we must have a distribution of electricity at some of the surfaces, whether we have any in the space included in the $dx\,dy\,dz$ integration or not. Analytically, it means that we are no longer to make

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0$$

throughout the entire space of the integration, but only up to an infinitesimal distance from the boundaries where there is electricity, just as in the case of the straight line, we are not to make

$$d^2y/dx^2=0$$

everywhere throughout the integration, but only up to an infinitesimal distance from the limiting points P and Q.

What occurs in the potential problem is probably typical of what happens in multiple integrals generally. For instance, given two continuous closed curves, the solution of the problem to join them by the surface of least area is analytically as well as physically continuous. But, if we replace the two curves by two lines with any number of angular points, we shall evidently have a physically continuous surface giving a solution which has analytical discontinuities at the boundaries. In general, we may expect that the effect of a superabundant number of limiting conditions is merely to introduce discontinuities of some kind *at the boundary*, and to leave the solution continuous within the general extent of the integration, and not by any means to render the function incapable of a maximum or a minimum value.

On those Orthogonal Substitutions that can be Generated by the Repetition of an Infinitesimal Orthogonal Substitution. By HENRY TABER. Received May 1st, 1895. Read May 9th, 1895.

§ 1.

In the following I show what are the conditions necessary and sufficient that a given orthogonal substitution of n variables may be generated by the repetition of an infinitesimal orthogonal substitution of the same number of variables (that is, by the repetition of an orthogonal substitution of n variables infinitely near to the identical substitution).

An orthogonal substitution may be designated as of the first or second kind according as it is or is not the second power of an orthogonal substitution. Improper orthogonal substitutions are then of the second kind. All real proper orthogonal substitutions, and all imaginary proper orthogonal substitutions of two or three variables, are of the first kind; but there are imaginary proper orthogonal substitutions of n variables of the second kind for any value of $n \geqq 4$. Thus the imaginary proper orthogonal substitution of four variables given on p. 255 of the *Bulletin of the New York Mathematical Society*, for July, 1894, is not the second power of any orthogonal substitution whatever; and, from the existence for four variables of an orthogonal substitution of the second kind, it follows that, for any number of variables greater than four, there are proper orthogonal substitutions of the second kind. In the number of the *Bulletin* referred to above, I have shown that any orthogonal substitution of the first kind can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution; but that no orthogonal substitution of the second kind can be generated thus. (See § 3.) The conditions necessary and sufficient that a given orthogonal substitution may be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution are, then, the same as the conditions necessary and sufficient that a given orthogonal substitution shall be the second power of an orthogonal substitution, that is, that an orthogonal substitution shall be of the first kind.

In Vol. XVI., p. 130, of the *American Journal of Mathematics*, I have shown that certain conditions, presently to be named, are satisfied by every orthogonal substitution of the first kind, that is, by every orthogonal substitution which is the second power of an orthogonal substitution. I now find that these conditions are sufficient as well as necessary.

That these conditions are sufficient may be most readily shown, if, in accordance with Cayley's "Memoir on the Theory of Matrices," *Philosophical Transactions*, 1858, we regard the operations of *addition* and *subtraction* as capable of extension to linear substitutions or their *matrices*, that is, the square array of their coefficients.* *Multiplication*

* Denoting by $(\phi)_{rs}$ the coefficient of the linear substitution ϕ of n variables in the r^{th} row and s^{th} column of its matrix, the sum or difference of two linear substitutions ϕ and ψ of n variables is defined as follows :—

$$(\phi \pm \psi)_{rs} = (\phi)_{rs} \pm (\psi)_{rs} \quad (r, s = 1, 2, \dots n).$$

is, of course, taken as equivalent to the composition of linear substitutions, and is associative and distributive. Multiplication is not in general commutative; but, if $f(\phi)$ and $F(\phi)$ are two polynomials in the linear substitution ϕ, we have

