ON A PLANE QUINTIC CURVE

By F. MORLEY.

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IN a memoir in the *American Journal* (Vol. x., "On Critic Centres") I pointed out that a curve of order 5, through the nine flexes of a cubic curve and the twelve other intersections of the lines of flexes, would have itself flexes at the nine points, the stationary lines thereat meeting on the quintic. And I mentioned that a curve of order 4, through the twelve points, would have them as flexes. This quartic appears in a paper by Caporali, published first in his *Works* (p. 336), whence it appears that the twenty-four flexes of the quartic fall into two such sets of twelve. Proofs of Caporali's statements are supplied in a memoir by Ciani (*Nap. Rendiconti*, Ser. 3, Vol. II., p. 126, 1896).

The object of the present paper is now clear—to prove that the forty-five flexes of the quintic fall into five sets of nine, or rather that *the forty-five stationary lines pass by nines through five points on the curve;* and to discuss the quintic with reference to this set of five points, which are singular points of a novel kind.

1. *First Equation of the Curve.*

If we denote a cubic by $(ax)^3$, its Hessian by $(hx)^3$, and draw from a point y tangents to the pencil $a+\lambda h = 0$, we have as locus of points of contact the quintic in question

(1 . 1) $$Q \equiv (ax)^3(hx)^2(hy)-(hx)^3(ax)^2(ay) = 0;$$

the polar quartic of y is

(1 . 2) $$Q_1 \equiv (ax)^3(hx)(hy)^2-(hx)^3(ax)(ay)^2 = 0;$$

the polar cubic of y is either

(1 . 3) $$Q_{11} \equiv (ax)^3(hy)^3-(hx)^3(ay)^3 = 0$$

or

(1 . 3′) $$(ax)^2(ay)(hx)(hy)^2-(hx)^2(hy)(ax)(ay)^2 = 0,$$

the two being equivalent by Salmon's identity (*Higher Plane Curves*, p. 206).

The polar conic of y is

(1 . 4) $$Q_{111} \equiv (ax)^2 (ay) (hy)^3 - (hx)^2 (hy) (ay)^3 = 0,$$

and the polar line of y is

(1 . 5) $$Q_{1111} \equiv (ax) (ay)^2 (hy)^3 - (hx) (hy)^2 (ay)^3 = 0.$$

It appears from these that (1) the points where a meets h are on Q, Q_1, Q_{11}; whence they are flexes on Q, the stationary lines meeting at y. (2) The points t, where $(ax)^2 (ay)$ meets $(hx)^2 (hy)$, are on Q, Q_{11}, Q_{111}. (3) The point where $(ax)(ay)^2$ meets $(hx)(hy)^2$ is on Q_1, Q_{11}, Q_{1111}. Thus, in particular, *y and any of the four points t are each on the polar conic of the other.*

2. *Mutuality of the Five Points.*

Use now the canonic form of Hesse, and let a term in parentheses be subjected to permutation of the suffixes 1, 2, 3, and then to summation of distinct terms. Thus let (x_1^3) stand for $x_1^3 + x_2^3 + x_3^3$, $(x_1 y_2 t_3)$ for the sum of six terms, and so on.

The pencil of cubics is $(x_1^3) + 6m x_1 x_2 x_3 = 0$, the polar conic of y is $(y_1 x_1^2) + 2m (y_1 x_2 x_3) = 0$. These give on elimination of m the quintic Q:

(2 . 1) $$(x_1^3)(y_1 x_2 x_3) = 3 x_1 x_2 x_3 (y_1 x_1^2).$$

The polar cubic of a point t as to Q is

(2 . 2) $$(y_1 t_2 t_3)(x_1^3) + 3 (y_1 t_2 x_3)(t_1 x_1^2) + 3 (y_1 x_2 x_3)(t_1^2 x_1)$$
$$= 3 (t_1 t_2 x_3)(y_1 x_1^2) + 6 (t_1 x_2 x_3)(y_1 t_1 x_1) + 3 x_1 x_2 x_3 (y_1 t_1^2).$$

I say that this equation holds when x and t are distinct points such that

(2 . 3) $$(y_1 x_1^2) = 0, \quad (y_1 x_2 x_3) = 0, \quad (y_1 t_1^2) = 0, \quad (y_1 t_2 t_3) = 0.$$

