $T$ ; so that, when the word "infinitesimal" is used in its ordinary sense, two infinitesimal operations are necessarily permutable; but this, of course, does not involve that the corresponding finite operations, which result from repeating the infinitesimal operations an infinite number of times, are permutable.

The group contains  $n$  independent infinitesimal operations, in the sense that every infinitesimal operation of the group can be obtained by a finite combination of them.

Every non-infinitesimal operation of the group can be generated by an infinite number of repetitions of an infinitesimal operation, suitably chosen, and thus the group is completely defined by a set of  $n$  independent infinitesimal operations.

On the other hand,  $n$  arbitrarily given infinitesimal operations will not in general generate a finite continuous group of  $\infty$  operations, but an infinite continuous group, *i.e.*, one whose individual operations depend on an infinite number of continuously varying quantities.

The simply-infinite set of operations obtained by repeating an infinitesimal operation (and its inverse) form a group, which is contained within the original group. It is a group whose individual operations are determined by a single continuously varying parameter; and is spoken of by Lie as an "*ein-gliedrige Untergruppe*" of the original group.

In addition to such simply-infinite sub-groups, the original group will in general contain other sub-groups.

Thus, it may happen that  $1, \mathfrak{Z}, \mathfrak{Z}', \mathfrak{Z}'', \dots$

a set of operations contained in the original group, possess among themselves the group-property defined in the first paragraph. If the number of operations in this set is finite, the sub-group formed by their totality is necessarily discontinuous; if the number is infinite, the sub-group may be either discontinuous or continuous. The latter will be the case, when the individual operations of the sub-group are determined by a number  $r$  (necessarily less than  $n$ ) of continuously varying quantities. The sub-group may then be spoken of as a continuous sub-group of  $\infty$  operations. Lie's phrase is "*r-gliedrige Untergruppe*." Such a sub-group again necessarily contains a set of  $r$  independent infinitesimal operations, from

tains  $\infty^0$  operations has no sub-group containing  $\infty^0$  operations, while the only two types of sub-groups which contain  $\infty^4$  operations are those arising from the combination of the group of right- (or left-) vectors with those left- (or right-) vectors which keep a given set of left- (or right-) parallels unchanged.

These two groups are analogous to, but not isomorphous with, that group of motions in Euclidean space which consists of all possible screw-motions about a set of parallel axes. Each of the two types contains  $\infty^3$  conjugate sub-groups.

Of sub-groups containing  $\infty^3$  operations, there are three types. Two of these are the groups of right-vectors and left-vectors which are self-conjugate in the main group. The third is the group of

which it can be generated, and the  $n$  infinitesimal operations of the original group can always be chosen so that these  $r$  occur among them. It is not, however, generally the case that any  $r$  of the  $n$  independent infinitesimal operations will generate a sub-group of  $\infty^r$  operations; they will, when  $r > 1$ , generally generate the original group itself.

If now  $T$  is any operation of the group, the operations  $S$  and  $T^{-1}ST$  are called conjugate [(Lie) *gleichberechtigte*] operations, when they are not identical, and  $T$  is said to transform  $S$  into  $T^{-1}ST$ .

Similarly,

$$1, \Sigma, \Sigma', \dots$$

and

$$1, T^{-1}\Sigma T, T^{-1}\Sigma' T, \dots$$

are called conjugate sub-groups when they are not identical with each other. If these two sub-groups are identical, whatever operation of the original group  $T$  may be, the sub-group

$$1, \Sigma, \Sigma', \dots$$

is called a self-conjugate sub-group. Lie uses the phrase "*invariante Untergruppe*" to denote a sub-group with this property, while Klein writes "*ausgezeichnete Untergruppe*." Lie uses the word "*ausgezeichnete*" only in connexion with "*ein-gliedrige Untergruppe*"; an "*ausgezeichnete ein-gliedrige Untergruppe*" being, in his phraseology, a simply-infinite continuous sub-group, each of whose operations is permutable with all the operations of the group.

If a continuous sub-group  $I$ , containing  $\infty^r$  operations, is contained as a self-conjugate sub-group within another more extensive continuous sub-group  $H$ , containing  $\infty^s$  ( $s > r$ ) operations, but is not self-conjugate within any continuous sub-group of the original group  $G$  that is more extensive than  $H$ , then  $I$  is transformed into itself by all the  $\infty^s$  operations of  $H$ .

When  $I$  is transformed by all the  $\infty^s$  operations of  $G$ , there result  $\infty^{s-r}$  sub-groups, all of them conjugate to  $I$ ; and then  $I$  is said to form one of a set of  $\infty^{s-r}$  conjugate sub-groups [(Lie) *gleichberechtigte Untergruppen*] within  $G$ .