$$f(\phi) \cdot F(\phi) = F(\phi) \cdot f(\phi).$$

In what follows the *identical substitution* will be denoted by δ; the linear substitution which, multiplied by or into ϕ, gives the identical substitution will be denoted by ϕ^{-1}; and the linear substitution *transverse* or *conjugate* to ϕ will be denoted by $\breve{\phi}$.* We then have

$$(\phi\psi)^{-1} = \psi^{-1}\phi^{-1}, \quad \overbrace{\phi+\psi} = \breve{\phi}+\breve{\psi}, \quad \overbrace{(\phi\psi)} = \breve{\psi}\,\breve{\phi},$$

and

$$\overbrace{(\phi^{-1})} = (\breve{\phi})^{-1}.$$

The linear substitution ϕ is *symmetric*, if $\breve{\phi} = \phi$; is *skew symmetric*, if $\breve{\phi} = -\phi$; and is *orthogonal*, if $\breve{\phi} = \phi^{-1}$. Finally, the *determinant* of the linear substitution ϕ will be denoted by $|\phi|$. The characteristic equation of ϕ is then

$$|\phi - z\delta| = 0.$$

Further, following Sylvester, I shall employ the term *nullity* to denote the complement relative to n, the number of variables, of the order of the non-evanescent minor formed from the rows and columns of the determinant or matrix of a linear substitution. Thus, the nullity of the linear substitution ϕ of n variables is m, if the $(m-1)^{\text{th}}$ minors of $|\phi|$ (the minors of order $n-m+1$) all vanish, but not all the m^{th} minors (the minors of order $n-m$). In particular, if

$$|\phi| \neq 0,$$

the nullity of ϕ is zero. If g is a root of multiplicity m of the characteristic equation of ϕ, the nullity of $\phi - g\delta$ is at least 1, and the nullity of successive integer powers of $\phi - g\delta$ increases until a power of index $\mu \leqq m$ is attained, whose nullity is m. The nullity

* With the notation of the preceding note, we have r and s taking all integer values from 1 to n,

$$(\delta)_{rr} = 1, \quad (\delta)_{rs} = 0 \quad (r \neq s),$$

$$(\breve{\phi})_{rs} = (\phi)_{sr},$$

and if, as in what follows, we denote the determinant of ϕ by $|\phi|$,

$$(\phi^{-1})_{rs} = \frac{1}{|\phi|}\frac{d\,|\phi|}{d\,(\phi)_{sr}}.$$

of the $(\mu+1)^{\text{th}}$ and higher powers of $\phi - g\delta$ is then also m. And if we designate respectively by

$$m_1,\ m_2,\ \dots\ m_{\mu-1},\ m_\mu = m,$$

the nullities of $(\phi-g\delta)$, $(\phi-g\delta)^2$, ... $(\phi-g\delta)^{\mu-1}$, $(\phi-g\delta)^{\mu}$,

we have $$m_2 - m_1 \geqq m_3 - m_2 \geqq \dots \geqq m_\mu - m_{\mu-1} \geqq 1.$$

The numbers m_1, m_2, &c., may be termed the numbers *belonging* to the root g of the characteristic equation of ϕ. If g is not a root of the characteristic equation of ϕ, that is, if the multiplicity of g is zero, the nullity of $\phi - g\delta$ [and of $(\phi-g\delta)^2$, &c.] is zero, and we may say that the number belonging to g is zero. Again, by the "corollary of the law of nullity," if g_1 and g_2 are distinct roots of the characteristic equation of ϕ, the nullity of $(\phi-g_1)^{p_1}(\phi-g_2)^{p_2}$ is the sum of the nullities of the two factors.