With these suppositions, (2 . 2) reduces at once to

(2 . 2′) $$(y_1 t_2 x_3)(t_1 x_1^2) = 2 (t_1 x_2 x_3)(y_1 t_1 x_1).$$

But, from (2 . 3), $$\lambda y_1 = t_2^2 x_3^2 - t_3^2 x_2^2,$$

$$\mu y_1 = t_1 x_1 (t_2 x_3 - t_3 x_2);$$

whence $$\frac{\lambda}{\mu} = \frac{t_2 x_3 + t_3 x_2}{t_1 x_1} = \frac{(y_1 t_2 x_3)}{(y_1 t_1 x_1)}$$

and also $$= \frac{2 (t_1 x_2 x_3)}{(t_1 x_1^2)};$$

whence (2.2′) follows. That is to say, the four points t_α are related among themselves as each was with y. The five points are mutual—we call them all t_α ($\alpha = 1$ to 5); the polar conic of any one goes through the rest, or, in other words, $t_\alpha^2 t_\beta^3$ is apolar with Q; or, again, the join of any two cuts from Q three other points of which the two are the Hessian.

It follows that what was true for the one point is true for all; in particular, that *the forty-five stationary lines pass, by nines, through the five points.*

3. *The Equation in terms of the Five Points.*

The quintic depended on a Hesse configuration (eight numbers) and a point (two numbers). Hence we may expect the five points to determine Q, and we next obtain a symmetric equation.

Let C be the conic on the five points, t_α; it touches the quintic at each point. Taking for coordinates of t_α

$$x_1 = t_\alpha^2, \quad x_2 = 2t_\alpha, \quad x_3 = 1,$$

let the conic be

$$C \equiv x_1 x_3 - x_2^2/4 = 0, \tag{3.1}$$

and let the tangent at t_α be

$$C_\alpha \equiv x_1 - x_2 t_\alpha + x_3 t_\alpha^2 = 0. \tag{3.2}$$

Also let $$C_{\alpha\beta} = (t_\alpha - t_\beta)^2,$$

and let the conic in lines be

$$\Gamma \equiv \xi_1 \xi_3 - \xi_2^2 = 0.$$

The polar of Γ as to $C_1 C_2 C_3 C_4 C_5$ is the cubic

$$\Sigma C_{12} C_3 C_4 C_5 = 0, \tag{3.3}$$

which cuts out from the conic the Hessian of the five points; and the second polar of Γ as to the five lines is the line

$$2\Sigma C_{12} C_3 C_{45} = 0, \tag{3.4}$$

which cuts out the fourth transvectant of the five points.

Assume now

$$Q \equiv C_1 C_2 C_3 C_4 C_5 - \lambda C \overset{10}{\Sigma} C_{12} C_3 C_4 C_5 + \mu C^2 \Sigma C_{12} C_3 C_{45}. \tag{3.5}$$

Excluding C_1 from the summations, we have

$$Q = C_1 C_2 C_3 C_4 C_5 - \lambda C C_1 \overset{6}{\Sigma} C_2 C_3 C_{45} - \lambda \overset{4}{\Sigma} C_{12} C_3 C_4 C_5$$
$$+ \mu C^2 C_1 \overset{3}{\Sigma} C_{23} C_{45} + \mu C^2 \overset{12}{\Sigma} C_{12} C_3 C_{45}.$$

Operating with $\frac{1}{2}(t_1^2\xi_1+2t_1\xi_2+\xi_3)^2$, we get the second polar of t_1,

$$Q_{11} = C_1\sum^{6} C_{12}C_{13}C_4C_5-\lambda C_1^2\sum^{12} C_{12}C_3C_{45}-2\lambda C_1\sum^{6} C_{12}C_{13}C_4C_5$$
$$+\mu C_1^3\sum^{3} C_{23}C_{45}+\mu C_1^2\sum^{12} C_{12}C_3C_{45}$$
$$+\text{terms with } C \text{ as factor.}$$

This vanishes at t_2 if

$$(1-2\lambda)\, C_{12}^2\sum^{3} C_{13}C_{24}C_{25}+(\mu-\lambda)\, C_{12}^2\left\{C_{12}\sum^{3} C_{23}C_{45}+2\sum^{3} C_{13}C_{24}C_{25}\right\}$$
$$+\mu C_{12}^3\sum^{3} C_{23}C_{45} = 0;$$

hence $1-2\lambda+2(\mu-\lambda)=0$ and $2\mu-\lambda=0$, whence $\lambda=\frac{1}{3}$, $\mu=\frac{1}{6}$. Thus the *symmetric equation of the quintic* is

