

*Note on the Algebraic Theory of Elliptic Transformation.* By  
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In a former note, published in the *Proceedings of the London Mathematical Society* (Vol. xviii., pp. 377-88), it was proved by the present writer that the transformations with regard to the  $\Theta$  functions can be easily deduced from an elementary formula, viz., if  $f$  is a function of  $x$  and  $k$ , then

$$\frac{d^2 f}{dx dk} = \frac{d^2 f}{dk dx}.$$

I now propose to show that we can form the differential equation satisfied by the numerator and denominator of a transformation equation

$$y = \frac{P(x, k)}{Q(x, k)}$$

by means of the elementary theorem in question.

#### SECTION I. *Elementary Theorem.*

If 
$$\frac{M dy}{\sqrt{1-y^2} \cdot 1-\lambda^2 y^2} = \frac{dx}{\sqrt{1-x^2} \cdot 1-k^2 x^2},$$

$$M^{-2} = \frac{nk k'^2}{\lambda \lambda'^2} \frac{d\lambda}{dk} \quad (n \text{ being a number}),$$

$$\Omega \equiv nk k'^2 \frac{d}{dk} + \Delta x G(x, k) \frac{d}{dx}, \quad \Delta x = \sqrt{1-x^2} \cdot 1-k^2 x^2,$$

$$\Omega \left( \frac{1-y^2}{1-\lambda^2 y^2} \right) = 0,$$

i.e., 
$$\Omega y = M^{-2} \lambda^2 y (1-y^2);$$

then 
$$\Delta x \frac{dG}{dx} + \frac{nk^2 k'^2 x^2}{1-k^2 x^2} - \frac{nk k'^2}{M} \frac{dM}{dk} = M^{-2} \lambda^2 (1-y^2) \dots \dots \dots (1).$$

This result is practically the same as that given in my former note (Vol. xviii., p. 383), and depends upon the identity

$$\frac{d^2 y}{dx dk} = \frac{d^2 y}{dk dx}.$$

By taking  $G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}}}$ ,

we arrive immediately, by means of this proposition, at the  $\Theta$  transformation equation

$$\Theta \{M^{-1}(n+u_0), \lambda\} = Oe^{\mu u + v u} \Theta^n(u) Q(\operatorname{sn} u),$$

where  $O, \mu, v$  are  $k$ -functions.

Assuming then that

$$G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}}},$$

the equation (1) may be written in the form

$$\begin{aligned} nk^2(1-x^2) - x(1+k^2-2k^2x^2) \frac{Q'}{Q} + (\Delta x)^2 \left\{ \frac{Q''}{Q} - \left( \frac{Q'}{Q} \right)^2 \right\} - \frac{nk^2}{M} \frac{dM}{dk} \\ = M^{-2} \lambda^2 (1-y^2) \dots \dots \dots (2), \end{aligned}$$

where

$$Q' = \frac{dQ}{dx}, \quad Q'' = \frac{d^2Q}{dx^2}.$$

## SECTION 2. *The Method of Squares.*

If  $P$  and  $Q$  be rational and integral functions of  $x$  and  $k$ , the equation (2) of the previous section leads to what may be called the method of squares.

In fact we have

$$\begin{aligned} \left( M^{-2} \lambda^2 + n \frac{kk'^2}{M} \frac{dM}{dk} - nk^2 \right) Q^2 + nk^2 x^2 Q^2 + x(1+k^2-2k^2x^2) Q'Q \\ + (\Delta x)^2 (Q'' - Q'^2/Q) = M^{-2} \lambda^2 P^2. \end{aligned}$$

In a similar manner, by writing  $\frac{1}{\lambda y}$  for  $y$ , and taking

$$G(x, k) = \Delta x \frac{d}{dx} \log \frac{P}{(1-k^2x^2)^{\frac{1}{2}}},$$

in the formulæ of Section I., it follows that

$$\begin{aligned} \left( M^{-2} \lambda^2 + n \frac{kk'^2}{M} \frac{dM}{dk} - nk^2 \right) P^2 + nk^2 x^2 P^2 + x(1+k^2-2k^2x^2) P'P \\ + (\Delta x)^2 (P'' - P'^2/P) = M^{-2} Q^2. \end{aligned}$$

For example, if  $n = 3$ ,

$$P = b_1x + b_3x^3, \quad Q = 1 + a_1x^2,$$

we find that  $a_3 - 6k^2a_1 - 4k^3(1+k^2)a_1 - 3k^4 = 0$ .

