ON ABSOLUTELY CONVERGENT IMPROPER DOUBLE INTEGRALS

By E. W. HOBSON.

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THE theory of those improper double integrals in which the domain of integration is limited has been developed on the basis of two distinct definitions: that of Jordan and that of de la Vallée-Poussin. In the first part of the present paper these two definitions are compared, and are shown to be completely equivalent to one another. These definitions only admit of the existence of such improper double integrals as are absolutely convergent. A definition of a less stringent character is required in order to admit of the existence of non-absolutely convergent improper double integrals, but in the present communication those double integrals only will be taken into account which exist in accordance with either of the two definitions above cited. A wider definition due to Lebesgue has also been considered. The second part of the paper is concerned with the conditions under which the double integrals can be replaced by repeated integrals. This matter has been elaborately treated by de la Vallée-Poussin, who has obtained, in the case of a function which is never negative, a necessary and sufficient condition for the equivalence in question. His theorem has, however, only been hitherto established under certain restrictive conditions, which impair the generality of the result. In the present communication, the theorem is established without any such restrictive assumption. The recent development of the theory of the measure of sets of points, by Borel and Lebesgue, has made this more general treatment of the question possible. A more general definition of regular convergence than that of de la Vallée-Poussin is here introduced, including the latter as a special case.

The definition here introduced is of a very general character, and is applicable to cases in which the functions of a sequence, and also the limiting function defined by the sequence, are not restricted to be limited functions, or to have definite values for every value of the variable, but may be indefinite between limits of indeterminacy, either of which may be finite or infinite. It is shown that Arzelà's "convergenza uniforme a tratti in generale" is equivalent to that particular case of regular convergence except at the points of a set of zero measure, which arises when all the functions of the sequence are limited functions, and have definite values for each value of the variable.

A short discussion is given of the view which has been maintained by Schönflies, that every improper double integral can be replaced by the corresponding repeated integrals, and it is shown that this view is untenable.

As regards notation, the double integral of $f(x, y)$ is denoted by $\int f(x, y)(dx\,dy)$, in recognition of the fact that a proper double integral is defined as a single limit: the word "double" must be taken to refer to the two dimensions of the domain for which the integral is defined. On the other hand, a repeated integral $\int dx \int f(x, y)\,dy$, or $\iint f(x, y)\,dx\,dy$ is properly represented by a double use of the sign of integration, since it is defined as a repeated limit.

The Definition of Improper Double Integrals.

1. Let G be a finite plane domain, that is a set of points (x, y) such that $|x|$, $|y|$ have definite upper limits when all the points of G are taken into account. Let it further be assumed that the domain G has a frontier which is of plane content zero, the term frontier being used in the sense employed by Jordan, as consisting of the set of points each of which is either a point of G which is also a limiting point of a sequence of points not belonging to G, or else a point not belonging to G which is also a limiting point of a sequence of points belonging to G. Let $f(x, y)$ be a function defined for all points of the domain G; this function may be replaced by a function defined for all points in a fundamental rectangle with sides parallel to the coordinate axes, and containing G. The function is defined to have the same value as $f(x, y)$ at all points of G, and to be zero at all points of the fundamental rectangle that do not belong to G. We may denote the function so extended by $f(x, y)$, as before. If the function $f(x, y)$ is such that at each point of a certain closed set K_∞ the function has an infinite discontinuity, the integral of $f(x, y)$ taken over the fundamental rectangle is said to be an improper double integral.

The following definition of an improper double integral is substantially that given by Jordan:—

Let $D_1, D_2, \ldots, D_n, \ldots$ denote a sequence of domains contained in the fundamental rectangle, each one of which consists of a finite number of connex closed portions with its frontier of zero plane content, and in which the number of portions may increase indefinitely with n. Let it be assumed that the function $f(x, y)$ is such that the closed set of points K_∞ of infinite discontinuity of the function has zero content, and that none of the domains D_n contain, in their interiors or on their boundaries, any point of K_∞. Let the sequence $\{D_n\}$ be such that the content of D_n converges to that of the fundamental rectangle. Then, if $f(x, y)$ is integrable in every domain such as D_n, and the integrals

$$\int_{D_1} f(x, y)(dx\,dy), \quad \int_{D_2} f(x, y)(dx\,dy), \quad \ldots, \quad \int_{D_n} f(x, y)(dx\,dy), \quad \ldots$$

converge to a definite limit independent of the particular sequence $\{D_n\}$ chosen, subject to the conditions stated, this limit is said to define the improper double integral $\int f(x, y)(dx\,dy)$ of $f(x, y)$ in the domain G.

It has been shown by Jordan that whenever $\int_G f(x, y)(dx\,dy)$ exists, in accordance with the above definition, then $\int_G |f(x, y)|\,(dx\,dy)$ also exists, so that every improper double integral, so defined, is absolutely convergent.

The following definition, different from that of Jordan, has been given by de la Vallée-Poussin :—

Let $f_n(x, y)$ be that function which is such that $f_n(x, y) = f(x, y)$ at every point (x, y) at which $M_n \geqslant f(x, y) \geqslant -N_n$, where M_n, N_n are two positive numbers, and that $f_n(x, y) = M_n$ at every point where $f(x, y) > M_n$; and also $f_n(x, y) = -N_n$ at every point where $f(x, y) < -N_n$. If $f(x, y)$ be such that the proper integral $\int f_n(x, y)(dx\,dy)$ over the fundamental rectangle exists whatever positive values M_n, N_n may have, then, if the sequence

$$\int f_1(x, y)(dx\,dy), \quad \int f_2(x, y)(dx\,dy), \quad \ldots, \quad \int f_n(x, y)(dx\,dy), \quad \ldots$$

has a definite limit, provided the sequences $\{M_n\}$, $\{N_n\}$ have no upper limits, and if this limit is independent of the particular sequences $\{M_n\}$, $\{N_n\}$ chosen, subject to the condition stated, then this limit defines the improper double integral $\int f(x, y)(dx\,dy)$ over the fundamental rectangle.

It has been shown by Schönflies that when the integral exists, in accordance with this definition, the set of points K_∞ must have zero content.

It is easily seen that every improper integral so defined is absolutely convergent.

The theory of absolutely convergent improper integrals has been developed on two independent lines from the two definitions given above as starting points. It will here be shown that the two definitions are completely equivalent to one another.*

2. In accordance with either of the definitions the existence of the absolutely convergent improper integral of $f(x, y)$ implies that of each of the two functions $f^+(x, y)$, $f^-(x, y)$, where $f^+(x, y)$ is defined by the conditions that $f^+(x, y) = f(x, y)$, at any point at which $f(x, y)$ is positive, and everywhere else $f^+(x, y) = 0$; and similarly

$$f^-(x, y) = -f(x, y),$$

where $f(x, y)$ is negative, and everywhere else $f^-(x, y) = 0$. It is consequently clear that, in order to show the complete equivalence of the two definitions, it is sufficient to consider the case in which $f(x, y)$ is everywhere either positive or zero. Let us then assume that the function $f(x, y)$, which is never negative, has an improper integral in accordance with Jordan's definition.

Let the set of points K_∞ be enclosed in a finite set of rectangles $\{\delta\}$, and let the remaining part of the fundamental rectangle consist of a set of non-overlapping rectangles $\{\eta\}$. The sum $\Sigma\delta$ can be chosen so small that the integral of $f(x, y)$ through the rectangles $\{\eta\}$ is less than the improper integral by less than an arbitrarily chosen positive number ρ.