When a one-to-one correspondence can be established between the individual operations of two continuous groups, each of which contains  $\infty^n$  operations, in such a way that to the product of any two operations of one group in a certain order corresponds the product of the two homologous operations of the other group in the same order, the two groups are said to be holohedrally isomorphous [(Lie) *holoedrisch isomorph*]. Abstractly considered, i.e., when the laws of combination of the individual operations only are taken into account, and not the nature of the operations themselves or of the operand, two holohedrally isomorphous groups are identical. Where the word "isomorphous" is used in the present paper without qualification it is to be regarded as an abbreviation for "holohedrally isomorphous."

rotations round a given point (or the general group of motions in a plane), which is isomorphous with the preceding, but, unlike them, forms one of  $\infty^3$  conjugate sub-groups. It may be regarded as arising from an isomorphous correspondence between the groups of right- and left-vectors established as follows. Through a given point one of every set of right-parallels and one of every set of left-parallels will pass. If, then, a right-vector and a left-vector correspond when their amplitudes are equal and their axes which pass through the given point are identical, the resultant of two right-vectors will correspond to the resultant of the corresponding two left-vectors.

When therefore the group of right-vectors is combined with the group of left-vectors by multiplying together corresponding operations in the two groups, the new group is isomorphous with either of the groups from which it is formed, while it keeps the given point fixed.

Of sub-groups containing  $\infty^3$  operations there is one type, namely, the group of motions which consist of arbitrary rotations round any pair of conjugate lines, and this type contains  $\infty^4$  conjugate sub-groups. It would appear at first sight that the sub-group arising from combining those right-vectors which keep an arbitrarily chosen set of right-parallels unchanged with a similar group of left-vectors would give rise to a new type; but it is an immediate deduction from the constructions in the earlier part of this paper that any set of right-parallels and any set of left-parallels have just two lines in common, these lines being conjugate.

Of sub-groups containing  $\infty^1$  operations there are three types. Of these those right- (or left-) vectors which keep a given set of right- (or left-) parallels unchanged form two types each containing  $\infty^3$  conjugate sub-groups, while the third type consists of screw-motions of given pitch round a given line, and contains  $\infty^4$  conjugate sub-groups.

All discontinuous groups of motions of finite order in elliptic space, corresponding to which there are divisions of the whole of space into a finite number of congruent portions, may be derived in a precisely similar manner from the known finite discontinuous groups of rotations about a point, *i.e.*, from the groups of the regular solids. Owing to the greater number of types of group involved, there is a very much greater variety of such discontinuous groups than of the continuous groups that have just been considered. They need not here be enumerated, as they have been in effect completely classified, though from a rather different point of view, by M. Goursat, in a memoir with the title, "*Sur les substitutions orthogonales et les*

divisions régulières de l'espace" (*Ann. de l'Ecole Norm. Sup.*, 3<sup>me</sup> série, tome vi.). Except for the simplest of such discontinuous groups, it is a matter of considerable difficulty to realize the nature of the corresponding division of space into congruent parts; and in the simplest case of all, that namely in which the group consists of a rotation through two right angles round a line and identity, the solution for simple elliptic space is by no means obvious. Before dealing with this particular case, take the case of a cyclical group generated by a right-vector,  $n$  repetitions of which lead to identity.

If  $n$  planes be drawn through any axis of the right-vector, each of which makes angles  $\frac{\pi}{n}$  with the planes on either side of it, the whole of space is divided into  $n$  congruent figures which may be called biangles, the space between any two adjacent planes being easily seen to be continuous with the vertically opposite space between them. The right-vector, consisting of a rotation  $\frac{\pi}{n}$  round the line, and a translation through  $\frac{1}{n}$ th of its length, transforms any one of these biangles successively into each of the others, and  $n$  repetitions of it, being equivalent to a rotation  $\pi$  round the line, and a translation through its whole length, which is the same as another rotation  $\pi$ , brings back the original biangle to coincidence with itself, point for point.

If now  $n$  is odd, and the generating operation a rotation  $\frac{2\pi}{n}$  round the line, the same construction will give  $n$  congruent spaces which are transformed into each other by the operations of the cyclical group, though the correspondence of points is not the same as in the former case. If, however,  $n$  is even, the  $n$  congruent biangles are not transformed into each other, but the original biangle is transformed only into  $\frac{n}{2}$  of the biangles, and into each by two operations in two different ways; a different division of space is therefore necessary in this case. When  $n$  is 2, it might appear sufficient at first sight to draw a single plane through the line; but in simple elliptic space the two sides of a plane are continuous with each other, so that this would not effect a division of space into two parts.

