Certain Concomitant Determinants. *By* J. W. Russell, M.A.
Received and read April 8th, 1897.

1. The object of this note is to give a simple proof of the invariancy of certain differential operators, viz., in the case of binary quantics of

$$\begin{vmatrix} \dfrac{d}{dx_1}, & \dfrac{d}{dy_1} \\ \dfrac{d}{dx_2}, & \dfrac{d}{dy_2} \end{vmatrix}, \qquad \begin{vmatrix} \dfrac{d^2}{dx_1^2}, & \dfrac{d^2}{dx_1 dy_1}, & \dfrac{d^2}{dy_1^2} \\ \dfrac{d^2}{dx_2^2}, & \dfrac{d^2}{dx_2 dy_2}, & \dfrac{d^2}{dy_2^2} \\ \dfrac{d^2}{dx_3^2}, & \dfrac{d^2}{dx_3 dy_3}, & \dfrac{d^2}{dy_3^2} \end{vmatrix},$$

the constituents of any row in the general case being the several terms in the expansion of

$$\left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}+\dots\right)^n.$$

Calling these operators $\Delta_1, \Delta_2, \dots$, it is shown that Δ_n^2 gives us in the form of a determinant a covariant of any quantic, which reduces in the case of a $2n$-ic to the catalecticant. It is then shown that, to some extent, we can operate on Δ_n with several lower operators of the same form, so that the result may still remain a determinant. It is hoped that, in this way, many old theorems may be proved more easily, and that possibly some new facts may be obtained which have hitherto escaped notice.

2. To show that the determinant whose successive rows are made up of the several terms in the expansions

$$\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\dots+\frac{d}{dx_q}\right)^n u_1,$$

$$\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\dots+\frac{d}{dx_q}\right)^n u_2,$$

$$\dots \quad \dots \quad \dots \quad \dots$$

is a covariant of the q-ary quantics $u_1, u_2, \dots u_r$, where r is the number of terms in each of these expansions.

For brevity, take the very simple case when $n = 2$ and $q = 2$.

Then we have to prove that

$$\begin{vmatrix} \frac{d^2u}{dx^2}, & \frac{d^2u}{dx\,dy}, & \frac{d^2u}{dy^2} \\ \frac{d^2v}{dx^2}, & \frac{d^2v}{dx\,dy}, & \frac{d^2v}{dy^2} \\ \frac{d^2w}{dx^2}, & \frac{d^2w}{dx\,dy}, & \frac{d^2w}{dy^2} \end{vmatrix}$$

is a covariant of the binary quantics u, v, and w.

First Method.—Let $\frac{d}{dx}$ operating on u be denoted by a_1, on v by a_2, and on w by a_3; so let $\frac{d}{dy}$ be denoted in these cases by b_1, b_2, b_3; then the operator

$$\begin{vmatrix} a_1^2, & a_1b_1, & b_1^2 \\ a_2^2, & a_2b_2, & b_2^2 \\ a_3^2, & a_3b_3, & b_3^2 \end{vmatrix} = -(a_1b_2-a_2b_1)(a_2b_3-a_3b_2)(a_3b_1-a_1b_3),$$

as we see by considering the identity

$$\begin{vmatrix} 1, & \alpha, & \alpha^2 \\ 1, & \beta, & \beta^2 \\ 1, & \gamma, & \gamma^2 \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

Hence the operator is invariant; and therefore produces a covariant.

Second Method.—Since

$$\frac{d}{dX} = l\,\frac{d}{dx} + l'\,\frac{d}{dy},$$

and
$$\frac{d}{dY} = m\,\frac{d}{dx} + m'\frac{d}{dy};$$

therefore $A_1^2 = (la_1 + l'b_1)^2 = l^2a_1^2 + 2ll'a_1b_1 + l'^2b_1^2,$

$$A_1B_1 = (la_1 + l'b_1)(ma_1 + m'b_1) = lma_1^2 + (lm' + l'm)\,a_1b_1 + l'm'b_1^2,$$

and so on. Hence

$$\begin{vmatrix} A_1^2, & A_1B_1, & B_1^2 \\ A_2^2, & A_2B_2, & B_2^2 \\ A_3^2, & A_3B_3, & B_3^2 \end{vmatrix},$$

multiplying rows by rows,

$$= \begin{vmatrix} a_1^2, & a_1b_1, & b_1^2 \\ a_2^2, & a_2b_2, & b_2^2 \\ a_3^2, & a_3b_3, & b_3^2 \end{vmatrix} \times \begin{vmatrix} l^2, & 2ll', & l'^2 \\ lm, & lm'+l'm, & l'm' \\ m^2, & 2mm', & m'^2 \end{vmatrix}.$$

Hence (see Prof. Elliott's *Algebra of Quantics*, § 23) the operator is invariant; and therefore produces a covariant.

