ON A THEOREM OF CLEBSCH'S.

BY PAUL SAUREL.

CLEBSCH has shown[1] that the integrals of the equations of motion of a vibrating elastic solid can be put into a very simple and elegant form. His demonstration, while by no means long or difficult, seems, nevertheless, to admit of some simplification.

The equations in question are :[2]

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= a^2 \Delta u + (b^2 - a^2)\frac{\partial \sigma}{\partial x} \\ \frac{d^2 v}{\partial t^2} &= a^2 \Delta v + (b^2 - a^2)\frac{\partial \sigma}{\partial y} \\ \frac{\partial^2 w}{\partial t^2} &= a^2 \Delta w + (b^2 - a^2)\frac{\partial \sigma}{\partial z} \end{aligned}\right\} , \qquad 1$$

in which

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\sigma = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \cdot$$

From the above equations, it follows without difficulty that

$$\frac{\partial^2 \sigma}{\partial t^2} = b^2 \Delta \sigma. \qquad 2$$

Let u, v, w be any given solution of the above equations and let us write

$$\left.\begin{aligned} u &= \frac{\partial P}{\partial x} + u' \\ v &= \frac{\partial P}{\partial y} + v' \\ w &+ \frac{\partial P}{\partial z} + w' \end{aligned}\right\} . \qquad 3$$

[1] Clebsch, Ueber die Reflexion an einer Kugelfläche. Borchardt's Journal für die reine u. angew Math., Bd. 61.

[2] Kirchhoff, Vorlesungen über Mechanik, XI Vorlesung.

From these we obtain

$$\sigma = \varDelta P + \sigma' \qquad 4$$

in which
$$\sigma' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\,.$$

If we substitute the above values of u, v, w in the equations of motion, we get

$$\left.\begin{aligned}
\frac{\partial^2 u'}{\partial t^2} + \frac{\partial}{\partial x}\left[\frac{\partial^2 P}{\partial t^2} - b^2 \varDelta P\right] &= a^2 \varDelta u' + (b^2 - a^2)\frac{\partial \sigma'}{\partial x} \\
\frac{\partial^2 v'}{\partial t^2} + \frac{\partial}{\partial y}\left[\frac{\partial^2 P}{\partial t^2} - b^2 \varDelta P\right] &= a^2 \varDelta v' + (b^2 - a^2)\frac{\partial \sigma'}{\partial y} \\
\frac{\partial^2 w'}{\partial t^2} + \frac{\partial}{\partial z}\left[\frac{\partial^2 P}{\partial t^2} - b^2 \varDelta P\right] &= a^2 \varDelta w' + (b^2 - a^2)\frac{\partial \sigma'}{\partial z}
\end{aligned}\right\} \qquad 5$$

Suppose that
$$\sigma' = 0. \qquad 6$$

The necessary and sufficient condition for this, is (equation 4) that

$$\varDelta P = \sigma. \qquad 7$$

Suppose moreover that P satisfies the equation

$$\frac{\partial^2 P}{\partial t^2} = b^2 \varDelta P. \qquad 8$$

Equations 5 then become

$$\left.\begin{aligned}
\frac{\partial^2 u'}{\partial t^2} &= a^2 \varDelta u' \\
\frac{\partial^2 v'}{\partial t^2} &= a^2 \varDelta v' \\
\frac{\partial^2 w'}{\partial t^2} &= a^2 \varDelta w'
\end{aligned}\right\} \qquad 9$$

If, then, it be possible to choose P so that equations 7 and 8 are simultaneously verified, equations 3 will determine u', v', w'. We shall thus have established that every solution of the given equations can be written in the form of equations 3, P being a common integral of equations 7 and 8, and u', v' w' being three solutions of the equation

$$\frac{\partial^2 \varphi}{\partial t^2} = a^2 \varDelta \varphi, \qquad 10$$

these solutions, moreover, being connected by equation 6:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \qquad 11$$

In order to show that we can determine P so as to satisfy equations 7 and 8, let us write the equation obtained from them:

$$\frac{\partial^2 P}{\partial t^2} = b^2 \sigma. \qquad 12$$

The pair of equations 7 and 8 may be replaced by the pair of equivalent equations 7 and 12. The integral of equation 12 is

$$P = b^2 \int dt \int \sigma dt + A(x, y, z) \cdot t + B(x, y, z),$$

A and B being two arbitrary functions. From this equation we obtain

$$\Delta P = b^2 \int dt \int \Delta \sigma dt + \Delta A(x, y, z) \cdot t + \Delta B(z, y, z).$$

Choosing the arbitrary functions so that

$$\Delta A(x, y, z) = 0, \quad \Delta B(x, y, z) = 0$$

and making use of equation 2, we obtain

$$\Delta P = \int dt \int \frac{\partial^2 \sigma}{\partial t^2} dt = \sigma.$$

Thus equation 7 is also verified and the theorem stated above is established.

This result may be put into a more striking form if we make use of a theorem tacitly admitted by Clebsch and rigorously established by Duhem.[1] The theorem in question is the following: If u', v', w' are three integrals of equation 10, which also verify equation 11, we may write

$$u' = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},$$
$$v' = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x},$$
$$w' = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y},$$

in which U, V, W are three solutions of equation 10.

[1] Duhem, Sur l'intégrale des équations des petits mouvements d'un solide isotrope. Mémoires de la Société des Sciences de Bordeaux. 5e Série, t. III., p. 325, 1899.

Thus finally we obtain Clebsch's theorem: Every solution of the equations of motion of a vibrating elastic solid can be written in the form:

$$\left.\begin{aligned} u &= \frac{\partial P}{\partial x} + \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} \\ v &= \frac{\partial P}{\partial y} + \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \\ w &= \frac{\partial P}{\partial z} + \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \end{aligned}\right\}, \qquad 13$$

U, V, W being three solutions of the equation

$$\frac{\partial^2 \varphi}{\partial t^2} = a^2 \Delta \varphi \qquad 14$$

and *P* being a solution common to the equations

$$\begin{aligned} \frac{\partial^2 P}{\partial t^2} &= b^2 \Delta P \\ \Delta P &= \sigma. \end{aligned} \qquad 15$$

Conversely, if *U, V, W* are any three solutions of equation 14, and *P* any solution of equation 15, equations 13 furnish a solution of the equations of motion. This is very easily verified by substituting in the original equations the values of *u, v, w* given by equations 13.

BORDEAUX, April, 1899.