The requisite division of space into two congruent parts may, however, be obtained as follows. Let  $A$  and  $B$  be two points taken one on each of two conjugate lines  $a$  and  $b$ , and bisect the straight segment  $AB$  in  $C$ . When  $A$  and  $B$  take all possible positions on  $a$  and  $b$  respec-

tively, the locus of  $O$  is an equidistant of each of these lines, whose "radius" relative to each line is the same, namely, one quarter of the complete straight line. It is easy to verify that on this particular equidistant the generators are at right angles; and, since it is impossible to draw a line from a point on  $a$  to a point on  $b$  which does not cut the equidistant once, it must divide the whole of space into two parts. Consider now the motion which consists of a rotation through two right angles about one of the generators of this equidistant. Every such line as  $ACB$ , used in the construction just given, which meets the generator is brought into the position  $BCA$ , so that the rotation interchanges the two conjugate lines  $a$  and  $b$ . It must, therefore, since only one such equidistant can be drawn with two given conjugate lines, bring the equidistant again into congruence with itself, while the two parts into which space is divided by the equidistant are interchanged. The two parts into which space is divided by this equidistant are therefore congruent with each other, and can be interchanged by a rotation through two right angles about any one of the generators of the equidistant. The construction for the division of space into  $2n$  congruent portions, any one of which can be brought to coincidence with each of the others by successive rotations through  $\frac{2\pi}{2n}$  round a line, is now almost obvious. With the given line as a generator, such an equidistant as is under consideration is described, and from it  $n-1$  more are formed by rotating it through angles  $\frac{\pi}{n}$ ,  $\frac{2\pi}{n}$ , ...  $\frac{(n-1)\pi}{n}$  round the given line. The  $n$  equidistants so formed divide space into  $2n$  parts with the required properties.

### III.

Returning now to the motions of hyperbolic space, it is to be noticed that the construction that has been given for the resultant of any two displacements fails in one case to lead to a definite result; viz., when the axes of the two displacements meet at infinity. This difficulty may be obviated by introducing between the two displacements whose resultant is required two arbitrarily chosen equal and opposite displacements; and combining, to begin with, the first given displacement with the first of the two thus introduced, and the second of these with the second given displacement. The axes of the two displacements thus obtained will not, unless the introduced displacements are specially chosen, meet at infinity, and with them the

original construction may be carried out. The axis of the resultant displacement will necessarily pass through the same point at infinity as the axes of the two given displacements; for this point, being undisplaced by each of the given displacements, is undisplaced by their resultant, and must therefore be one of the two points at infinity on the axis of their resultant. If the two given displacements are translations, the resultant is, as has already been seen, since the axes intersect, also a translation, and in this case a simple construction may be given for the axis and magnitude of the resulting translation. For this purpose I first recall the construction in the case when the axes intersect in a finite point. If  $AOA'$ ,  $BOB'$  are the intersecting axes, and if  $AO$ ,  $OB'$  be in direction and magnitude half the translations, then  $AB'$  is the axis of the resultant translation, and  $2AB'$  its magnitude. Now, let  $AI$ ,  $OI$  be the axes of the two given translations meeting in  $I$  at infinity. Then, if the translations are equal in magnitude and opposite in sense as regards  $I$  along the two lines, the construction just given shows that the axis of the resultant translation can have no finite point upon it, and therefore in this case it is useless to attempt to construct this axis. In any other case, the axis is a finite line passing through  $I$ , and therefore having a second point at infinity on it, say  $J$ . Draw a line  $AC$  meeting the two given lines, and not passing through the point  $J$ ; and take  $B$  such that  $AB$  is half the translation along  $AI$ . Join  $BO$  and produce it to  $B'$ , so that  $BO$  is equal to  $OB'$ , and then join  $B'$  to  $D$  on  $OI$ , where  $OD$  is half the translation along  $OI$ . Then the translation along  $AI$  is equivalent to translations  $2AO$  and  $2OB$  along  $AO$  and  $OB$  successively, and the translation along  $OD$  is equivalent to translations  $2OB'$  and  $2B'D$  along  $OB'$  and  $B'D$  successively.

Hence the two given translations are equivalent to  $2AC$  and  $2B'D$  along these lines, and from the construction it is impossible for these lines to meet at infinity; for, if they did, the axis of the resultant translation would pass through the point in which they met, while neither of the points  $I$  and  $J$ , at infinity on this axis lies on  $AC$ . Hence these two equivalent translations can be compounded in the ordinary way.