(3.6) $$C_1C_2C_3C_4C_5-\tfrac{1}{3}C\sum C_{12}C_3C_4C_5+\tfrac{1}{6}C^2\sum C_{12}C_3C_{45} = 0,$$

or, in a convenient notation,

(3.7) $$(1-\tfrac{1}{3}C.\Gamma+\tfrac{1}{12}C^2.\Gamma^2)\, C_1C_2C_3C_4C_5 = 0.$$

4. *Common Lines of the Quintic and the Five-fold Polar Conic.*

In the symmetric equation (3.6), let $t_4=0$, $t_5=\infty$, $x_1/2x_2=x$, $x_3/2x_2=y$, so that x and y will be conjugate coordinates if the conic be a circle and t_4 and t_5 the points at infinity on it. The equation becomes

$$3xy\,\Pi(x+yt_1^2-2t_1)$$
$$-(xy-1)\left\{\Sigma(t_2-t_3)^2(x+yt_1^2-2t_1)xy\right.$$
$$\left.+\Sigma(x+yt_1^2)(x+yt_2^2-2t_2)(x+yt_3^2-2t_3)+\Pi(x+yt_1^2-2t_1\right\}$$
$$+\tfrac{1}{2}(xy-1)^2\left\{\Sigma(t_2-t_3)^2(x+yt_1^2-2t_1)+\Sigma(t_2^2+t_3^2)(x+yt_1^2-2t_1)\right.$$
$$\left.+\Sigma(t_2-t_3)^2(x+yt_1^2)\right\} = 0;$$

or, if $$\prod^{3}(t-t_1) = t^3-s_1t^2+s_2t-s_3,$$

(4.1) $$-xy(x^3+y^3s_3^2)-6x^2y^2s_3+4(x^3+y^3s_3^2)+6xy(xs_2+ys_1s_3)$$
$$-6(x^2s_1+y^2s_2s_3)-4xy(s_1s_2+s_3)+3(xs_1^2+ys_2^2)$$
$$-2(s_1s_2-2s_3) = 0.$$

From this it appears immediately that, if τ be a cube root of s_3^2, the real asymptotes are $x+y\tau^2 = 2\tau$. These are lines of the circle. Hence :

The tangents of the quintic at the points where the join of two of the five points meets it again are also tangents of the conic.

There are then, in addition to the common tangents at the five points, thirty other common tangents, or forty in all. Hence the quintic is of class 20, and *is of full genus* 6.

Selecting a zero of direction such that $s_3 = 1$, the polar cubic of t_4 and t_5 as to (4 . 1) is

$$(4.2) \qquad x^3+y^3+6xy-3(xs_2+ys_1)+s_1s_2+1 = 0.$$

If we operate on this (written homogeneously) with

$$(\xi t_1^2+\eta+\zeta t_1)(\xi t_2^2+\eta+\zeta t_2),$$

the result is zero. Hence :

Any four of the five points are apolar with the quintic ; or, otherwise expressed, the polar conic of three of the five points is the double join of the other two.

In the notation of § 3, this double line is $C_{12}C = C_1C_2$.