Let N be a positive number not less than the upper limit of $f(x, y)$ in all the rectangles $\{\eta\}$, and let $f_n(x, y)$ be the function, corresponding to N, employed in de la Vallée-Poussin's definition. Let another set of non-overlapping rectangles $\{\delta'\}$ interior to the set $\{\delta\}$ also enclose all the points of K_∞, and let $\{\eta'\}$ be the finite set of rectangles complementary to $\{\delta'\}$. The integral of $f(x, y)$ over $\{\eta'\}$ lies between the value of the integral over $\{\eta\}$ and that of Jordan's improper integral, and therefore differs from the latter by less than ρ. It follows that the integral of $f(x, y)$ through the area obtained by removing the set $\{\delta'\}$ from the set $\{\delta\}$ is also $< \rho$; and, since $f_n(x, y) \leqslant f(x, y)$, we see that the integral of $f_n(x, y)$ over the same area is $< \rho$. From this we deduce that $\int f_n(x, y)\,(dx\,dy)$ taken through the rectangles $\{\delta\}$ is $< \rho + N\Sigma\delta'$; and, since this holds for an arbitrarily small value of $\Sigma\delta'$, N being fixed, we see that $\int f_n(x, y)\,(dx\,dy)$ taken over the rectangles $\{\delta\}$ is $\leqslant \rho$.

It now follows that the difference of the integrals of $f_n(x, y)$ taken over the fundamental rectangle and over the rectangles $\{\eta\}$ is $\leqslant \rho$; and, since ρ is arbitrarily small, N being sufficiently increased, it follows that

* Stolz seems to imply that the definition of de la Vallée-Poussin is in some sense more general than that of Jordan, which Stolz has himself adopted as the basis of his own treatment. See the *Grundzüge*, Vol. III., p. 124.

the integral of $f_n(x, y)$ over the fundamental rectangle has a definite limit when n is indefinitely increased, and that this limit is Jordan's improper integral. It has thus been shewn that a function which has an improper integral in accordance with Jordan's definition has one also in accordance with the definition of de la Vallée-Poussin ; the integrals having the same value in the two cases.

To prove the converse, we assume that

$$\int_{\{\delta\}} f_n(x, y)(dx\,dy) + \int_{\{\eta\}} f(x, y)(dx\,dy)$$

has a definite limit as n is indefinitely increased and $\Sigma\delta$ is indefinitely diminished, the value of N being fixed, as before, for each set $\{\eta\}$. Since both the integrals are positive, it follows that $\int_{\{\eta\}} f(x, y)(dx\,dy)$, which increases as $\Sigma\delta$ is diminished, is less than a fixed finite number, and therefore has a definite upper limit. It has thus been shewn that there exists a special class of domains $\{D_n\}$ such that $\int_{D_n} f(x, y)(dx\,dy)$ has a definite upper limit as the content of D_n converges, with increasing n, to the content of the fundamental rectangle. These domains D_n are complementary to a finite set of rectangles enclosing the points K_∞. It remains to be shewn, in order to establish the existence of Jordan's improper integral, that, if any other set of domains $\{D'_n\}$ be chosen such that the content of D'_n converges to that of the fundamental rectangle, but such that D'_n is not restricted to be complementary to a finite set of rectangles $\{\delta\}$, then $\int_{D'_n} f(x, y)(dx\,dy)$ converges to the same limit as $\int_{D_n} f(x, y)(dx\,dy)$ does. Denoting the content of D'_n by $m(D'_n)$, and that of the fundamental rectangle by A, let D'_n be such that $A - m(D'_n) = \epsilon_n$. The domain D_n can be so chosen as to contain D'_n in its interior. For, since D'_n does not contain, in its interior or on its boundary, any points of K_∞, it follows* that for each point of K_∞ the distance from all the points of D_n has a minimum greater than zero. Hence each point of K_∞ can be enclosed in a rectangle which contains no points of D'_n in its interior or on its sides. The set K_∞ being closed, a finite set of these rectangles can, in accordance with a known extension of the Heine-Borel theorem, be chosen so as to enclose the whole set of points K_∞ ; and the complement of this finite set

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* This is a consequence of the connexity of the domains D_n, D'_n. Jordan, in his definition (*Cours d'Analyse*, Vol. II., p. 76), does not explicitly state that D_n is made up of a finite number of connex portions. He describes it as "mesurable et parfait."

of rectangles may be taken as D_n. This may be done for each value of n. If $m(D'_n)$ converges to A, it is clear that $m(D_n)$, which as just chosen is $> m(D'_n)$, also converges to A. Also a number $n' > n$ can be determined such that $D'_{n'}$ encloses D_n in its interior; we have then

$$\int_{D_{n'}} f(x, y)(dx\,dy) \geqslant \int_{D'_{n'}} f(x, y)(dx\,dy) \geqslant \int_{D_n} f(x, y)(dx\,dy)$$
$$\geqslant \int_{D'_n} f(x, y)(dx\,dy).$$

If then $\int_{D_n} f(x, y)(dx\,dy)$ converges to a definite limit $\int_A f(x, y)(dx\,dy)$, n may be taken so great that

$$\int_A f(x, y)(dx\,dy) - \int_{D_n} f(x, y)(dx\,dy) < \eta,$$

where η is an arbitrarily chosen positive number; then also

$$\int_A f(x, y)(dx\,dy) - \int_{D'_{n'}} f(x, y)(dx\,dy) < \eta,$$

and it thus follows that $\int_{D'_{n'}} f(x, y)(dx\,dy)$ also converges to the limit $\int_A f(x, y)(dx\,dy)$. It has now been shewn that the existence of Jordan's improper double integral is a necessary consequence of the existence of that of de la Vallée-Poussin; and the two definitions have thus been shewn to be completely equivalent.

Lebesgue's Definition of an Improper Integral.

3. A limited function $f(x, y)$ which is everywhere positive or zero in the fundamental rectangle being defined, the integral $\int f(x, y)(dx\,dy)$ has been defined by Lebesgue* as follows:—

Denoting by U the upper limit of $f(x, y)$ in its domain, let $u_0, u_1, u_2, \ldots, u_n$, where $u_0 = 0$, $u_n = U$, be a set of numbers such that $u_1 - u_0$, $u_2 - u_1$, ..., $u_n - u_{n-1}$ are all positive, the greatest of them being η. Let e_r denote the set of points (x, y) for which $f(x, y) = u_r$, and $\bar{e}_r$ the set of points for which $u_r < f(x, y) < u_{r+1}$, and let $m(e_r)$, $m(\bar{e}_r)$ denote

* See his memoir "Intégral, Longueur, Aire," *Annali di Matematica*, Ser. 3A, Vol. VIII., 1902.

the plane measure of e_r and $\bar{e}_r$ respectively. Consider the two sums

$$\sigma = \sum_{r=0}^{r=n} u_r m(e_r) + \sum_{r=0}^{r=n-1} u_r m(\bar{e}_r),$$

$$\sigma' = \sum_{r=0}^{r=n} u_r m(e_r) + \sum_{r=0}^{r=n-1} u_{r+1} m(\bar{e}_r).$$

It can be shewn that, if the number n be increased indefinitely, whilst at the same time η converges to zero, then σ and σ' converge to one and the same number; and that this number is independent of the mode in which the interval $(0, U)$ is divided by the set of numbers $u_1, u_2, \ldots, u_{n-1}$, and is independent of the mode in which the successive further sub-division of the interval $(0, U)$ proceeds, subject to the condition that η, the greatest of the differences $u_r - u_{r-1}$, converges to zero as the sub-division is continued indefinitely. The common limit of σ and σ' is defined to be the double integral $\int f(x, y)(dx\,dy)$, and it is shewn that the integral so defined always exists provided only that $f(x, y)$ is a *summable* function, *i.e.*, a function such that the set of points (x, y) for which $A < f(x, y) < B$ is a measurable set, for every pair of values of A and B.