Returning now to the case in which the two translations are equal in magnitude and opposite in sense, the resultant motion might be characterized as a translation whose axis is at infinity. This is not intended to imply that in hyperbolic space it is correct to speak of lines at infinity, but the phrase is used to describe shortly a motion in hyperbolic space which has nothing completely analogous

to it in Euclidean or elliptic space. Indeed this motion may be equally well described as a rotation whose axis is at infinity. To verify this statement, and to bring out as clearly as possible the nature of this motion, I give the following construction. Draw that line  $OI$  meeting the axes  $AI, CI$  of the equal and opposite translations at infinity, with respect to which they are symmetrically situated. From  $O$ , let fall perpendiculars  $OA, OC$  on  $AI, CI$ , and take equal lengths  $AB, CD$  on  $AI, CI$ , equal in magnitude to half the translations, and measured either both towards or both from  $I$ , according as the translation along  $AI$  is in the direction  $AI$  or  $IA$ . From  $B$  and  $D$  draw  $BP, DP$  in the plane of the figure perpendicular to  $AI, CI$ , and meeting in  $P$ , which necessarily lies on  $OI$ . From  $P$  draw a perpendicular  $PQ$  to  $OA$ , and produce it to  $P'$  so that  $QP'$  is equal to  $PQ$ ; then join  $P'O$ , and through  $P'$  draw a line  $P'K$ , such that the angle  $KP'O$  is equal to the angle  $DPB$ , while equal rotations which bring  $P'K$  to  $P'O$  and  $PD$  to  $PB$  are in the same sense. Now the two translations are equivalent to successive half-turns round  $OA, PB, PD, OC$ . Successive half-turns round  $PB, PD$  are equivalent to a rotation round  $Pp$ , perpendicular to the plane of the figure, through twice the angle  $BPD$ . The half-turn round  $OA$  followed by this rotation is equivalent to an equal rotation in the opposite sense round  $P'p'$ , perpendicular to the plane of the figure, followed by a half-turn round  $OA$ . This equal rotation in the opposite sense round  $P'p'$  is equivalent to successive half-turns round  $P'K$  and  $P'O$ ; while, since the angle  $AOO$  is equal to the angle  $P'OP$ , successive half-turns round  $OA, OC$  are equivalent to successive half-turns round  $OP', OP$ . Hence, finally, the two translations are equivalent to successive half-turns round  $P'K$  and  $OP$ . Now  $OP$  passes through  $I$ , and the resultant displacement leaves  $I$  unchanged, so that  $P'K$  must also pass through  $I$ . The motion under consideration is therefore equivalent to successive half-turns about two lines in the same plane with the two original axes, and passing through the same point at infinity with them; in other words, it may be regarded as a rotation about an axis at infinity perpendicular to the plane of the figure. By such a motion every point in the plane  $AIC$  is displaced along the circle of infinite radius described through it with  $I$  as centre. This brings out in a striking manner the fact that in hyperbolic space a circle of infinite radius is not the same as a straight line. If through the lines  $AI, CI, \dots$  passing through the same point  $I$  in the original plane, planes be drawn perpendicular to this plane, the motion in question displaces each such plane into

another of the set; and if in these planes lines  $A'I$ ,  $O'I$ , ... be drawn, so that each pair of lines such as  $AI$ ,  $A'I$  or  $OI$ ,  $O'I$  is congruent with each other pair, then  $A'I$ ,  $O'I$ , ... lie in a plane, and are displaced among themselves by the motion. It is also easy to see that any two motions which keep the point  $I$  unchanged and displace every finite line passing through  $I$  are permutable with each other.

When the two component displacements about axes intersecting at infinity are rotations, the axis of the resultant rotation may be found at once by the same construction as that used when the axes intersect in a finite point.

It is now possible to analyse the general group of real motions in hyperbolic space, so far as concerns the complete enumeration of all types of sub-group contained within it. Owing to the fact, which will be proved immediately, that the group contains no self-conjugate sub-groups, it does not appear possible to present the structure of the group itself, without the consideration of imaginary motions, in a form in any way analogous to that in which the group of motions in elliptic space has been presented.

It has been proved in the earlier part of this paper that no infinitesimal motion in hyperbolic space transforms more than one line into itself. Now any continuous sub-group must contain some infinitesimal displacement, an infinitesimal screw-motion of given pitch, about some line. If then the sub-group is self-conjugate it must contain every conjugate operation within the main group, and therefore must contain a similar infinitesimal screw-motion about every line in space. But from such a set of motions, infinitesimal screw-motions of any pitch whatever can be constructed, and therefore the group in question must coincide with the main group of motions.