Let, now, ϕ be an orthogonal substitution which is the second power of an orthogonal substitution ψ; that is, let $\phi = \psi^2$, ψ being orthogonal. The roots of the characteristic equation of ϕ are the squares of the roots of the characteristic equation of ψ. Therefore, if -1 is a root of the characteristic equation of ϕ, $\sqrt{-1}$ is a root of the characteristic equation of ψ; that is, the determinant of $\psi - \sqrt{-1}\,\delta$ is zero. But then the determinant of the transverse of $\psi - \sqrt{-1}\,\delta$, namely, $\breve{\psi} - \sqrt{-1}\,\delta$, obtained from $\psi - \sqrt{-1}\,\delta$ by interchanging the rows and columns of its matrix, is also zero; and, since

$$\breve{\psi} - \sqrt{-1}\,\delta = \psi^{-1} - \sqrt{-1}\,\delta = -\sqrt{-1}\,\psi^{-1}(\psi + \sqrt{-1}\,\delta),$$

therefore, because $$|\,\psi^{-1}\,| \neq 0,$$

$$|\,\psi + \sqrt{-1}\,\delta\,| = 0,$$

that is, $-\sqrt{-1}$ is also a root of the characteristic equation of ψ.

If the nullity of $\psi - \sqrt{-1}\,\delta$ is m_1, the nullity of its transverse,

namely, $\breve{\psi} - \sqrt{-1}\,\delta = \psi^{-1} - \sqrt{-1}\,\delta) = -\sqrt{-1}\,\psi^{-1}(\psi + \sqrt{-1}\,\delta)$

is also m_1; and, since the nullity of ψ^{-1} is zero, the nullity of $\psi + \sqrt{-1}\,\delta$ is m_1. Therefore, the nullity of

$$(\phi+\delta) = (\psi - \sqrt{-1}\,\delta)(\psi + \sqrt{-1}\,\delta)$$

is $2m_1$.

Similarly, if the nullity of $(\psi-\sqrt{-1})^p$ is m_p, the nullity of its transverse

$$(\psi-\sqrt{-1}\,\delta)^p=(\psi^{-1}-\sqrt{-1}\,\delta)^p=(-\sqrt{-1}\,\psi^{-1})^p\,(\psi+\sqrt{-1}\,\delta)^p$$

is also m_p. Therefore, the nullity of

$$(\phi+\delta)^p=(\psi-\sqrt{-1})^p\,(\psi+\sqrt{-1})^p$$

is $2m_p$.

§ 2.

Conversely, if ϕ is orthogonal, and if for any positive integer p the nullity of $(\phi+\delta)^p$ is even, ϕ is the second power of an orthogonal substitution.

In Vol. LXXXIV. of *Crelle's Journal*, Frobenius has given substantially the following theorem, namely, that an orthogonal substitution ψ can always be formed of whose characteristic equation any given quantities (other than zero) are roots of any given multiplicities, provided that, if $g \neq \pm 1$ is a root of the characteristic equation, g^{-1} is also a root of the same multiplicity as g; moreover, that we may take any set of numbers $m_1, m_2, \dots m_\mu$, subject to the conditions

$$m_2-m_1 \geqq m_3-m_2 \geqq \dots \geqq m_\mu-m_{\mu-1} \geqq 1,$$

as the numbers belonging to the root $g \neq \pm 1$, provided that the same set of numbers belongs to g^{-1}. Further, ψ may have $+1$ as a root of its characteristic equation, and the numbers belonging to $+1$ may be taken the same as the numbers belonging to the root $+1$ of the characteristic equation of any other orthogonal substitution.

Let, now, the roots of the characteristic equation of ϕ be $+1$ of multiplicity m, -1 of multiplicity $2m$, and g_r, g_r^{-1} each of multiplicity $m^{(r)}$, r taking all integer values from 1 to ν. Let the numbers belonging respectively to $+1$ and -1 be

$$(\mathrm{m}_1, \mathrm{m}_2, \dots \mathrm{m}_{\mu_0})\,(2m_1, 2m_2, \dots 2m_\mu),$$

and the numbers belonging to g_r, g_r^{-1}, for $r=1, 2, \dots, \nu$, be

$$(m_1^{(r)}, m_2^{(r)}, \dots m_{\mu_r}^{(r)}).$$

Let us now form an orthogonal substitution ψ whose characteristic equation shall have as roots $+1$ of multiplicity m, $\pm\sqrt{-1}$ each of multiplicity m, and for