The integral of a limited function which is not necessarily positive is then defined by

$$\int f(x, y)(dx\,dy) = \int f^+(x, y)(dx\,dy) - \int f^-(x, y)(dx\,dy),$$

where $f^+(x, y)$ is equal to $f(x, y)$ or 0 according as $f(x, y) \geqslant 0$ or < 0, and $f^-(x, y)$ is equal to $-f(x, y)$ or 0 according as $f(x, y)$ is negative or not. This definition being applicable to every summable function, it is wider than the ordinary Riemann definition of a double integral, and includes all the functions defined by ordinary processes. The condition that the plane measure of all the points of discontinuity of the function shall be zero, which is necessary for the existence of the Riemann integral, is not necessarily satisfied by a function which possesses a Lebesgue integral. It is not definitely known whether every limited function is summable or not. Lebesgue has shewn that, when the Riemann integral exists, the integral in accordance with his own definition also exists, and that the two are identical in value. When the Riemann integral does not exist, Lebesgue's integral lies between the upper and lower integrals of the function as defined by Darboux.

Lebesgue has extended his definition so as to afford a definition of an absolutely convergent improper integral. It is clearly sufficient to take the case of an unlimited function $f(x, y)$ which is nowhere negative in the fundamental rectangle. The definition is substantially as follows :—

Let $u_0, u_1, u_2, \ldots, u_n, \ldots$ be a sequence of increasing numbers, such that $u_0 = 0$, and that u_n has no upper limit as the index n is indefinitely increased; also let the differences u_1-u_0, u_2-u_1, ..., $u_{n+1}-u_n$, ... be limited, having η as their upper limit. Consider the two series

$$\sigma = \sum_{r=0}^{\infty} u_r m(e_r) + \sum_{r=0}^{\infty} u_r m(\bar{e}_r),$$

$$\sigma' = \sum_{r=0}^{\infty} u_r m(e_r) + \sum_{r=0}^{\infty} u_{r+1} m(\bar{e}_r).$$

Since the difference of the two series is

$$\sum_{r=0}^{\infty} (u_{r+1}-u_r)\, m(\bar{e}_r),$$

which is less than $\eta \sum_{r=0}^{\infty} m(\bar{e}_r)$, it is clear that the two series are either both convergent or are both divergent. Let us suppose that the series are both convergent; it can then be shewn that they are still convergent when further numbers are interpolated between each consecutive pair of the numbers u_0, u_1, u_2, ..., and the corresponding new series are formed. It can then be further shewn that, as the process of successive sub-division of the interval $(0, \infty)$ proceeds in any manner consistent with the continual diminution of η to the limit zero, the sums σ, σ' both converge to a single number, for $\lim \eta = 0$; in fact σ constantly increases, and σ' constantly decreases. The value to which σ, σ' converge can be shewn to be independent of the original mode of sub-division of the interval $(0, \infty)$, and of the precise mode in which the further sub-division proceeds. The common limit of σ, σ', when it exists, is then defined to be the value of the improper integral $\int f(x, y)(dx\, dy)$.

In order that an improper integral may exist, it is necessary, though not sufficient, that $f(x, y)$ be a summable function, and also that the measure of those points (x, y) at which $f(x, y)$ is greater than or equal to an arbitrarily great number N shall be arbitrarily small for a sufficiently great value of N. For it is a necessary consequence of the convergence of the above series that $\sum_{r=n}^{\infty} \{m(e_r)+m(\bar{e}_r)\}$, which is the plane measure of the set of points at which $f(x, y) \geqslant u_n$, should have a value which converges to zero, as n and u_n are indefinitely increased. It is, however, not necessary that the content of the set K_∞ of all the points of infinite discontinuity should be zero; in fact it is even possible that the improper integral may exist whilst every point of the fundamental rectangle is a point of infinite discontinuity.

It will now be shewn that Lebesgue's definition of an improper integral can be replaced by one which differs from that of de la Vallée-Poussin, only in the one respect that the convergent sequence of proper integrals $\int f_n(x, y)(dx\,dy)$ consists of Lebesgue integrals, which are not necessarily Riemann integrals.

From the condition of convergence of the second series, corresponding to an arbitrarily chosen positive number ϵ, we may determine s so that

$$\sigma' = \sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + R,$$

where $R < \epsilon$, whilst at the same time η is so small that σ' differs from $\int f(x, y)(dx\,dy)$ by less than ϵ. Now let $u_s = N$, and let $f_n(x, y)$ be that function which $= f(x, y)$, for $f(x, y) < N$, and $= N$, for $f(x, y) \geqslant N$.

The Lebesgue proper integral $\int f_n(x, y)(dx\,dy)$ is then the limit when η converges to zero of the sum

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + u_s \sum_{r=s+1}^{\infty} m(e_r) + u_s \sum_{r=s}^{\infty} m(\bar{e}_r),$$

and this sum is equal to

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} u_{r+1} m(\bar{e}_r) + S,$$

where $S < R < \epsilon$. Keeping $u_r = N$ fixed, we may now, if necessary, diminish η by interpolating further numbers between the pairs of numbers $u_0, u_1, u_2, \ldots, u_r, \ldots$, until we have the new sum which corresponds to

$$\sum_{r=0}^{r=s} u_r m(e_r) + \sum_{r=0}^{s-1} m(\bar{e}_r) + S$$

differing from $\int f_n(x, y)(dx\,dy)$ by less than ϵ, the part S not having been increased by any diminution of η. We thus find that σ' differs from $\int f_n(x, y)(dx\,dy)$ by less than ϵ, when N is sufficiently great and η sufficiently small; also σ' has been taken to differ from $\int f(x, y)(dx\,dy)$ by less than ϵ, η having been chosen sufficiently small. Since ϵ is arbitrarily small, it is clear that $\int f_n(x, y)(dx\,dy)$ converges to $\int f(x, y)(dx\,dy)$ as N is increased indefinitely.

It has therefore been shewn that de la Vallée-Poussin's definition of an improper double integral may be extended to the case in which the integrals $\int f_n(x, y)(dx\,dy)$ exist only in the sense defined by Lebesgue. This definition is then equivalent to that of Lebesgue. It is clear that Jordan's definition is only capable of extension, in the case in which K_∞ has zero content; for otherwise the measures of the domains D_n do not converge to that of the fundamental rectangle; and, in fact, in case K_∞ contains every point of the fundamental rectangle, no such domains as the D_n exist.

If the condition that K_∞ have zero content be satisfied, the whole of the reasoning in § 2 is applicable without essential change, and in that case Jordan's definition of an improper integral can be extended to the case in which the proper integrals $\int_{D_n} f(x, y)(dx\,dy)$ exist only in the sense defined by Lebesgue. Thus in this case all three definitions are equivalent to one another.*

The Regular Convergence of a Sequence of Functions.

