Second Note on a Quaternary Group of 51840 *Linear Substitutions.* *By* Dr. G. G. MORRICE. Received and Read April 14th, 1892.

In considering complex numbers of the form

$$a_0\rho_0 + a_1\rho_1 + a_2\rho_2 + \&c.,$$

we may attribute to the symbols ρ_0, ρ_1, ρ_2, &c. any significance we please, provided that they conform to the proper multiplication-table. They may be steps along lines, rotations, strains, substitutions, or what not.

In the matter now under consideration they are quaternary matrices.

Let us start from a quaternion $d+ai+bj+ck$; to multiply this by a second quaternion $d'+a'i+b'j+c'k$ is to subject the parameters c, b, a, d to the matrix

$$\begin{pmatrix} d', & a', & -b', & c' \\ -a', & d', & c', & b' \\ b', & -c', & d', & a' \\ -c', & -b', & -a', & d' \end{pmatrix} \qquad (1),$$

viz., we produce the quaternion

$$d''+a''i+b''j+c''k,$$

where

$$\begin{aligned} a'' &= ad'+a'd-(bc'-b'c), \\ b'' &= bd'+b'd-(ca'-c'a), \\ c'' &= cd'+c'd-(ab'-a'b), \\ d'' &= -aa'-bb'-cc'+dd'. \end{aligned}$$

The matrix (1) may be exhibited as a linear function of the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{............}(2),$$

as appears at once if we multiply these severally by the scalar parameters a, b, c, d and add. Moreover these four matrices form a group, and indeed we might regard the symbols i, j, k, 1 as being nothing else than symbols for these matrices. That a quaternion is a binary matrix has long been recognised, but it appears to me that its connection with quaternary matrices is even more obvious, and has a better claim to notice, because without it the composition formulæ

$$a'' = ad' + a'd - (bc' - b'c), \text{ \&c.}$$

present no definite idea to our minds.

The process of exhibiting a matrix as a linear function of matrices of special forms occurs in kinematics. A strain is split up into the sum of a uniform dilatation, a skew strain, and a wry shear. The fact that the components form a group is not emphasized.

I now recur to the note on this subject which I had the honour of reading to this Society on December 12th, 1889.

We have four functions z_1, z_2, z_3, z_4 connected with the multiplication by 3 of the normal periods of the double theta-functions, viz.,

$$z_1 = X_{01} - X_{02},$$
$$z_2 = X_{10} - X_{20},$$
$$z_3 = X_{11} - X_{22},$$
$$z_4 = X_{12} - X_{21},$$

where

$$X_{\alpha\beta}(v_1, v_2;\ \tau_{11}, \tau_{12}, \tau_{22})$$

$$= p_{12}^{\frac{1}{2}(k-1)} \cdot e^{i\pi/k\,.\,\phi(\alpha,\beta) + (2\alpha v_1 + 2\beta v_2)\,i\pi} \cdot \vartheta\begin{pmatrix} kv_1 + \alpha\tau_{11} + \beta\tau_{12} \\ kv_2 + \alpha\tau_{12} + \beta\tau_{22} \end{pmatrix};\ k\tau_{11}, k\tau_{12}, k\tau_{22}\Big)$$

$$\times \frac{e^{ki\pi\,.\,\Phi(v_1, v_2)}}{\vartheta(\tau_{11}, \tau_{12}, \tau_{22})^k}.$$

I found a sub-group of 4 linear substitutions of the periods ω, all of which were also to be found in the z-group, *i.e.*, the group of 51840 linear substitutions which the functions z_1, z_2, z_3, z_4 undergo; but I did not notice that the simplicity of the group was accounted for by its isomorphism with quaternions. In fact my ω sub-group is exactly (2).

Following Heinrich Burkhardt, "Untersuchungen aus dem Gebiete der Hyperelliptischen Modulfunctionen," *Math. Annalen*, XXXVIII., 2, our matrices (2) are

$$B^3D, \quad (BD)^2B^2, \quad BDB, \quad 1,$$

to which we have, as corresponding matrices in the z-group,

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If then we multiply by the scalar parameters $a, b, c, d,$ and add, we find that when the periods ω are subjected to the matrix

$$\begin{pmatrix} d, & a, & -b, & c \\ -a, & d, & c, & b \\ b, & -c, & d, & a \\ -c, & -b, & -a, & d \end{pmatrix},$$

the functions z_1, z_2, z_3, z_4 are subjected to the matrix

$$\begin{pmatrix} d-b, & c-a, & -b+c, & b+c \\ c+a, & d-b, & b+c, & b-c \\ -b+c, & b+c, & d-c, & -b-a \\ b+c, & b-c, & -b+a, & d+c \end{pmatrix}.$$

We should have expected to arrive at a matrix of the same form, but with the signs of the four elements in the top corner on the righthand side reversed, viz.,

$$\begin{pmatrix} d-b, & c-a, & b-c, & -b-c \\ c+a, & d+b, & -b-c, & -b+c \\ -b+c, & b+c, & d-c, & -b-a \\ b+c, & b-c, & -b+a, & d+c \end{pmatrix} \qquad (3),$$

for we can easily verify that

$$\begin{pmatrix} d-b, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \end{pmatrix} \begin{pmatrix} d'-b', & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \end{pmatrix}$$

$$= \begin{pmatrix} d''-b'', & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \end{pmatrix} \qquad (4),$$

where $$d'' = -aa'-bb'-cc'+dd', \text{ \&c.};$$

that is: the matrix (3) has the same law of composition as quaternions.

We seem to require a notation for expressing the fact that the matrix (4) arises from the matrix (3) in two distinct ways: either by subjecting the letters c, b, a, d in the matrix to the matrix

$$\begin{pmatrix} d', & a', & -b', & c' \\ -a', & d', & c', & b' \\ b', & -c', & d', & a' \\ -c', & -b', & -a', & d \end{pmatrix},$$

or multiplying (3) in the ordinary way by the matrix

$$\begin{pmatrix} d'-b', & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \end{pmatrix}.$$

It is only an extension to complex numbers generally of what is familiar in the case of vectors, viz., that the linear and vector function of a vector is a ternary matrix, but there should be a notation independent of the representation of matrices by complex numbers.

An example may be cited for binary matrices. Cayley, *Messenger of Mathematics*, Vol. XIV., p. 178, gives, for a binary matrix Q such that

$$qQ - Qq' = 0,$$

the form

$$\left(\begin{array}{cc} -f\xi - g\eta - h\zeta, & b\xi - a\eta + h\omega \\ -c\xi + a\zeta + g\omega, & c\eta - b\zeta + f\omega \end{array}\right),$$

and we require a notation to show that this is derived by subjecting the elements of the matrix

$$\left(\begin{array}{cc} \xi, & \eta \\ \zeta, & \omega \end{array}\right)$$

to the matrix

$$\left(\begin{array}{cccc} -f, & -g, & -h, & 0 \\ b, & -a, & 0, & h \\ -g, & 0, & a, & g \\ 0, & g, & -b, & f \end{array}\right).$$

Note on the Skew Surfaces applicable upon a given Skew Surface.

By Prof. CAYLEY. Received March 26th, 1892. Read April 14th, 1892.

The question was considered by Bonnet—§ 7 of his "Mémoire sur la théorie générale des Surfaces," *Jour. Ecole Polyt.*, Cah. 32 (1848); I resume it here, making a greater use of the line of striction.