### SECTION 3. *Differential Equation satisfied by P and Q.*

Taking  $G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{1/2}},$

and writing  $\xi = M^{-2}\lambda^2(1-y^2),$

the results of Section I. are

$$\begin{aligned} \xi &= M^{-2}\lambda^2(1-y^2) \\ &= nk^2(1-x^2) + x(2k^2x^2 - 1 - k^2) \frac{Q'}{Q} + 1 - x^2 \cdot 1 - k^2x^2 \frac{d}{dx} \left( \frac{Q'}{Q} \right) - \frac{nk^2}{M} \frac{dM}{dk}, \end{aligned}$$

$$\Omega \left( \frac{1-y^2}{1-\lambda^2y^2} \right) = \Omega \left\{ \frac{\xi}{\lambda^2\xi + M^{-2}\lambda^2} \right\} = 0,$$

where

$$\lambda^2 = 1 - k^2,$$

$$\left. \begin{aligned} \Omega\xi &= 2\xi \left\{ \xi - \frac{nk^2}{M} \frac{dM}{dk} + M^{-2}(1-2\lambda^2) \right\} \\ \Omega &\equiv nk^2 \frac{d}{dk} + \left\{ nk^2x(1-x^2) + 1 - x^2 \cdot 1 - k^2x^2 \frac{Q'}{Q} \right\} \frac{d}{dx} \end{aligned} \right\} \dots (3).$$

We thus find ultimately that  $Q$  satisfies a differential equation of the form

$$\begin{aligned} &X_1Q^3 + X_2Q^2 + X_3Q \\ &+ x_3 \left\{ x_3 \frac{d^2Q}{dx^2} + (x_2 + 2x_1x_3) \frac{dQ}{dx} + 2nk^2x^2 \frac{dQ}{dk} \right\} \left( \frac{dQ}{dx} \right)^2 = 0 \dots (4), \end{aligned}$$

where  $x_1 = nk^2(1-x^2)$ ,  $x_2 = x(2k^2x^2 - 1 - k^2)$ ,  $x_3 = 1 - x^2 \cdot 1 - k^2x^2$ ,

and  $X_1$  is a rational and integral function of degree 4 in  $x$ , also  $X_2$ ,  $X_3$  are rational and integral functions of  $x$  involving the differential coefficients  $\frac{dQ}{dx}$ ,  $\frac{dQ}{dk}$ ,  $\frac{d^2Q}{dx^2}$ , &c.

If, then,  $Q$  is a rational and integral function of  $x$  of the form

$$Q = a_0 + a_1x + a_2x^2 + \dots,$$

which contains no square factor, it follows from (4) that the expression

$$x_3 \frac{d^3 Q}{dx^3} + (x_3 + 2xx_1) \frac{dQ}{dx} + 2nkk^2 \frac{dQ}{dk} \text{ involves the factor } Q.$$

In other words, we may take

$$x_3 \frac{d^3 Q}{dx^3} + (x_3 + 2xx_1) \frac{dQ}{dx} + 2nkk^2 \frac{dQ}{dk} = (\alpha + \beta x + \gamma x^2) Q \dots (5),$$

where  $x_3 = 1 - x^3$ ,  $1 - k^2 x^3$ ,  $x_2 = x(2k^2 x^3 - 1 - k^3)$ ,  $x_1 = nk^3(1 - x^3)$ ,

and  $\alpha, \beta, \gamma$  are  $k$ -functions. In general we have

$$\beta = 0 \quad \text{and} \quad \gamma = -n(n-1)k^3.$$

#### SECTION 4. Example of the result of Section 3.

If

$$P = b_1 x + b_3 x^3 + \dots + b_n x^n,$$

and

$$Q = 1 + a_3 x^3 + a_4 x^4 + \dots + a_t x^t + \dots a_{n-1} x^{n-1},$$

where  $n$  is an odd number, the differential equation (5) of the last section gives the following result, viz.,

$$(t+2)(t+1) a_{t+2} + 2nkk^2 \frac{da_t}{dk} - \{2a_3 + t^2 - t(2n-t)k^2\} a_t \\ + (n-t+2)(n-t+1) k^3 a_{t-2} = 0,$$

where

$$2a_3 = M^{-2} \lambda^3 - nk^3 + \frac{nkk^2}{M} \frac{dM}{dk}.$$

It thus appears that, if  $a_3$  were given, we could find  $a_4, a_5, \dots a_{n-1}$ .