Let $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... be a sequence of functions defined for the interval (a, b). We shall suppose that, for each value of x, any one of these functions $\phi_n(x)$ has either a definite value or is multiple-valued, and is then regarded as indeterminate, between limits of indeterminacy,† of which the upper limit may be denoted by $\overline{\phi_n(x)}$, and the lower limit by $\underline{\phi_n(x)}$. For any value of x for which $\phi_n(x)$ is determinate, we have $\overline{\phi_n(x)} = \underline{\phi_n(x)}$. When either $\overline{\phi_n(x)}$ or $\underline{\phi_n(x)}$ is to be taken indifferently, we may use the notation $\overline{\underline{\phi_n(x)}}$. The consideration of a function $\phi_n(x)$ which, for a particular set of values of x, is indeterminate between limits of indeterminacy, as a single function, involves an extension of Dirichlet's definition of a function, which is justified by its convenience for use in investigations such as the present one. This extension is convenient when the functional value of $\phi_n(x)$ at a point x is defined by means of a limit, say $(\phi_n x) = \lim_{m=\infty} \psi_n(x, m)$, such that, for a particular value of x, $\lim_{m=\infty} \psi_n(x, m)$ has no single value, but may be multiple-valued between finite or infinite limits $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$. The function $\phi_n(x)$, for such a value of x, may be capable of having a finite number of values, or an infinite number, and possibly all values between $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$: but in the application of the theory we need only attend to these upper and lower limits of indeterminacy, it being indifferent whether $\phi(x)$ has all values between these limits or some values only. The fluctuation of $\phi_n(x)$ in any interval (α, β) is the excess of the upper limit of the numbers $\overline{\phi_n(x)}$ for all points of (α, β) over the lower limit of the numbers $\underline{\phi_n(x)}$ in the same interval. The saltus (*Sprung*) of $\phi_n(x)$ at the point x is the limit of the fluctuation in an interval $(x-\delta, x+\delta)$, when δ is indefinitely

* In the remainder of the paper, it will be assumed that all proper integrals exist in accordance with Riemann's definition.

† In the application of this definition made in this paper, all the functions $\phi_n(x)$ are limited functions. This is, however, not necessary for the validity of the definition. In general $\phi_n(x)$ may be unlimited, and the values of $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$ for particular values of x may have the improper values ∞ or $-\infty$.

diminished, and this saltus is $\geqslant \overline{\phi_n(x)} - \underline{\phi_n(x)}$. Riemann's theory of integration is applicable to such a function $\phi_n(x)$, in case it is limited in (a, b), just as in the case of a single-valued function.

For any fixed value of x, the numbers

$$\overline{\phi_1(x)},\ \overline{\phi_2(x)},\ \ldots,\ \overline{\phi_n(x)},\ \ldots;\quad \underline{\phi_1(x)},\ \underline{\phi_2(x)},\ \ldots,\ \underline{\phi_n(x)},\ \ldots$$

form a set which we may denote by G.

Let us consider the derivative G' of G; then, if G' is limited, since it is a closed set, it has a greatest value A and a least value B, and these numbers A and B are such that, for a given ϵ, there are an infinite number of values of n such that $|\overline{\phi_n(x)} - A| < \epsilon$, and also an infinite number of values of n such that $|\underline{\phi_n(x)} - B| < \epsilon$. If G' is unlimited in one direction or in both directions, either A or B or both may be regarded as having one of the improper values ∞, $-\infty$.

We now define a function $\phi(x)$ for the interval (a, b) in the following manner:—When, for a particular value of x, the numbers A and B are equal and finite, their value is taken to be that of $\phi(x)$. If A and B are unequal and finite, we regard $\phi(x)$ as multiple-valued, with $\overline{\phi(x)} = A$, $\underline{\phi(x)} = B$. If either A or B has one of the improper values $+\infty$, $-\infty$, the point x is taken to be a point of infinite discontinuity of $\phi(x)$. The function $\phi(x)$ is regarded as a single function, not necessarily limited, and it may have an improper integral in (a, b) in accordance with Harnack's definition of the improper integral of an unlimited function. This function $\phi(x)$ is said to be *the limiting function defined by the sequence* $\{\phi_n(x)\}$, and the functions $\phi_n(x)$ may be said to *converge*, in an extended sense of the term, to the function $\phi(x)$; and thus $\phi(x) = \lim_{n=\infty} \phi_n(x)$.

In case the sequence $\{\overline{\phi_n(x)}\}$ is monotone and non-diminishing, so that, for every value of x and n, the condition $\overline{\phi_n(x)} \leqslant \overline{\phi_{n+1}(x)}$ is satisfied, the sequence $\{\overline{\phi_n(x)}\}$ has, for each particular value of x, either a definite upper limit A or the improper limit $+\infty$. If $\underline{\phi_n(x)} \geqslant \underline{\phi_{n+1}(x)}$, for every value of x and n, the sequence $\{\underline{\phi_n(x)}\}$ has, for each particular value of x, either a definite lower limit B or the improper lower limit $-\infty$.

Let a positive number ϵ and a positive integer n_1 be arbitrarily chosen, and let E be a set of points in (a, b) of which the measure is zero. Let us suppose that, for each point x_1 in (a, b) which does not belong to a certain component E_ϵ of E, this component depending on ϵ, an integer $m > n_1$, and also a neighbourhood $(x_1 - \delta, x_1 + \delta')$, can be determined, such that the four inequalities $|\underline{\overline{\phi(x)}} - \underline{\overline{\phi_m(x)}}| < \epsilon$ are all satisfied at every point in the interval $(x_1 - \delta, x_1 + \delta')$ which is in (a, b).

Then, provided this condition is satisfied for every value of ϵ, *and also* E *is such that each point of it belongs to* E_ϵ *for some sufficiently small value of* ϵ, *the convergence of the sequence* $\{\phi_n(x)\}$ *to* $\phi(x)$ *is said to be regular in* (a, b) *except for the set* E *of zero measure.**

If will be observed that, for a given ϵ, the integer n $(> n_1)$ depends in general upon the particular point x_1 which does not belong to E_ϵ. Moreover, since n_1 is arbitrary, there exists for a particular point x_1 an infinite number of values of n; the neighbourhood $(x_1 - \delta,\ x_1 + \delta')$ depending, however, in general upon the value of n chosen.

In case the sequence $\{\phi_n(x)\}$ is monotone and increasing, so that $\overline{\phi_n(x)} \leqslant \overline{\phi_{n+1}(x)}$, and also $\underline{\phi_n(x)} \leqslant \underline{\phi_{n+1}(x)}$, when the conditions

$$|\overline{\underline{\phi(x)}} - \overline{\underline{\phi_n(x)}}| < \epsilon$$

are satisfied for a particular value of n, they are also satisfied for every greater value of n. In the general case, however, this is no longer true.

It is easily seen that the set E_ϵ must, for each value of ϵ, be a non-dense closed set, although the set E is not necessarily non-dense, and may be everywhere dense in (a, b). For, if ξ be a limiting point of E_ϵ, then every neighbourhood of ξ contains points of E_ϵ, and it is impossible that the conditions $|\overline{\underline{\phi(x)}} - \overline{\underline{\phi_n(x)}}| < \epsilon$ can be satisfied for every point of such a neighbourhood. Therefore ξ must itself belong to E_ϵ, which must consequently be a closed set; and, since E_ϵ has the measure zero, it cannot contain all the points of any interval (α, β), and is therefore non-dense in (a, b).