Again, a continuous sub-group which does not coincide with the main group, must be such that all of its operations transform either some one point, some one line, or some one plane into itself. For, if not, the group must contain infinitesimal motions displacing every point in three directions which do not all lie in a plane; and from these may be compounded infinitesimal motions displacing every point in all possible directions, and therefore also finite motions which will displace every point to every other point of space. If, then, the group contains an infinitesimal operation whose axis passes through some chosen point, it must contain conjugate operations whose axes pass through every other point of space, and from this property it may easily be seen to coincide with the main group.

Now there are  $\infty^3$  points at infinity,  $\infty^3$  finite points,  $\infty^5$  planes, and  $\infty^4$  lines in space. Hence there can be no sub-group containing  $\infty^5$  operations; for, if there were there would be  $\infty^1$  such conjugate sub-groups, and therefore the point, line, or plane, which is undisplaced by the group, would have only  $\infty^1$  different positions. Also any sub-group containing  $\infty^4$  operations must keep a point at infinity fixed. Now it has been seen that any two displacements whose axes meet at infinity have for their resultant another displacement whose axis passes also through the same point at infinity. Hence the totality of displacements whose axes meet in a point at infinity do actually form a group, and since there are  $\infty^3$  such axes and  $\infty^2$  displacements corresponding to each axis, the group contains  $\infty^4$  operations. There is, then, one type of such sub-group, and the type contains  $\infty^2$  conjugate sub-groups.

In any type of sub-group containing  $\infty^5$  operations, there must be  $\infty^3$  or  $\infty^5$  conjugate sub-groups, and in the former case the sub-group must be self-conjugate within a sub-group containing  $\infty^4$  operations. Now the sub-group just considered has been seen to contain two sub-groups with  $\infty^5$  operations, namely, those sub-groups made up of all its translations and of all its rotations respectively; and from their nature these are self-conjugate within the larger sub-group. Hence arise two types of sub-groups containing  $\infty^5$  operations, one consisting entirely of translations, and the other entirely of rotations, each keeping a point at infinity fixed, and each forming one of a set of  $\infty^3$  conjugate sub-groups. The only other possible types of sub-group containing  $\infty^5$  operations must contain  $\infty^5$  conjugate sub-groups, and must therefore keep either a finite point or a plane unchanged. Now the group of rotations round a point does actually consist of  $\infty^5$  operations, as also does the general group of motions in a plane, so that these two types exist and are completely accounted for.

To simplify the discussion of the remaining sub-groups it may be pointed out that of the sub-groups containing  $\infty^5$  operations the last two types are simple and contain no self-conjugate sub-groups, while the first two types contain self-conjugate sub-groups of the same type, or rather identical self-conjugate sub-groups, consisting of those motions, which, as has been seen, may be indifferently regarded as translations or rotations, whose axes lie at infinity. These sub-groups, moreover, are self-conjugate within groups of  $\infty^4$  operations. Thus arises a single type of sub-group containing  $\infty^3$  operations, and consisting of  $\infty^3$  conjugate sub-groups. Every other type of sub-group containing  $\infty^3$  operations must contain within it  $\infty^4$  conjugate sub-

groups. Hence it must either keep a line unchanged, or else a point at infinity, and a plane passing through it. Both of these types actually exist, the first consisting of all possible displacements with a given line for axis, and the latter of translations in a plane along lines passing through the same point at infinity. Lastly, sub-groups containing  $\infty^1$  operations must occur in conjugate sets of  $\infty^1$  at most, and must therefore be contained self-conjugately in the two preceding types.

Now displacements with a given line for axis are all permutable with each other, so that every sub-group is contained in such a group self-conjugately. The first of the two preceding types, therefore, gives rise to an infinite number of types of simply-infinite sub-groups, each consisting of those screw-motions with a given line for axis which have a given pitch, and including as limiting cases simply-infinite sets of translations and rotations respectively. The second of the two preceding types contains a single self-conjugate sub-group, namely, the set of motions which have been spoken of as rotations about an axis at infinity. This forms the one other type of simply-infinite sub-group.