The first method applies only to binary quantics, and shows us that Δ_n is equivalent to $\overline{12}\,.\,\overline{23}\,.\,\overline{34}\,...$, where every number from 1 to $n+1$ is taken with every other. The second method applies in all cases.

A particular case arises when all the quantics are of the n^{th} order; in this case we get a joint invariant.

Also, since $a_1, b_1, ...,$ are any contragredient quantities, we may replace one row by $(\xi+\eta+...)^n$, or in the case of binary quantics by $(-y+x)^n$. In the latter case we may also replace $\left(\frac{d}{dx}+\frac{d}{dy}\right)^n u$ in any row by $\left(\frac{du}{dx}+\frac{du}{dy}\right)^n$.

Notice that we have indirectly proved an interesting algebraic identity, viz., that the determinant whose first row consists of the coefficients in

$$(l_1x_1+l_2x_2+...+l_qx_q)^n,$$

and whose second row of those in

$$(l_1x_1+l_2x_2+...)^{n-1}(m_1x_1+m_2x_2+...),$$

and so on, the general row being given by

$$(l_1x_1+...)^\lambda (m_1x_1+...)^\mu (n_1x_1+...)^\nu ...,$$

where $$\lambda+\mu+\nu+... = n,$$

is a power of the determinant $|\, l_1m_2n_3 ... \,|$.

This is proved directly for the case $q=2$ in Prof. Elliott's *Algebra of Quantics*, § 16.

3. To prove that the determinant whose successive rows are made up of the several terms in the expansions

$$\left(\frac{d}{dx_1}\right)^n \left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots + \frac{d}{dx_q}\right)^n u,$$

$$\left(\frac{d}{dx_1}\right)^{n-1} \left(\frac{d}{dx_2}\right) \left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots + \frac{d}{dx_q}\right)^n u,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the multipliers $\left(\frac{d}{dx_1}\right)^n$, $\left(\frac{d}{dx_1}\right)^{n-1} \left(\frac{d}{dx_2}\right)$, ... repeat the terms of $\left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots\right)^n$, is a covariant of the q-ary quantic u.

For brevity, consider the very simple case when $n = 2$ and $q = 2$. Then we have to show that

$$\begin{vmatrix} \frac{d^4u}{dx^4}, & \frac{d^4u}{dx^3\,dy}, & \frac{d^4u}{dx^2\,dy^2} \\ \frac{d^4u}{dx^3\,dy}, & \frac{d^4u}{dx^2\,dy^2}, & \frac{d^4u}{dx\,dy^3} \\ \frac{d^4u}{dx^2\,dy^2}, & \frac{d^4u}{dx\,dy^3}, & \frac{d^4u}{dy^4} \end{vmatrix}$$

is a covariant of u.

Consider first three quantics u, v, w. Then the following operator is invariant, viz.,

$$\begin{vmatrix} a_1^2, & a_1 b_1, & b_1^2 \\ a_2^2, & a_2 b_2, & b_2^2 \\ a_3^2, & a_3 b_3, & b_3^2 \end{vmatrix} = \Delta_2.$$

Take the first element $a_1^2 . a_2 b_2 . b_3^2$ of this determinant; and multiply the first row by a_1^2, the second row by $a_2 b_2$, and the third row by b_3^2. We get

$$a_1^2 a_2 b_2 b_3^2 \Delta_2 = \begin{vmatrix} a_1^4, & a_1^3 b_1, & a_1^2 b_1^2 \\ a_2^3 b_2, & a_2^2 b_2^2, & a_2 b_2^3 \\ a_3^2 b_3^2, & a_3 b_3^3, & b_3^4 \end{vmatrix}.$$

Next, multiply Δ_2 in the same way by the element got by interchanging the first and second rows, viz., $a_2^2 . a_1 b_1 . b_3^2$, having first interchanged the same two rows in Δ_2. We get

$$-a_2^2 \cdot a_1 b_1 \cdot b_3^2 \cdot \Delta_2 = \begin{vmatrix} a_2^4, & a_2^3 b_2, & a_2^2 b_2^2 \\ a_1^3 b_1, & a_1^2 b_1^2, & a_1 b_1^3 \\ a_3^2 b_3^2, & a_3 b_3^3, & b_3^4 \end{vmatrix}.$$