The set E, which consists of the points which belong to any of the sets $E_{\epsilon_1}, E_{\epsilon_2}, \ldots, E_{\epsilon_n}, \ldots$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ is a sequence of descending values of ϵ converging to zero, is a set of the *first category*.

The set E contains every point at which $\phi(x)$ has not a definite finite value; for, since $\overline{\phi(x)} - \overline{\phi_n(x)}$, $\underline{\phi(x)} - \underline{\phi_n(x)}$ are both numerically less than ϵ, at a point which does not belong to E, for some value of n, it follows that $\overline{\phi(x)} - \underline{\phi(x)}$ is less than 2ϵ; and, since ϵ is arbitrarily small, it follows that $\overline{\phi(x)} = \underline{\phi(x)}$. It is clear that the points of infinite discontinuity of $\phi(x)$ belong to the set E_ϵ, whatever be the value of ϵ.

Let the numbers ϵ and n_1 be fixed; then, since E_ϵ is closed and has its content zero, all its points may be enclosed in the interiors of a finite set of intervals of which the sum is η, an arbitrarily small number; let

* The term *measure* of a set is throughout used in the sense employed by Borel and Lebesgue. The term *content* is used in the sense employed by Cantor and Harnack. The measure and the content of a closed set are identical, but this is not in general true of an unclosed set.

these intervals be excluded from (a, b). There remains a finite set of intervals such that, for each point x_1 in any of them, a neighbourhood $(x_1-\delta,\ x_1+\delta')$ can be found, for the whole of which the conditions $|\phi(x)-\phi_n(x)| < \epsilon$ are satisfied for some value of n $(> n_1)$ dependent on x_1. Let us consider these intervals $(x_1-\delta,\ x_1+\delta')$ for every point of the finite set of intervals which remain in (a, b). Each point of this finite set is in the interior of some of the intervals $(x_1-\delta,\ x_1+\delta')$; and therefore, by employing the Heine-Borel theorem, we see that a finite number of the intervals $(x_1-\delta,\ x_1+\delta')$ can be selected so that every point of (a, b) not interior to the excluded intervals is interior to one at least of these selected intervals. It follows that the conditions $|\phi(x)-\phi_n(x)| < \epsilon$ are satisfied at every point x of (a, b) not interior to the excluded intervals whose sum is η, when n has one of a finite number of values

$$n_1+p_1,\ n_1+p_2,\ \ldots,\ n_1+p_r.$$

The particular number n_1+p which must be taken for a point x depends upon the position of that point, but the same number n_1+p is applicable to all the points of one or more continuous intervals.*

In the particular case in which $\phi_1(x),\ \ldots,\ \phi_n(x),\ \ldots$ are all definite in value for each value of x, and for which $\phi_n(x) \leqslant \phi_{n+1}(x)$ for every value of x and n, the condition $\phi(x)-\phi_n(x) < \epsilon$ is satisfied for every point not interior to the intervals enclosing E_ϵ, the value of n being everywhere the same. For we may take as the value of n the greatest of the numbers n_1+p. In this case the definition is equivalent to the definition of regular convergence given† by de la Vallée-Poussin for the case he considered. The definition given above is much more general than that of de la Vallée-Poussin, but is requisite for the purpose of a complete treatment of the conditions under which an absolutely convergent improper integral can be replaced by a repeated integral.

The Repeated Improper Integrals.

5. The function $f(x, y)$ being defined, as explained in § 1, for the fundamental rectangle bounded by $x = a,\ x = b,\ y = c,\ y = d$, and it being assumed that the absolutely convergent improper integral

$$\int f(x, y)\,(dx,\ dy),$$

taken over the fundamental rectangle, exists, necessary and also sufficient

* It thus appears that regular convergence, except for a set E of zero measure, is closely related to Arzelà's "convergenza uniforme a tratti in generale," which I have considered in *Proceedings*, Ser. 2, Vol. 1, p. 380. In fact, in the case in which the functions are all single-valued at every point there is precise equivalence between the two definitions.

† *Liouville's Journal*, Ser. 4, Vol. VIII., 1892, pp. 435, 436.

conditions will now be investigated that the repeated integral

$$\int_a^b dx \int_c^d f(x, y)\, dy$$

may exist and have the same value as the double integral.

We shall consider a sequence $f_1(x, y), f_2(x, y), \ldots, f_n(x, y), \ldots$ of functions obtained from $f(x, y)$ as in de la Vallée-Poussin's definition given in § 1.

The integral $\int_c^d f_n(x, y) dy$ will be denoted by $\phi_n(x)$, where $\phi_n(x)$ may either have a determinate value or may have as limits of indeterminacy $\overline{\phi_n(x)}$, $\underline{\phi_n(x)}$, the upper and lower values of the integral $\int_c^d f_n(x, y)\, dy$, in accordance with Darboux's definition of the upper and lower integral of a limited function. The existence of $\int f_n(x, y)(dx\, dy)$ does not ensure the determinacy of $\phi_n(x)$ for all values of x. The integral $\int_c^d f(x, y)\, dy$ will be denoted by $\phi(x)$; a similar remark applies to the determinacy of $\phi(x)$, as in the case of $\phi_n(x)$. Moreover, $\phi(x)$ may have the improper value ∞ or $-\infty$, or may have one of these as a limit of indeterminacy; for $f(x, y)$ does not necessarily, for each value of x, possess either a proper or an improper integral in the interval (c, d). In de la Vallée-Poussin's investigation the restrictive assumptions are made that the functions $\phi_n(x)$ are everywhere definite and that $\phi(x)$ is everywhere finite or definite. Moreover, in part of his work it is assumed that the functions $\phi_n(x)$ are all essentially positive or zero.

It will first be shewn on the assumption of the existence of the double integral $f(x, y)(dx, dy)$ to be necessary, in order that

$$\int_a^b dx \int_c^d f(x, y)\, dy$$

may exist, that the sequence $\{\phi_n(x)\}$ should converge regularly to the limit $\phi(x)$, except for a set of points E of the first category and of zero measure.

When, for a fixed x, the function $f(x, y)$ has points of infinite discontinuity with respect to the variable y, in the interval (c, d), the value of $\overline{\int_c^d} f(x, y) dy$ or $\overline{\phi(x)}$ is the upper limit of $\overline{\int_c^d} f_n(x, y)\, dy$, that is of $\overline{\phi_n(x)}$, when all values of n are taken into account, and $\underline{\int_c^d} f(x, y) dy$, or $\underline{\phi(x)}$, is the limit of $\underline{\int_c^d} f_n(x, y)\, dy$, or $\underline{\phi_n(y)}$. In case the $\overline{\phi_n(x)}$ have no upper limit, $\overline{\phi(x)}$ has the improper value ∞, and a similar remark applies to $\underline{\phi}(x)$.