$$r=1, 2, \dots \nu, \quad h_r=\sqrt{g_r}, \quad h_r^{-1}=\frac{1}{\sqrt{g_r}},$$

each of multiplicity $m^{(r)}$. Further, let the numbers belonging to $+1$ be

$$(\mathrm{m}_1,\ \mathrm{m}_2,\ \dots\ \mathrm{m}_{\mu_0});$$

let the numbers belonging to $+\sqrt{-1}$ and to $-\sqrt{-1}$ be

$$(m_1,\ m_2,\ \dots\ m_\mu);$$

and, for $r = 1, 2, \dots \nu$, let the numbers belonging to h_r and h_r^{-1} be

$$(m_1^{(r)},\ m_2^{(r)},\ \dots\ m_{\mu_r}^{(r)}).$$

The roots of the characteristic equation of ψ^2 are then $+1$ of multiplicity m, -1 of multiplicity $2m$, and (for $r = 1, 2, \dots \nu$) g_r and g_r^{-1} each of multiplicity $m^{(r)}$. Further, since -1 is not a root of the characteristic equation of ψ, the nullity of $\psi+\delta$ is zero; therefore, the nullity of

$$(\psi^2-\delta)^p = (\psi+\delta)^p\,(\psi-\delta)^p$$

is equal to the nullity of $(\psi-\delta)^p$. Consequently, the numbers belonging to the root $+1$ of the characteristic equation of ψ^2 are

$$(\mathrm{m}_1,\ \mathrm{m}_2,\ \dots\ \mathrm{m}_{\mu_0}).$$

Again, the nullity of

$$(\psi^2+\delta)^p = (\psi-\sqrt{-1}\,\delta)^p\,(\psi+\sqrt{-1}\,\delta)^p$$

is equal to the sum of the nullities of

$$(\psi-\sqrt{-1}\,\delta)^p \quad \text{and} \quad (\psi+\sqrt{-1}\,\delta)^p,$$

since both $\pm\sqrt{-1}$ are roots of the characteristic equation of ψ. Therefore, the numbers belonging to the root -1 of the characteristic equation of ψ^2 are

$$(2m_1,\ 2m_2,\ \dots\ 2m_\mu).$$

Finally, the nullity of

$$(\psi^2+g_r\delta)^p = (\psi+h_r\delta)^p\,(\psi-h_r\delta)^p$$

is equal to the nullity of $(\psi-h_r\delta)^p$, since $-h_r$ is not a root of the characteristic equation of ψ; and, therefore, the nullity of $(\psi+h_r\delta)^p$ is zero. Similarly, the nullity

$$(\psi^2-g_r^{-1}\,\delta)^p = (\psi+h_r^{-1}\,\delta)^p\,(\psi-h_r^{-1}\,\delta)^p$$

is equal to the nullity of $(\psi-h_r^{-1}\,\delta)^p$, since $-h_r^{-1}$ is not a root of the characteristic equation of ψ. Whence it follows that, for $r = 1, 2, \dots \nu$, the numbers belonging to the root g_r, g_r^{-1} of the characteristic equation of ψ^2 are

$$(m_1^{(r)},\ m_2^{(r)},\ \dots\ m_{\mu_r}^{(r)}).$$

Since ϕ and ψ^2 are similar, that is, the roots of the characteristic equations of ϕ and ψ and the numbers belonging to these roots are the same, a linear substitution ϖ of non-zero determinant can be found such that

$$\phi = \varpi\psi^2\varpi^{-1}.$$

Since both ϕ and ψ are orthogonal, we can always so choose ϖ that it shall be orthogonal. For we have

$$\delta = \breve{\phi}\phi = \breve{\varpi}^{-1}\breve{\psi}^2\breve{\varpi}\,\varpi\psi^2\varpi^{-1}.$$