The only discontinuous groups of motion of finite order in hyperbolic space are the known finite groups of rotations round a point; for such a group cannot contain any displacement other than a rotation, as otherwise it could not be of finite order, and for the same reason it cannot contain rotations about non-intersecting axes. On the other hand, of discontinuous groups of motion, whose orders are not finite, there is in hyperbolic space an infinite variety. The truth of this statement may be made clear as follows, by considering certain discontinuous groups of plane motions. If from a point  $O$  three equal lines  $OA, OB, OC$  are drawn in a plane and making equal angles with each other, and through  $A, B, C$  lines are drawn perpendicular respectively to  $OA, OB, OC$ , these lines, when  $OA$  is infinitesimal, will form an infinitesimal equilateral triangle, whose angles are infinitesimally less than  $\frac{\pi}{3}$ . As  $OA$  is taken greater and greater

the angles of the triangle become less and less, and for a certain length of  $OA$  each pair of sides will meet at infinity, and the angles of the triangles will be zero. Hence, equilateral triangles can be constructed in hyperbolic space, whose angles are  $\frac{\pi}{n}$ , where  $n$  is any integer greater than 3.

If, now, such a triangle be drawn in a plane, and on each of its sides an equal triangle be constructed, and if this process be continued

indefinitely, the whole plane will be divided into an infinite number of such congruent equilateral triangles without gaps or overlapping,  $2n$  triangles being ranged round every angular point. When planes are drawn through the sides of the triangles perpendicular to their plane, the whole of space is divided into what may be described as equilateral prisms, all of which are congruent with each other. Moreover, by rotations  $\frac{2\pi}{3}$ ,  $\pi$ ,  $\frac{2\pi}{n}$  about perpendiculars to the plane of the triangles through  $O$ , the middle point of a side and an angular point respectively, this infinite set of equilateral prisms is brought to congruence with itself. Hence, of necessity, these three rotations generate a discontinuous group of motions.

Another very simple, but interesting, illustration of the division of space into congruent parts, and of the corresponding discontinuous group of motions, arises in connexion with the regular solids. From a point  $O$ , perpendicular to a line  $OI$ , draw  $n$  lines  $OO_1, OO_2, \dots OO_n$ , equal and equally inclined to each other; and through their extremities draw lines  $O_1I, O_2I, \dots$  to meet  $OI$  at infinity. By taking a section of the prismatic figure so formed at a sufficiently great distance from  $O$ , the size of the section can be made as small as desired, and, therefore, the dihedral angles at the edges must be the same as for an infinitesimal figure. Each of these dihedral angles is, therefore,  $\frac{n-2}{n}\pi$ , which, for  $n = 3, 4, 5$ , gives  $\frac{\pi}{3}$ ,  $\frac{\pi}{2}$ ,  $\frac{3\pi}{5}$ . Hence if a regular solid be described with its vertices at infinity, the internal dihedral angle between two adjacent faces will be  $\frac{\pi}{3}$  for a tetrahedron, cube, or dodecahedron,  $\frac{\pi}{2}$  for an octohedron, and  $\frac{3\pi}{5}$  for an icosahedron.

With the exception of the last, these angles are all submultiples of four right angles; and, therefore, in the first four cases, if the original solid is rotated about its edges, through the dihedral angle, the new figures so formed rotated about their edges, and so on indefinitely, the whole of space will be exactly filled, without gaps, with congruent figures. It may be added here, without proof, as the result depends only on certain simple inequalities, that there are only four\* other ways

\* While these pages are passing through the press, I have become acquainted with a paper by Signor L. Bianchi: "Sulle divisioni regolari dello spazio non euclideo in poliedri regolari" (*Rendiconti, Accademia dei Lincei*, 1893), in which it is stated that there are only two modes of division of hyperbolic space into congruent regular polyhedra. It appears to me that Signor Bianchi has introduced an unnecessary limitation into his discussion; but it is impossible to discuss this point adequately in a footnote, and I shall hope to return to it in a future paper.

of dividing hyperbolic space into equal and congruent regular solids. These are: (i) cubes, there being twenty cubes arranged round each vertex with icosahedral symmetry; (ii) and (iii) dodecahedra, there being either eight or twenty dodecahedra arranged round each vertex with octohedral and icosahedral symmetry respectively; (iv) icosahedra, there being twelve icosahedra arranged round each vertex with dodecahedral symmetry.

There is a marked difference, as regards discontinuous groups of motion, between hyperbolic space, on the one hand, and elliptic and Euclidean space, on the other. It has been seen above that for elliptic space there are only a finite number of types of such groups, and in Euclidean space Herr Schönflies (*Krystallsysteme und Krystallstruktur*: Teubner, 1891), among others, has shown that there are just 65 types.

#### IV.

Returning now again to the motions of elliptic space, it is interesting to point out that it is only necessary to investigate some analytical form of the group of rotations round a point (a problem of group-theory) in order to pass on from the foregoing purely synthetical considerations to the complete metrical system for elliptic space.