Since $f(x, y)$ is integrable in the fundamental rectangle, all the functions $f_n(x, y)$ have proper integrals in that domain. The proper integral $\int f_n(x, y).(dx, dy)$ is, by a known theorem, replaceable by the repeated integral $\int_a^b dx \int_c^d f_n(x, y)\,dy$, and thus $\phi_n(x)$ is integrable in the linear interval (a, b). It follows that the points of discontinuity of $\phi_n(x)$ form a set of linear measure zero. The set of all points of discontinuity of any of the functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$ is consequently, in accordance with the theory of measurable sets, also a set of zero measure. If $\phi(x)$ be integrable in the interval (a, b), its points of discontinuity must form a set of zero measure. Let us suppose that $\phi(x)$ is integrable in (a, b), and thus that $\int_a^b dx \int_c^d f(x, y)\,dy$ exists; and let us assume, if possible, that the set E_ϵ, referred to in the definition of regular convergence in § 4, has its measure greater than zero. Remove from E_ϵ those points at which one or more of the functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$ is discontinuous, and also remove all those points at which $\phi(x)$ is discontinuous; we have then left a set F_ϵ of measure equal to that of E_ϵ, and therefore, by hypothesis, greater than zero. At every point of F_ϵ all the functions $\phi_n(x)$ are definite and continuous, and $\phi(x)$ is also definite and continuous. If ξ be a point of F_ϵ, the number n $(> n_1)$ can be so chosen that

$$|\phi(\xi)-\phi_n(\xi)| < \tfrac{1}{3}\epsilon;$$

also δ can then be so chosen that, for every point x in the interval $(\xi-\delta, \xi+\delta)$, the four inequalities

$$|\phi(\xi)-\overline{\underline{\phi(x)}}| < \tfrac{1}{3}\epsilon,$$

$$|\phi_n(\xi)-\overline{\underline{\phi_n(x)}}| < \tfrac{1}{3}\epsilon$$

are all satisfied. From these inequalities we deduce that the four inequalities $|\overline{\underline{\phi(x)}}-\overline{\underline{\phi_n(x)}}| < \epsilon$ are all satisfied for all points x in the interval $(\xi-\delta, \xi+\delta)$. But this is contrary to the hypothesis that ξ is a point belonging to E_ϵ. It therefore follows that, on the assumption that $f(x, y)$ has an improper integral in the fundamental rectangle, the repeated integral $\int_a^b dx \int_c^d f(x, y)\,dy$ cannot exist unless E_ϵ has the measure zero. Since this holds for every value of ϵ, we have obtained the following theorem:—

If $f(x, y)$ has an improper (absolutely convergent) integral in the *fundamental rectangle, a necessary condition for the existence of the repeated integral* $\int_a^b dx \int_c^d f(x, y)dy$ *is that the convergence of* $\int_c^d f_n(x, y)dy$ *to* $\int_c^d f(x, y)dy$ *should be regular except for a set of points E of the first category and of zero measure.*

The special case of this theorem which arises when $f(x, y)$ is restricted to be everywhere positive or zero has been established by de la Vallée-Poussin * under certain restrictive hypotheses. He assumed that

$$\int_c^d f(x, y)dy \quad \text{and} \quad \int_c^d f_n(x, y)dy$$

both have definite finite values at all points x which do not belong to a set of points of zero content; this is equivalent to the assumption that all those points x, such that the set of points on the ordinate through x at which the saltus of $f(x, y)$ is $\geqslant \alpha$, where α is an arbitrarily chosen positive number, have content zero, form a set of linear content zero. It is true that the set of such points x forms a set of zero measure, but, as it is not necessarily non-dense in (a, b), the content is not necessarily zero. In a later memoir,† de la Vallée-Poussin states that he has not been able to remove the restrictive hypothesis made in the first memoir. He then proves that, when $f(x, y) \geqslant 0$, the double integral can be replaced by $\int_a^b dx \int_c^d mf(x, y)dy$, where $mf(x, y)$ denotes the minimum of the function $f(x, y)$ at the point (x, y); but he gives no general investigation of the conditions that the equality

$$\int_a^b dx \int_c^d mf(x, y)dy = \int_a^b dx \int_c^d f(x, y)dy$$

may hold.

6. Let it now be assumed that at every point (x, y) the function $f(x, y)$ is either positive or zero, but never negative. It will be shewn that in this case the condition of regular convergence of the sequence $\{\phi_n(x)\}$ to $\phi(x)$, at all points except a set of the first category and of zero measure, is sufficient to ensure that $\int_a^b dx \int_c^d f(x, y)dy$ exists and is equal to $\int f(x, y)(dx, dy)$; it being assumed that the double integral exists.

In this case the four inequalities $|\overline{\underline{\phi(x)}} - \overline{\underline{\phi_n(x)}}| < \epsilon$ are equivalent to the one $\overline{\phi(x)} - \underline{\phi_n(x)} < \epsilon$; and, if at any point x this is satisfied for a value of n, then it is also satisfied for all greater values of n. Including

* *Loc. cit.*, pp. 448–450. † *Liouville's Journal*, Ser. 5, Vol. v., 1899.

all the points of E_ϵ in the interior of intervals of a finite set, such that the sum of these intervals is the arbitrarily small number η, we see that the condition $\overline{\phi}(x)-\underline{\phi_n(x)} < \epsilon$ is satisfied for one and the same value of n ($> n_1$) at all points x not interior to the intervals whose sum is η. For we have only to take for n the greatest of the numbers

$$n_1+p_1,\ n_1+p_2,\ \ldots,\ n_1+p_r$$

defined in § 4.

The number ϵ being fixed, we can choose the number η so small that the double integral $\int f(x, y)(dx\,dy)$ over those rectangles of which the height is $d-c$ and the sum of the breadths η is less than an arbitrarily fixed positive number ζ; this follows from Jordan's definition of an improper double integral. The number η being fixed, a number m exists, such that, for $n \geqslant m$, we have $\overline{\phi(x)}-\underline{\phi_n(x)} < \epsilon$, except in the intervals which enclose E_ϵ. We have therefore

$$\overline{\underline{\int}} \phi(x)\,dx-\int \phi_n(x)\,dx < \epsilon(b-a-\eta) < \epsilon(b-a),$$

the integration being taken along the parts of (a, b) which remain when the enclosing intervals are removed. Hence we have

$$\overline{\underline{\int}} \phi(x)\,dx-\int f_n(x, y)\,(dx\,dy) < \epsilon(b-a),$$

where the double integral is taken over the fundamental rectangle with the exception of those parts of which the breadths are the enclosing intervals. Also, if ζ' is an arbitrarily chosen number, we can choose n so great that

$$\int f(x, y)(dx\,dy)-\int f_n(x, y)\,(dx\,dy) < \zeta',$$

where both the double integrals are taken over the same region as before. We now see that

$$\left|\overline{\underline{\int}} \phi(x)\,dx-\int f(x, y)(dx\,dy)\right| < \zeta'+\epsilon(b-a),$$

and from this we see that

$$\left|\int f(x, y)(dx\,dy)-\overline{\underline{\int}} \phi(x)\,dx\right| < \zeta+\zeta'+\epsilon(b-a),$$

where the double integral is taken over the fundamental rectangle, and the single integral over the points of (a, b) which remain when the intervals enclosing E_ϵ are removed. Now ζ' is arbitrarily small, and ζ, η converge together to zero. It follows that $\int_a^b \phi(x)\,dx$, whether definite or not, lies between $\int f(x, y)(dx\,dy) \pm \epsilon(b-a)$; and, since ϵ is arbitrarily small, it follows that $\int_a^b \phi(x)\,dx$ exists as a definite proper or improper integral, and is equal to $\int f(x, y)\,(dx\,dy)$.