That is, denoting $\breve{\varpi}\varpi$ by ω,

$$\breve{\psi}^2\omega\psi^2 = \omega;$$

or, since ψ^2 is orthogonal, $\omega\psi^2 = \psi^2\omega.$

The linear substitution ω is of non-zero determinant; there are, therefore, one or more polynomials in ω whose second power is equal to ω. Let $\omega^{\frac{1}{2}}$ denote any one of these polynomials. Then, since ω is symmetric, $\omega^{\frac{1}{2}}$ is also symmetric; and, moreover, since ψ^2 is commutative with ω, it is also commutative with $\omega^{\frac{1}{2}}$, that is,

$$\omega^{\frac{1}{2}}\psi^2 = \psi^2\omega^{\frac{1}{2}}.$$

Any linear substitution ϖ satisfying the equation

$$\breve{\varpi}\varpi = \omega$$

is given by the expression $\omega^{\frac{1}{2}}\chi$, in which χ is an orthogonal substitution. For the last equation may be written

$$(\omega^{\frac{1}{2}})^{-1}\breve{\varpi}\varpi(\omega^{\frac{1}{2}})^{-1} = \delta;$$

and, if we put $\chi = \varpi(\omega^{\frac{1}{2}})^{-1},$

it becomes $\breve{\chi}\chi = \delta.$

Conversely, if χ is orthogonal, and

$$\varpi = \chi\omega^{\frac{1}{2}},$$

we have $\breve{\varpi}\varpi = \omega^{\frac{1}{2}}\breve{\chi}\chi\,\omega^{\frac{1}{2}} = \omega.$

We therefore have

$$\phi = \varpi\psi^2\varpi^{-1} = \chi\,\omega^{\frac{1}{2}}\psi^2\omega^{-\frac{1}{2}}\chi^{-1} = \chi\psi^2\chi^{-1}.$$

And, if we put $$\Psi = \chi\,\psi\,\chi^{-1},$$

Ψ is orthogonal, since both ψ and χ are orthogonal; and

$$\phi = \chi\psi^2\chi^{-1} = \Psi^2;$$

that is, ϕ is the second power of an orthogonal substitution.

§ 3.

We may show as follows that any orthogonal substitution of the first kind can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution, but that no orthogonal substitution of the second kind can be generated thus. Let e^{ϑ} denote the infinite series

$$\delta + \vartheta + \frac{1}{2!}\vartheta^2 + \frac{1}{3!}\vartheta^3 + \ldots + \frac{1}{m!}\vartheta^m + \ldots,$$

convergent for any linear substitution ϑ. We then have

$$(e^{\vartheta})^{-1} = e^{-\vartheta},$$

$$(\breve{e^{\vartheta}}) = e^{\breve{\vartheta}};$$

and, if m is any positive integer,

$$(e^{\vartheta})^m = e^{m\vartheta}.$$

Moreover, if ϑ and ϑ' are commutative,

$$e^{\vartheta}\,e^{\vartheta'} = e^{\vartheta+\vartheta'}.$$

Finally, for any linear substitution ϕ of non-zero determinant, we can always find a polynomial in ϕ,

$$\vartheta = f(\phi),$$

such that $$\phi = e^{\vartheta}.$$

If, now, ϕ is orthogonal, we have

$$\breve{\vartheta} = f(\breve{\phi}) = f(\phi^{-1}),$$

that is, $\breve{\vartheta}$ is also a polynomial in ϕ;* and, consequently, $\breve{\vartheta}$ is commutative with ϑ.

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* The reciprocal of a linear substitution ϕ is expressible as a polynomial in ϕ.