For example, if  $n$  is a square number and  $\lambda = k$ , then  $a_3 = 0$  and  $a_4, a_5, \dots$  are also known functions of  $k$ .

It is clear that in general we have a differential equation of a certain order to determine  $a_3$ .

For instance, if  $n = 3$ , so that  $a_4 = 0$ ,  $a_5 = 0$ , ..., we have, by putting  $t = 2$ ,  $a_0 = 1$ ,

$$3kk^2 \frac{da_3}{dk} - a_3^2 - 2(1 - 2k^2) a_3 + 3k^3 = 0.$$

Similarly, if  $n = 5$ , so that  $a_6 = 0$ , ... there results a differential equation of the second order in  $k$  to determine  $a_3$ .

The above differential equation connecting  $a_{t+2}$ ,  $a_t$ ,  $a_{t-2}$  is also true when  $n$  is any number, and  $Q$  is an integral function of the form

$$Q = 1 + a_1x + a_2x^2 + \dots + a_nx^n,$$

the general term being  $a_tx^t$ .

If  $Q$  be written

$$Q = 1 + \sqrt{k} a_1x + ka_2x^2 + \dots + k^{1t} a_tx^t + \dots + k^{1n} a_nx^n,$$

and

$$k + \frac{1}{k} = p,$$

we have

$$(t+4)(t+3) a_{t+4} + 2nk^2 \frac{d}{dk} (a_{t+2}) - \{2a_2 - (t+2)(n-t-2)p\} a_{t+2} \\ + (n-t)(n-t-1) a_t = 0.$$

For instance, if  $Q = 1 + \sqrt{k} a_1x + ka_2x^2$ ,

we have  $n = 2$ ,  $a_3 = 0$ ,  $a_4 = 0$ , ...; so that, when  $t = -1$ ,  $a_{-1} = 0$ , we obtain

$$4k^2 \frac{da_1}{dk} = (2a_2 - p) a_1.$$

Again, when  $t = 0$ ,  $4k^2 \frac{da_2}{dk} - 2a_2^2 + 2 = 0$ .

It thus appears that  $a_2 = 1$ ,  $a_1 = ck^{-1} (1+k)^{\frac{1}{2}}$  (where  $c$  is a constant independent of  $k$ ) are solutions.

#### SECTION 5. *Deduction of the P Formulæ.\**

The following is an example of a method by which the  $P$  formulæ may be derived from those for  $Q$ .

Let  $Q = 1 + a_1x + a_2x^2 + \dots + a_tx^t + \dots + a_nx^n$ ,

$$P = b_0 + b_1x + b_2x^2 + \dots + b_tx^t + \dots + b_nx^n,$$

and suppose the transformation equation  $y = \frac{P}{Q}$  to be such that  $y$  is

\* This section has been added at the suggestion of one of the referees.

changed into  $\frac{1}{\lambda y}$  when  $\frac{1}{kx}$  is written for  $x$ ; we then have

$$(1) \quad (t+2)(t+1) a_{t+2} + 2nk k^2 \frac{da_t}{dk} - \{2a_3 + t^2 - t(2n-t)k^3\} a_t \\ + (n-t+2)(n-t+1) k^3 a_{t-2} = 0,$$

$$(2) \quad (t+2)(t+1) b_{t+2} + 2nk k^2 \frac{db_t}{dk} - \{2a_3 - M^{-2}\lambda^2 + t^2 - t(2n-t)k^3\} b_t \\ + (n-t+2)(n-t+1) k^3 b_{t-2} = 0.$$

In the particular example under consideration it can be proved without any difficulty that either of these formulæ may be transformed into the other.