The following theorem has now been established :—

If $f(x, y)$ is never negative, and has an improper double integral in the fundamental rectangle, then the condition that the integrals $\int_c^d f_n(x, y)\,dy$ converge to $\int_c^d f(x, y)\,dy$ regularly, except for a set of points of the first category and zero measure, is a sufficient condition that $\int_a^b dx \int_c^d f_n(x, y)\,dy$ exists and is equal to $\int f(x, y)\,(dx\,dy)$.

The sufficiency of the same condition, for the case in which $f(x, y)$ is not restricted to have one sign only, does not appear to be capable of establishment, because it is in this case impossible to shew that the conditions $|\overline{\underline{\phi(x)}} - \overline{\underline{\phi_n(x)}}| < \epsilon$ are satisfied at all points except in the enclosing intervals, for one and the same value of n; it having been only established that it holds when n has one of a finite number of values.

Combining the present results with that of § 5, we see that—

If $f(x, y)$ is never negative, and has an absolutely convergent improper integral in the fundamental rectangle, the necessary and sufficient condition that $\int_a^b dx \int_c^d f(x, y)\,dy$ exists and is equal to $\int f(x, y)(dx\,dy)$ is that the sequence $\int_c^d f_n(x, y)\,dy$ $(n = 1, 2, 3, \ldots)$ converges regularly to $\int_c^d f(x, y)\,dy$, except for a set E of the first category and of zero measure.

It has also been shewn that *when $\int f(x, y)\,(dx\,dy)$ exists then, if $\int_a^b dx \int_c^d f(x, y)\,dy$ have a definite meaning, it is equal to the double integral.*

For it has been shewn that the repeated integral cannot have a definite meaning, $\phi(x)$ being integrable in (a, b), unless the convergence is of the kind specified.

7. Returning to the case in which $f(x, y)$ is not restricted to be of one sign, the following theorem will be established :—

If $f(x, y)$ have an absolutely convergent improper integral in the fundamental rectangle, a sufficient condition that $\int_a^b dx \int_c^d f(x, y)\,dy$ may *exist and have the same value as the double integral* $\int f(x, y)(dx\,dy)$ *is that* $\int_c^d |f_n(x, y)|\,dy$ *should converge regularly to* $\int_c^d |f(x, y)|\,dy$, *except for a set of the first category and of zero measure.*

Using
$$f(x, y) = f^+(x, y) - f^-(x, y),$$
$$f_n(x, y) = f_n^+(x, y) - f_n^-(x, y),$$

as in § 2, and denoting $\int_c^d f_n^+(x, y)\,dy$, $\int_c^d f_n^-(x, y)\,dy$ by $\phi_n^+(x)$, $\phi_n^-(x)$ respectively, we see that the condition stated in the theorem is that $\phi_n^+(x) + \phi_n^-(x)$ should converge regularly to $\phi^+(x) + \phi^-(x)$. In order that this condition may be satisfied, we must have

$$\overline{\phi^+(x)} + \overline{\phi^-(x)} - \underline{\phi_n^+(x)} - \underline{\phi_n^-(x)} < \epsilon,$$

for a sufficiently great value of n, at every point not interior to a finite set of intervals of arbitrarily small sum η enclosing the points of E_ϵ, a set of zero content. From this condition we deduce that

$$\overline{\phi^+(x)} - \underline{\phi_n^+(x)} < \epsilon \quad \text{and} \quad \overline{\phi^-(x)} - \underline{\phi_n^-(x)} < \epsilon,$$

at every point not in the interior of the intervals; and hence $\phi_n^+(x)$ converges regularly to $\phi^+(x)$, and also $\phi_n^-(x)$ converges regularly to $\phi^-(x)$, at all points except a set E of zero measure. It follows that

$$\int_a^b dx \int_c^d f^+(x, y)\,dy$$

exists and is equal to $f^+(x, y)(dx\,dy)$, and also that $\int_a^b dx \int_c^d f^-(x, y)\,dy$ exists and is equal to $f^-(x, y)(dx\,dy)$; and therefore $\int_a^b dx \int_c^d f(x, y)\,dy$ exists and is equal to $\int f(x, y)(dx\,dy)$.

The condition stated in the theorem, though sufficient, is not necessary; for the integral $\int_a^b \{\phi^+(x) - \phi^-(x)\}\,dx$ may exist only as a non-absolutely convergent improper integral, in which case $\int_a^b \{\phi^+(x) + \phi^-(x)\}\,dx$ does not exist. In this case, $\int_a^b dx \int_c^d |f(x, y)|\,dy$ not existing, the convergence of $\int_c^d |f_n(x, y)|\,dy$ to $\int_c^d |f(x, y)|\,dy$ cannot be regular. An example will be given below in which this case actually arises.

8. Whether the double integral $\int f(x, y)(dx\,dy)$ exist or not, the proof in § 5 suffices to shew that, if all the double integrals $\int f_n(x, y)(dx\,dy)$ exist, then it is a necessary condition for the existence of the repeated integral $\int_a^b dx \int_c^d f(x, y)\,dy$ that the integrals $\int_c^d f_n(x, y)\,dy$ should converge regularly to $\int_c^d f(x, y)(dy)$, except for a set of points x, of the first category and of zero measure.

Moreover, if it be known that $\int_c^d f(x, y)\,dy$ is a function of x which is limited in the interval (a, b), we can infer the existence of the double integral $\int f(x, y)(dx\,dy)$. For, since

$$\int f_n(x, y)(dx\,dy) = \int_a^b dx \int_c^d f_n(x, y)\,dy,$$

we have

$$\left| \int f_n(x, y)(dx\,dy) \right| < (b-a)\,U_n,$$

where U_n is the upper limit of $\left| \int_c^d f_n(x, y)\,dy \right|$ in the interval (a, b). It is thus seen that $\int f_n(x, y)(dx\,dy)$ cannot increase indefinitely in numerical value as n is indefinitely increased, since U_n does not increase indefinitely.

The following theorem has therefore been established:—

If all the functions $f_n(x, y)$ have double integrals in the fundamental rectangle, and $\int_c^d f_n(x, y)\,dy$ converges to $\int_c^d f(x, y)\,dy$ regularly, except for a set of points x of zero measure and of the first category, then, if $\int_c^d f(x, y)\,dy$ be a limited function of x in the interval (a, b), the double integral $\int f(x, y)(dx\,dy)$ exists and is equal to $\int_a^b dx \int_b^d f(x, y)\,dy$.

Combining this theorem with that of § 7, we have the following theorem:—

If all the functions $f_n(x, y)$ have double integrals in the fundamental rectangle, and either $\int_c^d f(x, y)\,dy$ is limited in the interval (a, b) of x, or $\int_a^b f(x, y)\,dx$ is limited in the interval (c, d) of y, and if the conditions are *satisfied that each of the sequences*

$$\int_c^d |f_n(x, y)|\, dy, \quad \int_a^b |f_n(x, y)|\, dx$$

converges to the limits

$$\int_c^d |f(x, y)|\, dy, \quad \int_a^b |f(x, y)|\, dx$$

in each case regularly, except for a set of points of zero measure and of the first category, then the double integral exists, and

$$\int f(x, y)\,(dx\,dy) = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx.$$

9. It has been maintained* by Schönflies that an absolutely convergent improper double integral can always be replaced by either of the corresponding repeated integrals; no condition beyond that of the existence of the double integral in accordance with the definition of de la Vallée-Poussin being required for the validity of this equivalence. In the case in which the integrand is essentially positive or zero, this view could only be correct in case the regular convergence of the functions $\phi_n(x)$ to the limiting function $\phi(x)$ followed as a necessary consequence of the existence of the double integral. That Schönflies' view is incorrect can be shewn by means of an example.