Proceeding in this way, and noticing that in each case the interchanges are the same and therefore the sign of the element is correct, we finally obtain, by permuting every two rows and adding, the identity

$$\Delta_2^2 = \Sigma \begin{vmatrix} a_1^4, & a_1^3 b_1, & a_1^2 b_1^2 \\ a_2^3 b_2, & a_2^2 b_2^2, & a_2 b_2^3 \\ a_3^2 b_3^2, & a_3 b_3^3, & b_3^4 \end{vmatrix}.$$

Hence this sum, when operating on $u_1 u_2 u_3$, gives a covariant; *i.e.*,

$$\Sigma \begin{vmatrix} \frac{d^4 u_1}{dx^4}, & \frac{d^4 u_1}{dx^3 dy}, & \frac{d^4 u_1}{dx^2 dy^2} \\ \frac{d^4 u_2}{dx^3 dy}, & \frac{d^4 u_2}{dx^2 dy^2}, & \frac{d^4 u_2}{dx dy^3} \\ \frac{d^4 u_3}{dx^2 dy^2}, & \frac{d^4 u_3}{dx dy^3}, & \frac{d^4 u_3}{dy^4} \end{vmatrix}$$

is a covariant of u_1, u_2, u_3.

Now, on putting $u_1 = u_2 = u_3 = u$, each of the determinants included in the sum becomes identical with the given determinant, which is therefore a covariant of u.

Notice that this covariant is $\overline{12}^2 . \overline{23}^2 . \overline{34}^2 \ldots$ for a binary quantic.

If the quantic u is of the $2n^{\text{th}}$ order, the covariant reduces to what we may call, by analogy, the catalecticant of the q-ary $2n$-ic, which is therefore an invariant.

4. We have just shown how to express $\Delta_n \times \Delta_n$ as a determinant. We shall now proceed to show how to operate on Δ_n with Δ_{n-1}. For brevity, take a very simple example, viz., to show that

$$\begin{vmatrix} \frac{d^3 u}{dx^3}, & \frac{d^3 u}{dx^2 dy}, & \frac{d^3 u}{dx dy^2} \\ \frac{d^3 u}{dx^2 dy}, & \frac{d^3 u}{dx dy^2}, & \frac{d^3 u}{dy^3} \\ y^2, & -xy, & x^2 \end{vmatrix}$$

is a covariant of u.

We have seen that

$$\begin{vmatrix} \frac{d^2u}{dx^2}, & \frac{d^2u}{dx\,dy}, & \frac{d^2u}{dy^2} \\ \frac{d^2v}{dx^2}, & \frac{d^2v}{dx\,dy}, & \frac{d^2v}{dy} \\ y^2, & -xy, & x^2 \end{vmatrix}$$

or

$$\begin{vmatrix} a_1^2, & a_1 b_1, & b_1^2 \\ a_2^2, & a_2 b_2, & b_2^2 \\ y^2, & -xy, & x^2 \end{vmatrix} uv \quad (=\Delta, \text{ say})$$

is a covariant of u and v.

Consider

$$\Delta_1 = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

Take the first element of Δ_1, viz., $a_1 . b_2$, and multiply the first row of Δ by a_1, and the second row by b_2, permute the first and second row in each determinant, and add the results. Then, as before,

$$\Delta_1 . \Delta = \Sigma \begin{vmatrix} a_1^3, & a_1^2 b_1, & a_1 b_1^2 \\ a_2^2 b_2, & a_2 b_2^2, & b_2^3 \\ y^2, & -xy, & x^2 \end{vmatrix} uv.$$

Now put $u = v$, and we obtain the theorem stated.

Similarly, we get a concomitant of any binary quantic, viz.,

$$\begin{vmatrix} \frac{d^{2n-1}u}{dx^{2n-1}}, & & \frac{d^{2n-1}u}{dx^{2n-2}dy}, & \dots \\ \frac{d^{2n-1}u}{dx^{2n-2}dy}, & & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \xi^n, & & \xi^n\eta, & \dots \end{vmatrix},$$

where ξ, η are any contragredient quantities. As a particular case, we may replace ξ, η by $\frac{d}{dx}$, $\frac{d}{dy}$. Thus we can replace the last row by

$$\frac{d^n v}{dx^n}, \quad \frac{d^n v}{dx^{n-1}y}, \quad \dots,$$

where again we may replace v by u.