The most symmetrical analytical form of the group of rotations round a point is that in which it is regarded as that group of homogeneous projective transformations of three variables  $q_1, q_2, q_3$  which keep the form

$$q_1^2 + q_2^2 + q_3^2$$

unchanged. Hence, if  $q_1, q_2, q_3, q_4, q_5, q_6$  are six independent variables, the group of motions in elliptic space can be expressed as that group of homogeneous projective transformations of these variables which keep the two forms

$$q_1^2 + q_2^2 + q_3^2 \quad \text{and} \quad q_4^2 + q_5^2 + q_6^2$$

unchanged.

If, now, new variables  $p_1, p_2, p_3, p_4, p_5, p_6$  are introduced, such that

$$p_1 = q_1 + q_4, \quad p_2 = q_2 + q_5, \quad p_3 = q_3 + q_6,$$

$$p_4 = q_1 - q_4, \quad p_5 = q_2 - q_5, \quad p_6 = q_3 - q_6,$$

the group, expressed in terms of the  $p$ 's, is that homogeneous projective group which keeps unchanged

$$p_1 p_4 + p_2 p_5 + p_3 p_6$$

and

$$p^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2$$

The  $p$ 's may therefore be regarded as homogeneous line-coordinates in ordinary space, and when they are so regarded the equation

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 = 0$$

represents a quadric surface which contains no real points. The group of motions in elliptic space is, therefore, abstractly considered, identical with that group of projective transformations in ordinary space which preserves unchanged a purely imaginary quadric; and this is the starting-point from which the metrical relations of elliptic space are actually derived.

[*Added, December 28th.*

Since the group of real motions in hyperbolic space is a simple group, it is not possible to determine its analytical form by a process precisely analogous to that employed above for the group of elliptic motions. On the other hand, the group having been exhaustively analysed, it becomes a problem of pure group-theory to make this determination. It will be simplified by the following considerations. To every motion of hyperbolic space corresponds a transformation of the points at infinity, and no motion keeps more than two points at infinity unchanged. Hence between the group of motions in hyperbolic space and the group of transformations of the points at infinity, that they involve, there is a one-to-one correspondence; i.e., the groups are, abstractly considered, identical. Now, the points at infinity form a doubly-infinite set, and, therefore, a transformation group of the points at infinity may be represented as a transformation-group of points in an ordinary plane. Again, if  $IJ$ ,  $I'J'$  be any two lines of hyperbolic space, and  $PK$ ,  $P'K'$  any other two lines meeting the former two respectively at right angles, a single motion can be found which will bring  $IJ$ ,  $PK$  into the positions  $I'J'$ ,  $P'K'$ . Hence the group of transformations of the points at infinity, or the corresponding transformation-group of points in a plane, is such that it contains a single transformation which will bring any three arbitrarily chosen points into any other three arbitrarily chosen positions; or, in the phraseology of group-theory, the transformation-group is triply-transitive. The group of motions of hyperbolic space is, therefore, capable of being represented in the form of a triply-transitive group of  $\infty^2$  transformations of points in a plane. Now, it can be shown that of such groups there is one type, and one only—groups between which a one-to-one correspondence can be established, being, of course, regarded as identical (*cf.* Lie-Scheffers, *Vorlesungen über kontinuierliche Gruppen*, pp. 355, 356).

The particular form of the group which it is most convenient to consider here is that group of point-transformations which arises from an even number of inversions at all real circles of a plane. It is easy to see that this group contains  $\infty^6$  transformations, and that it contains one, and just one, transformation which will displace any three given points into any other three. Moreover, if the equation to any circle be written in the form

$$\alpha(x^2 + y^2 + 1) + 2\beta x + 2\gamma y + \delta(x^2 + y^2 - 1) = 0,$$

so that the square of its radius is

$$\frac{\alpha^2 + \beta^2 + \gamma^2 - \delta^2}{(\alpha + \delta)^2},$$

the group in question, when expressed in terms of the symbols  $\alpha, \beta, \gamma, \delta$ , is easily found to be that homogeneous projective group which keeps

$$\alpha^2 + \beta^2 + \gamma^2 - \delta^2$$

unchanged, this latter condition corresponding to the fact that, by inversion at real circles, a real circle necessarily remains a real circle.

If now, finally,  $\alpha, \beta, \gamma, \delta$  are regarded as homogeneous point-coordinates in ordinary space, the group of hyperbolic motions is seen to be identical with that projective group of ordinary space which transforms a real quadric with imaginary generators into itself.]

*Thursday, December 13th, 1894.*

Major MACMAHON, R.A., F.R.S., President, and subsequently  
A. E. H. LOVE, Esq., F.R.S., Vice-President, in the Chair.