Let the fundamental rectangle be bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$; and let the function $\psi(x)$ be defined† by the rule that, for every rational value of x of the form $\frac{2m+1}{2^n}$ $(n > 0)$, $\psi(x) = \frac{1}{2^n}$, and that, for every other value of x, $\psi(x) = 0$. Let

$$f(x, y) = \left| \frac{1}{y} \sin \frac{1}{y} \right| \psi(x);$$

then it is easily seen that the improper integral

$$\int \left| \frac{1}{y} \sin \frac{1}{y} \right| \psi(x)\,(dx\,dy),$$

taken over the rectangle, exists and has the value zero. The integral

* See the *Bericht über die Mengenlehre*, pp. 198–202.

† This function $\psi(x)$ was first given by Du Bois Reymond, *Crelle's Journal*, Vol. XCII., p. 278. See also Stolz's *Grundzüge*, Vol. III., p. 149, where the above method of constructing a double integral which cannot be replaced by the repeated integral is indicated.

$\int_0^1 \psi(x)\,dx$ exists and has the value zero. The repeated integral

$$\int_0^1 dx \int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$$

does not exist; for $\int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$

diverges for each value of x of the form $\frac{2m+1}{2^n}$, and is zero for other values of x. In this case the function $\phi(x)$ is infinite for the everywhere dense set of values $x = \frac{2m+1}{2^n}$; and therefore $\phi(x)$ is not integrable in the interval (0, 1); therefore $\int_0^1 \phi(x)\,dx$ has no meaning. In this case, the other repeated integral

$$\int_0^1 dy \int_0^1 \psi(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dx$$

exists, and is equal to zero. The integral

$$\int \left(\frac{1}{y} \sin \frac{1}{y}\right) \psi(x)\,(dx\,dy)$$

can be replaced by the repeated integral

$$\int_0^1 dx \int_0^1 \left(\frac{1}{y} \sin \frac{1}{y}\right) \psi(x)\,dy;$$

for $\int_0^1 \frac{1}{y} \sin \frac{1}{y}\,dy$ exists as a non-absolutely convergent single integral and has a value A; hence in this case

$$\phi(x) = A\,\psi(x);$$

and therefore the repeated integral has the value zero, the same as that of the double integral. This is an instance in which the repeated integral exists and is equal to the double integral, although the sufficient condition given in § 5 is not satisfied.

Schönflies has given an example (*loc. cit.*, pp. 201, 202) intended to illustrate his theorem that the condition of regular convergence is unnecessary for the equality of the double integral and repeated integral of a function which is never negative. It will, however, be seen that the example does not bear out his contention. He defines the function $f(x, y)$ as follows:—The rectangle for which the function is defined is bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, and in all points $x = \frac{2m+1}{2^s}$, $y \leqslant \frac{1}{2}$,

$f(x, y)$ has the improper value* $+\infty$, and everywhere else $f(x, y) = 0$. In this case the function $f_n(x, y)$ will be given by the conditions

$$f_n(x, y) = N_n,$$

at all points $x = \frac{2m+1}{2^s}$, $y \leqslant \frac{1}{2^s}$, and $f_n(x, y) = 0$, everywhere else.

It can be shewn that $\qquad \int f_n(x, y)(dx\,dy) = 0,$

for every value of N_n, and thus $\int f(x, y)(dx\,dy)$ exists and $= 0$. The condition

$$\int f(x, y)\,dy - \int f_n(x, y)\,dy < \epsilon$$

is not satisfied for any of the everywhere dense set of values $x = \frac{2m+1}{2^s}$;

and therefore the convergence of $\int f_n(x, y)\,dy$ to $\int f(x, y)\,dy$ is not regular. Schönflies maintains that, notwithstanding this, the repeated integral $\int_0^1 dx \int f(x, y)\,dy$ exists, and is also equal to zero; it will be shewn that this is not the case.

It is true that $\int_0^1 dx \int_0^1 f_n(x, y)\,dy$ is equal to zero, for every value of n, and thus that $\lim_{n=\infty} \int_0^1 dx \int_0^1 f_n(x, y)\,dy$ is zero. But $\int_0^1 dx \int_0^1 f(x, y)\,dy$ is not equivalent to $\lim_{n=\infty} \int_0^1 dx \int_0^1 f_n(x, y)\,dy$, but to $\int_0^1 dx \lim_{n=\infty} \int_0^1 f_n(x, y)\,dy$, since $\int_0^1 f(x, y)\,dy$ is defined to be $\lim_{n=\infty} \int_0^1 f_n(x, y)\,dy$, in accordance with de la Vallée-Poussin's definition of an improper single integral. Now $\int_0^1 f_n(x, y)\,dy$ is zero, unless $x = \frac{2m+1}{2^s}$, in which case it is $\frac{N_n}{2^s}$, and, for such values of x, $\lim_{n=\infty} \int_0^1 f_n(x, y)\,dy$ or $\phi(x)$ is ∞, and thus $\int_0^1 \phi(x)\,dx$ does not exist, because $\phi(x)$ is ∞ at the everywhere-dense set of points $x = \frac{2m+1}{2^s}$, in the interval (0, 1). It has thus been shewn that the repeated integral has no existence, and, since the condition of regular convergence is not satisfied, this is in accordance with the theorem of § 6.

* It may be objected to this definition that the function is not properly defined at the points of the specified set, since the functional *values* are there regarded as having the improper values ∞. The extension of Dirichlet's definition of a function involved in the admission of improper functional values ∞ or $-\infty$, as distinct from functional limits, leads, however, to no difficulty in relation to the theory of integration, and may therefore be admitted without modifying the theory.

This example throws light on the error in Schönflies' proof (*loc. cit.*, pp. 199, 200) of the theorem that the existence of the double integral necessarily entails that of the repeated integral, and the equality of the two. He replaces the function $f_n(x, y)$ by the most nearly continuous function $\phi_n(x, y)$ and then also $\int \phi_n(x, y)\,dy$ by the most nearly continuous function $\Phi_n(x)$, and argues that

$$\int \Phi_1(x)\,dx, \quad \int \Phi_2(x)\,dx, \quad \ldots, \quad \int \Phi_n(x)\,dx, \quad \ldots$$

form a sequence which defines $\int \Phi(x)\,dx$, in accordance with de la Vallée-Poussin's definition of an improper single integral. To establish this, he relies upon the insufficient fact that $\Phi_{n+1}(x) \geqslant \Phi_n(x)$, for every value of x, whereas $\Phi_n(x)$ in general differs from $\Phi_{n+1}(x)$, not merely for such values as are greater than some fixed number. In the above example

$$\int \Phi_1(x)\,dx, \quad \int \Phi_2(x)\,dx, \quad \ldots$$

are all zero, but $\int \Phi(x)\,dx$ is infinite. The error in the proof appears to depend essentially on an illegitimate identification of

$$\lim_{n=\infty} \int dx \int f_n(x, y)\,dy$$

with

$$\int dx \lim_{n=\infty} \int f_n(x, y)\,dy\,;$$

the former of these limits is always equal to $\int f(x, y)(dx\,dy)$, but the latter, which is the interpretation of $\int dx \int f(x, y)\,dy$, is not unconditionally equal to the former limit.