For, if 
$$y = \frac{b_0 + b_1 x + \dots + b_n x^n}{1 + a_1 x + \dots + a_n x^n},$$

and also 
$$\frac{1}{\lambda y} = \frac{b_0 + b_1 (kx)^{-1} + \dots + b_n (kx)^{-n}}{1 + a_1 (kx)^{-1} + \dots + a_n (kx)^{-n}},$$

or 
$$y = \frac{k^n x^n + a_1 k^{n-1} x^{n-1} + \dots + a_n}{\lambda \{b_0 k^n x^n + b_1 k^{n-1} x^{n-1} + \dots + b_n\}},$$

we must have

$$\begin{array}{ccc|c} b_n = \mu k^n & a_n = \mu \lambda k^n b_0 & & \\ b_{n-1} = \mu k^{n-1} a_1 & a_{n-1} = \mu \lambda k^{n-1} b_1 & & \\ \dots\dots\dots & \dots\dots\dots & & \\ b_t = \mu k^t a_{n-t} & a_t = \mu \lambda k^t b_{n-t} & \text{and } \mu^2 \lambda k^n = 1. \end{array}$$

Hence, from (2), it follows that

$$(t+2)(t+1) \mu k^{t+2} a_{n-t-2} + 2nk k^2 \frac{d}{dk} (\mu k^t a_{n-t}) \\ - \{2a_3 - M^{-2}\lambda^2 + t^2 - t(2n-t)k^3\} \mu k^t a_{n-t} \\ + (n-t+2)(n-t+1) k^3 \cdot \mu k^{t-2} a_{n-t-2} = 0,$$

or 
$$(n-t+2)(n-t+1) a_{n-t+2} + 2nk k^2 \frac{da_{n-t}}{dk} \\ - \left\{ 2a_3 - M^{-2}\lambda^2 + t^2 - t(2n-t)k^3 - \frac{2nk k^2}{\mu k^t} \frac{d}{dk} (\mu k^t) \right\} a_{n-t} \\ + (t+2)(t+1) k^3 a_{n-t-2} = 0;$$

where

$$\mu^2 \lambda k^n = 1,$$

and, consequently,

$$\mu^2 k^{2t} = \frac{1}{\lambda k^{n-2t}};$$

$$\begin{aligned} -nkk^2 \frac{d}{dk} \log (\mu^2 k^{2t}) &= nkk^2 \frac{d}{dk} \log (\lambda k^{n-2t}) = \frac{nkk^2}{\lambda} \frac{d\lambda}{dk} + n(n-2t)k^2 \\ &= M^{-2}\lambda^2 + n(n-2t)k^2. \end{aligned}$$

It thus appears that (2) is transformed into

$$\begin{aligned} (n-t+2)(n-t+1) a_{n-t+2} + 2nkk^2 \frac{da_{n-t}}{dk} \\ - \{2a_2 + (n-t)^2 - (n^2 - t^2)k^2\} a_{n-t} + (t+2)(t+1)k^2 a_{n-t-2} = 0. \end{aligned}$$

Now, if we write  $t$  for  $n-t$ , this is

$$\begin{aligned} (t+2)(t+1) a_{t+2} + 2nkk^2 \frac{da_t}{dk} - \{2a_2 + t^2 - t(2n-t)k^2\} a_t \\ + (n-t+2)(n-t+1)k^2 a_{t-2} = 0; \end{aligned}$$

in other words (2) can be transformed into (1).

*Further Note on Automorphic Functions. By W. BURNSIDE.*

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I propose in the present paper to continue the consideration of certain groups, and the automorphic functions connected with them, with which I dealt shortly in a previous paper, published in the current volume of the Society's *Proceedings*. In that paper it was shown that, for symmetrical fuchsian groups of the first class, automorphic functions taking every value twice only in the generating (or any) polygon can always be found, and therefore that the algebraic equation connecting two different automorphic functions is in this case an equation of the hyperelliptic class.

I shall here show how, when the group is given, to calculate the coefficients in the equation, incidentally expressing the two variables which the equation connects as uniform functions of a single parameter.