As u is a binary quantic, we may also replace ξ, η by $\frac{du}{dx}$, $\frac{du}{dy}$, or by $-y$, x.

The theorem in the case of a q-ary quantic is that the determinant whose successive rows are the terms of

$$\left(\frac{d}{dx_1}\right)^{n-1}\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\left(\frac{d}{dx_1}\right)^{n-2}\left(\frac{d}{dx_2}\right)\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the multipliers $\left(\frac{d}{dx_1}\right)^{n-1}$, $\left(\frac{d}{dx_1}\right)^{n-2}\frac{d}{dx_2}$, ... consist of the terms of $\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots\right)^{n-1}$, the rest of the rows being the terms of such expansions as $(\xi+\eta+\ldots)^n$, where ξ, η, ... are any contragredient quantities, is a concomitant of u. We may, of course, replace any row $(\xi+\eta+\ldots)^n$ by $\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots\right)^n v$, and so obtain a joint covariant of several quantics.

5. The above result is obtained by operating, say, with Δ_{n-1} on Δ_n. Other theorems may be obtained by operating with Δ_{n-r} on Δ_n. For example, to prove that

$$\begin{vmatrix} \frac{d^4u}{dx^4}, & \frac{d^4u}{dx^3dy}, & \frac{d^4u}{dx^2dy^2}, & \frac{d^4u}{dx\,dy^3} \\ \frac{d^4u}{dx^3dy}, & \frac{d^4u}{dx^2dy^2}, & \frac{d^4u}{dx\,dy^3}, & \frac{d^4u}{dy^4} \\ \frac{d^3u}{dx^3}, & \frac{d^3u}{dx^2dy}, & \frac{d^3u}{dx\,dy}, & \frac{d^3u}{dy^3} \\ y^3, & -x^2y, & xy^2, & -x^3 \end{vmatrix}$$

is a covariant of any binary quantic.

Start with the invariant operator

$$\begin{vmatrix} a_1^3, & a_1^2b_1, & a_1b_1^2, & b_1^3 \\ a_2^3, & \ldots & \ldots & \ldots \\ a_3^3, & \ldots & \ldots & \ldots \\ a_4^3, & \ldots & \ldots & b_4^3 \end{vmatrix},$$

and operate with $(a_1 b_2 - a_2 b_1)$; afterwards put $u_1 = u_2 = u_3$, and finally replace the last row by the terms of $(y-x)^3$.

6. We shall next show how to operate on Δ_n successively with operators of the same kind. Take a simple example.

To prove that

$$\left|\begin{array}{cccc} \dfrac{d^4u}{dx^4}, & \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3} \\ \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3}, & \dfrac{d^4u}{dy^4} \\ \dfrac{d^4v}{dx^4}, & \dfrac{d^4v}{dx^3dy}, & \dfrac{d^4v}{dx^2dy^2}, & \dfrac{d^4v}{dx\,dy^3} \\ \dfrac{d^4v}{dx^3dy}, & \dfrac{d^4v}{dx^2dy^2}, & \dfrac{d^4v}{dx\,dy^3}, & \dfrac{d^4v}{dy^4} \end{array}\right|$$

is a covariant of any two binary quantics.

Start with the invariant operator

$$\left|\begin{array}{cccc} a_1^3, & a_1^2b_1, & a_1b_1^2, & b_1^3 \\ a_2^3, & a_2^2b_2, & a_2b_2^2, & b_2^3 \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3^3 \\ a_4^3, & a_4^2b_4, & a_4b_4^2, & b_4^3 \end{array}\right|$$

As before, multiply by $(a_1b_2 - a_2b_1)$. This gives the invariant operator

$$\Sigma \left|\begin{array}{cccc} a_1^4, & a_1^3b_1, & a_1^2b_1^2, & a_1b_1^3 \\ a_2^3b_2, & a_2^2b_2^2, & a_2b_2^3, & b_2^4 \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3^3 \\ a_4^3, & a_4^2b_4, & a_4b_4^2, & b_4^3 \end{array}\right|,$$

where Σ means the sum of the determinants obtained by permuting the subscripts 1 and 2.

In this put $u_1 = u_2 = u$; then

$$\left|\begin{array}{cccc|c} a^4, & a^3b, & a^2b^2, & ab^3 & u \\ a^3b, & a^2b^2, & ab^3, & b^4 & u \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3 & u_3 \\ a_4, & a_4^2b_4, & a_4b_4^2, & b_4^3 & u_4 \end{array}\right|$$

is a covariant, where

$$| \; a^4, \quad a^3b, \quad a^2b^2, \quad ab^3 \; | \, u$$

is an abbreviation of

$$\left| \frac{d^4u}{dx^4}, \quad \ldots, \quad \frac{d^4u}{dxdy^3} \right|,$$

and so on.