Therefore, $$e^{\breve{\vartheta}+\vartheta} = e^{\breve{\vartheta}} e^{\vartheta} = \breve{\phi}\phi = \delta.$$

Let $$2\theta_0 = \vartheta + \breve{\vartheta}, \quad 2\theta = \vartheta - \breve{\vartheta}.$$

Since ϑ and $\breve{\vartheta}$ are commutative, θ_0 and θ, their half sum and difference, are commutative; and, consequently,

$$\phi = e^{\vartheta} = e^{\theta_0+\theta} = e^{\theta_0} e^{\theta} = e^{\theta} e^{\theta_0}.$$

Therefore, $$\phi^2 = (e^{\theta_0})^2 (e^{\theta})^2 = e^{2\theta_0} e^{2\theta} = e^{2\theta},$$

since $$e^{2\theta_0} = e^{\vartheta+\breve{\vartheta}} = \delta.$$

Let m be any positive integer, and let

$$\psi = e^{(2/m)\theta};$$

then, since θ is skew symmetric,

$$\breve{\psi}\psi = e^{(2/m)\breve{\theta}} e^{(2/m)\theta} = e^{-(2/m)\theta} e^{(2/m)\theta} = e^{-(2/m)\theta+(2/m)\theta} = \delta;$$

moreover, $$\psi^m = (e^{(2/m)\theta})^m = e^{2\theta} = \phi^2.$$

By taking m sufficiently great, we can make the coefficients of $\frac{2}{m}\theta$ as small as we please, and, consequently, we can make

$$\psi = e^{(2/m)\theta}$$

as nearly as we please equal to the identical substitution. But, however great m may be, we have

$$\breve{\psi}\psi = \delta \quad \text{and} \quad \psi^m = \phi^2.$$

Therefore, any orthogonal substitution, as ϕ^2, which is the second power of an orthogonal substitution can be generated by the repetition of an infinitesimal orthogonal substitution.

Every orthogonal substitution given by Cayley's expression is of the first kind. For, if

$$\phi = (\delta - \Upsilon)(\delta + \Upsilon)^{-1},$$

in which Υ is skew symmetric, and such that

$$|\,\delta + \Upsilon\,| \neq 0,$$

we can find a polynomial in Υ,

$$\vartheta = f(\Upsilon),$$

such that $$(\delta + \Upsilon) = e^{\vartheta}.$$

Equating the transverse of either side, we have

$$\delta - \Upsilon = e^{\breve{\vartheta}}.$$

And, since
$$\breve{\vartheta} = f(\breve{\Upsilon}) = f(-\Upsilon)$$
is also a polynomial in Υ, ϑ and $\breve{\vartheta}$ are commutative. Therefore, if we put
$$\theta = \vartheta - \breve{\vartheta} = f(-\Upsilon) - f(\Upsilon),$$
θ is skew symmetric, as it is a polynomial in odd powers of the skew symmetric matrix Υ, and
$$\phi = (\delta - \Upsilon)(\delta + \Upsilon) = e^{\breve{\vartheta}} e^{-\vartheta} = e^{\theta}.$$
If, now,
$$\psi = e^{\frac{1}{2}\theta},$$
since θ is skew symmetric,
$$\breve{\psi}\psi = e^{\frac{1}{2}\breve{\theta}} e^{\frac{1}{2}\theta} = e^{-\frac{1}{2}\theta} e^{\frac{1}{2}\theta} = e^{-\frac{1}{2}\theta + \frac{1}{2}\theta} = \delta;$$
moreover,
$$\psi^2 = (e^{\frac{1}{2}\theta})^2 = e^{\theta} = \phi.$$
That is, ϕ is an orthogonal substitution of the first kind.

If we take the orthogonal substitution ϕ sufficiently near to the identical substitution, -1 cannot be a root of the characteristic equation of ϕ; and ϕ is therefore given by Cayley's expression, and is consequently of the first kind. But the repetition of an orthogonal substitution of the first kind gives an orthogonal substitution of that kind. Whence it follows that no orthogonal substitution of the second kind can be generated by the repetition of an infinitesimal orthogonal substitution. Nevertheless, we can approximate as near as we please to any proper orthogonal substitution of the second kind by the repetition of an infinitesimal orthogonal substitution properly chosen. For we can obtain an orthogonal substitution of the first kind which shall be as nearly as we please equal to any proper orthogonal substitution of the second kind,* and the former