The following gentlemen were elected members of the Society:—  
William Henry Young, M.A., formerly Fellow of Peterhouse, Cambridge; William Montgomery Coates, M.A., Fellow and Assistant Tutor of Queens' College, Cambridge; Philip Herbert Cowell, B.A., Fellow of Trinity College, Cambridge; Gilbert Harrison John Hurst, B.A., Scholar of King's College, Cambridge; Horace J. Harris, B.A.,

University College, London; Ernest William Brown, M.A., Fellow of Christ's College, Cambridge, and Professor of Mathematics in Haverford College, Pennsylvania.

The Treasurer having read the Auditor's report, the adoption of the Treasurer's report was moved by Mr. Kempe, seconded by Prof. Rogers, and carried unanimously. A vote of thanks to the Auditor for the trouble he had taken was moved by Prof. Hill, seconded by Mr. Walker, and carried.

The following communications were made:—

On Maxwell's Law of Partition of Energy: Mr. G. H. Bryan.  
The Spherical Catenary; and The Transformation of Elliptic Functions: Prof. Greenhill.

On certain Definite Theta-Function Integrals: Prof. Rogers.  
Groups defined by Congruences (second paper): Prof. W. Burnside.

Vibrations in Condensing Systems: Dr. J. Larmor.

On the Integration of Allégret's Integral: Mr. A. E. Daniels.

On the Complex Number formed by two Quaternary Matrices  
Dr. G. G. Morrice.

The Chairmen, Messrs. Bryan, Greenhill, Rogers, Larmor, and Walker took part in the discussions on the papers.

The following presents were received:—

"The Imperial University of Japan Calendar," 1893-4.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xviii., St. 10, 11; Leipzig, 1894.

"Proceedings of the Cambridge Philosophical Society," Vol. viii., Part 3; 1894.

"Proceedings of the Royal Society," Vol. lvi., Nos. 338-339.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. viii., No. 3; 1893-4.

"Bulletin de la Société Mathématique de France," Tome xxii., No. 8; Paris.

"Bulletin des Sciences Mathématiques," Tome xviii., Oct., Nov.; 1894.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxviii., Livr. 3 and 4; Harlem, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. i., No. 2.

"Journal of the College of Science, Japan," Vol. viii., Pt. 1; Tokyo, 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. iii., Fasc. 8-9, Sem. 2<sup>a</sup>; Roma, 1894.

"Educational Times," December, 1894.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. viii., Fasc. 8-10; Napoli, 1894.

"Observations made during 1889 at the United States Naval Observatory," 4to; Washington, 1893.

Balbin, V.—“Tratado de Geometría Analítica,” 8vo; Buenos Ayres, 1888.  
 “Tratado de Estereometría Genética,” 8vo; Buenos Ayres, 1894. “Método de los Cuadrados Mínimos,” 8vo; Buenos Ayres, 1889. “Elementos de Cálculo de los Cuaterniones,” 8vo; Buenos Ayres, 1887. “Geometría Plana Moderna,” 8vo; Buenos Ayres, 1894.

D'Ocagne, M.—“Mémoire sur les Suites Récurentes,” 4to pamphlet.

“Annales de l'Ecole Polytechnique de Delft,” Tome VIII., Livr. 1-2; Leide, 1894.

“Annales de la Faculté des Sciences de Toulouse,” Tome VIII., Fasc. 4; Paris, 1894.

“Journal für die reine und angewandte Mathematik,” Bd. cxiv., Heft 2; Berlin, 1894.

“Transactions of the Royal Irish Academy,” Vol. xxx., Parts 13 and 14; Dublin, 1894.

“Indian Engineering,” Vol. xvi., Nos. 16-20; Oct. 20th-Nov. 17th.

*On a Class of Groups defined by Congruences. (Second Paper.)*

By W. BURNSIDE. Received December 7th, 1894. Read December 13th, 1894.

1. *Introduction.*

In a paper printed in Vol. xxv of the Society's *Proceedings*, I have discussed the groups defined by a congruence of the form

$$z' \equiv \frac{\alpha z + \beta}{\gamma z + \delta} \pmod{p},$$

where  $p$  is prime, and  $\alpha, \beta, \gamma, \delta$  are rational integral functions of the roots of an irreducible congruence of the  $n^{\text{th}}$  degree to the same prime modulus.

This discussion was greatly facilitated by the fact that the groups defined by a congruence of the same form in which the coefficients are ordinary integers had been already exhaustively analysed.

Now the corresponding group in two non-homogeneous variables, namely, the group defined by the congruences

$$x' \equiv \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad y' \equiv \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''} \pmod{p},$$

has not hitherto been the subject of any similar discussion. If the