Now, multiply by $(a_3b_4-a_4b_3)$, and put $u_3 = u_4 = v$, and we get the theorem stated.

If we start with Δ_n, we get similarly a covariant of two binary quantics which reduces to a joint invariant in certain cases.

7. In a similar way, we may obtain joint covariants and invariants of several binary quantics. For example,

$$\begin{vmatrix} a_1^7, & a_1^6b_1, & a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5 \\ a_1^6b_1, & a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5, & a_1b_1^6 \\ a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5, & a_1b_1^6, & b_1^7 \\ a_2^6, & a_2^5b_2, & a_2^4b_2^2, & a_2^3b_2^3, & a_2^2b_2^4, & a_2b \\ a_2^5b_2, & a_2^4b_2^2, & a_2^3b_2^3, & a_2^2b_2^4, & a_2b_2^5, & b_2^6 \\ a_3^5, & a_3^4b_3, & a_3^3b_3^2, & a_3^2b_3^3, & a_3b_3^4, & b_3^5 \end{vmatrix},$$

or

$$\Delta_1 \, . \, \Delta_2 \, . \, \Delta_5 \, ,$$

gives us a joint covariant of three binary quantics, and also a joint invariant of a binary 7-ic, a binary 6-ic, and a binary 5-ic.

We can also extend the method to quantics in several variables. For example,

$$\begin{vmatrix} a_1^3, & a_1^2b_1, & a_1^2c_1, & a_1b_1^2, & a_1b_1c_1, & a_1c_1^2 \\ a_1^2b_1, & a_1b \; , & a_1b_1c_1, & b_1^3, & b_1^2c_1, & b_1c_1^2 \\ a_1^2c_1, & a_1b_1c_1, & a_1c_1^2, & b_1^2c_1, & b_1c_1^2, & c_1^3 \\ a_2^3, & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_2^2b_2, & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_2^2c_2, & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

gives a joint covariant of two ternary quantics, and a joint invariant of two ternary cubics, as we see by operating upon the determinant operator derived from $(a+b+c)^2$ successively by $| a_1 b_2 c_3 |$ and $| a_4 b_5 c_6 |$.

Similarly, by starting with the determinant operator derived from $(a+b+c)^3$, which is of the tenth order, we can obtain a joint covariant of any three ternary quantics, and also a joint invariant of a ternary quintic, a ternary quartic, and a ternary cubic, viz., by operating with the operator derived from $(a+b+c)^2$ on the first six rows, with the operator $| a_7 b_8 c_9 |$ on the next three rows, and leaving the last row unaltered.

Note on the Potential of Rings. *By* A. L. Dixon.

Received and read April 8th, 1897.

In a paper recently read before this Society,* Dr. E. W. Hobson gave the following result, viz., that

$$\iiint \dots \frac{\left\{1-\sum\limits_{r=1}^{r=n} \frac{\xi_r^2}{a_r^2}\right\}^{\lambda-1}}{\left\{\sum\limits_{r=1}^{r=n} \overline{x_r-\xi_r}^2+h^2\right\}^{\frac{1}{2}m}} F\{\xi_1, \xi_2, \xi_3 \dots \xi_n\}\, d\xi_1\, d\xi_2 \dots d\xi_n$$

(where the integral is taken for all real values of the ξ's which make $1-\Sigma \frac{\xi^2}{a^2}$ positive or zero)

$$= \frac{m\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} U^{\frac{1}{2}(n-m)+\lambda-1}$$

$$\times \left\{1+\frac{U\theta D}{2\,.\,n-m+2\lambda}+\frac{U^2\theta^2 D^2}{2\,.\,4\,.\,n-m+2\lambda\,.\,n-m+2\lambda+2}\right.$$

$$\left.+\frac{U^3\theta^3 D^3}{2\,.\,4\,.\,6\,.\,n-m+2\lambda\,.\,n-m+2\lambda+2\,.\,n-m+2\lambda+4}+\dots\right\}$$

$$\times F\left\{\frac{a_1^2 x_1}{a_1^2+\theta}, \frac{a_2^2 x_2}{a_2^2+\theta}, \dots \frac{a_n^2 x_n}{a_n^2+\theta}\right\} d\theta,$$

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* *Proceedings*, Vol. XXVII., p. 524.