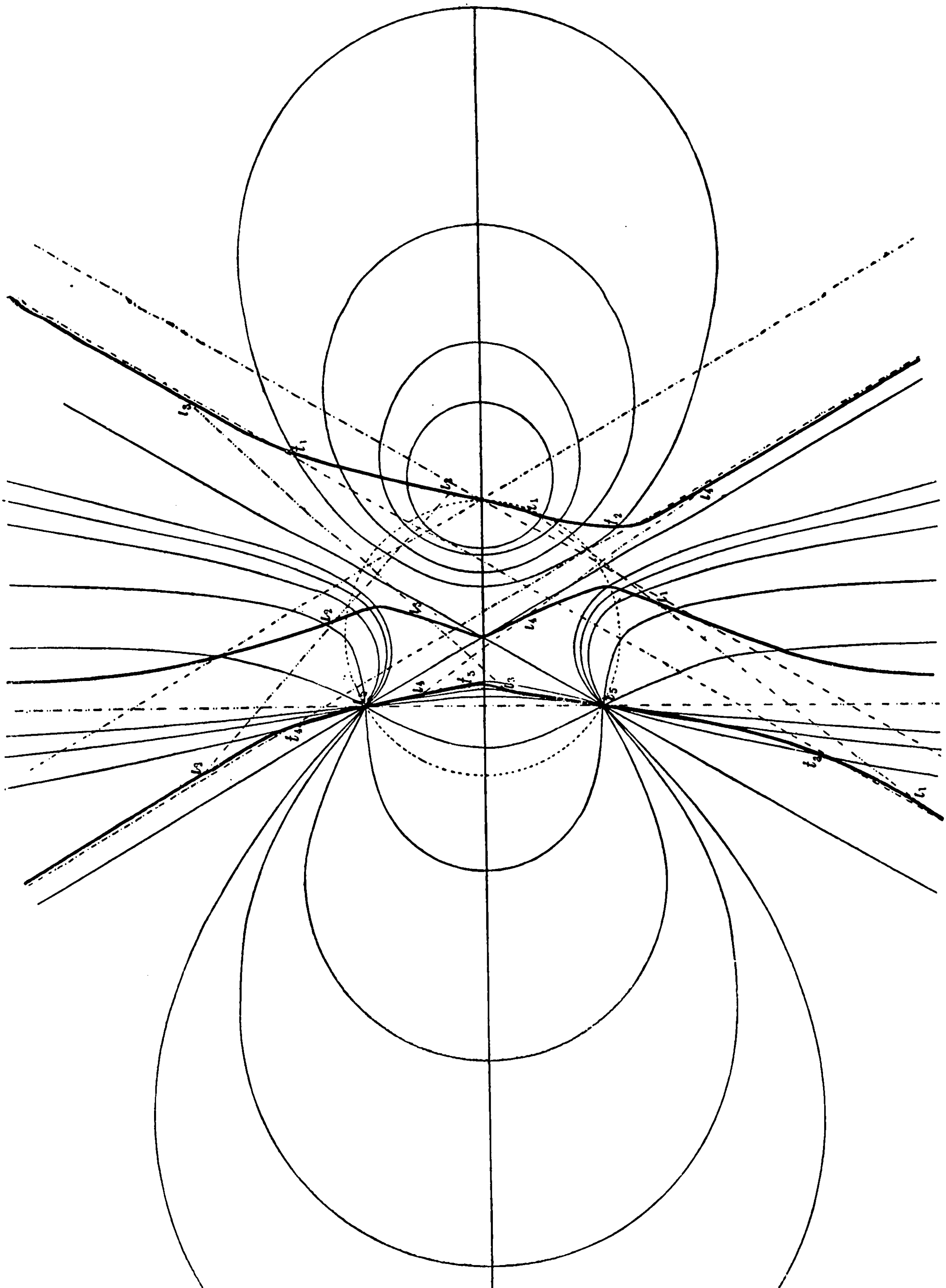
A number of conics and cubics associated with the curve might be written down ; I mention only the conic $xy = 4$, which from (4 . 1) osculates the quintic at t_4 and t_5, and passes through the intersections of the tangents where their join meets Q again.

5. *Nature of the Five Points.*

In the general plane quintic, through a point of the curve can be drawn six lines meeting the curve again in a self-apolar set of four points. This six-line for the quintic replaces the four tangents from a point of a cubic, which Salmon showed to have constant double ratios. And it may be—to this point I expect to return—that among the three six-lines from three points of a quintic there is a special trilineation.

At certain points of the curve specified invariants of the six-line will vanish : for instance, the six-line will be self-apolar in general at forty points.

The peculiarity of the points t, in the quintic of this memoir, is that *their six-lines are arbitrary ;* any line through a point t cuts out a self-apolar four-point. The proof is simply that, if in (4 . 1) we hold y,



we have a quartic in x, say $(ax)^4$, for which $|\alpha\beta|^4 = 0$. This is verified at once.

The same could be proved from the fact that the pencil of cubics

$$(ax)^3+\lambda(hx)^3 = 0$$

cuts a line in an involution any two triads of which are apolar; whence the four points where cubics of the pencil touch a line are self-apolar, and the four points in which these cubics cut the line again are also self-apolar, the two sets of four forming a cube-configuration.

6. *The Look of the Curve.*

The five points t may be all real, three real, or one real, so that we may have fifteen, nine, or three real flexes, and respectively 0, 3, 6 real isolated double lines, by Klein's rule.

The figure (drawn by Mr. J. F. Messick) indicates the most difficult case of fifteen real flexes, and may be otherwise of use as showing the pencil of cubics. In obtaining this figure I write the pencil, in conjugate co-ordinates,

$$xy(x+y)+\mu(x^2+xy+y^2)+1 = 0;$$

two flexes are at the circular points, the two real finite flexes are the complex cube roots of unity.