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* In particular, if ϕ is any proper orthogonal substitution of the second kind, we can find an orthogonal substitution ϕ_ρ of the first kind whose coefficients are rational functions of a parameter ρ such that, by taking ρ sufficiently small, the several coefficients of ϕ_ρ can be made as nearly as we please equal to the corresponding coefficients of ϕ. Consequently, if the rational functions are properly chosen, we shall have $(\phi_\rho)_{\rho=0} = \phi$. So long as $\rho \neq 0$, there exists an orthogonal substitution ψ_ρ whose coefficients are algebraic functions of ρ such that $\psi_\rho^2 = \phi_\rho$; and thus, by taking ρ sufficiently small, we may make ψ_ρ^2 as nearly as we please equal to ϕ. We thus have $\phi = \lim_{\rho=0}(\psi_\rho^2)$. But, for $\rho = 0$, ψ_ρ becomes illusory, as its coefficients are then infinite. (See *Bulletin of the New York Mathematical Society*, for July, 1894, p. 255.)

can be generated by the repetition of an infinitesimal orthogonal substitution.

Since an orthogonal substitution of the first kind, and only an orthogonal substitution of the first kind, can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution, we have, by § 1 and § 2, the following theorem:

The necessary and sufficient condition that a given orthogonal substitution may be generated by the repetition of an infinitesimal orthogonal substitution is that either -1 *shall not be a root of the characteristic equation of the substitution, or, if* -1 *is a root of this equation, that the numbers belonging to* -1 *shall all be even.*

§ 4.

The preceding division of the substitutions of the orthogonal group gives, of course, a corresponding division of the group of linear substitutions which transform automorphically a symmetric bilinear form with cogredient variables. Thus we may designate a substitution of this group as of the first or second kind according as it is or is not the second power of a substitution of the group; and then any substitution of the first kind may be generated by the repetition of an infinitesimal substitution of the group, but no substitution of the second kind can be generated thus.

For let the variables of the symmetric bilinear form

$$(\Omega \between x_1, x_2, \dots x_n \between y_1, y_2, \dots y_n)$$

be cogredient. The necessary and sufficient condition that the linear substitution ϕ shall transform the form automorphically is that ϕ shall satisfy the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

It is assumed that the determinant of the form is not zero, that is, that

$$|\,\Omega\,| \neq 0.$$

There are therefore one or more polynomials in Ω whose second power is equal to Ω. Let $\Omega^{\frac{1}{2}}$ denote any such polynomial in Ω. We may then write the preceding equation as

$$(\Omega^{\frac{1}{2}})^{-1}\breve{\phi}\,\Omega^{\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}\phi\,(\Omega^{\frac{1}{2}})^{-1} = \delta\,;$$

and, if
$$\psi = \Omega^{\frac{1}{2}}\phi\,(\Omega^{\frac{1}{2}})^{-1},$$
it becomes
$$\breve{\psi}\,\psi = \delta,$$

since $\Omega^{\frac{1}{2}}$ is symmetric, as it is a polynomial in the symmetric matrix Ω. Whence it follows that the most general expression for the linear substitution ϕ is

$$(\Omega^{\frac{1}{2}})^{-1}\,\psi\,\Omega^{\frac{1}{2}},$$

in which $\Omega^{\frac{1}{2}}$ is a symmetric square root of Ω, and ψ is an arbitrary orthogonal linear substitution or matrix.

If, now, ψ is an orthogonal substitution of the first kind, ϕ is also of the first kind, and conversely. For, if ψ_0 is orthogonal, and

$$\psi_0^2 = \psi,$$

then, if
$$\phi_0 = (\Omega^{\frac{1}{2}})^{-1}\,\psi_0\,\Omega^{\frac{1}{2}},$$
we have
$$\breve{\phi}_0\,\Omega\,\phi_0 = \Omega,$$
and
$$\phi_0^2 = (\Omega^{\frac{1}{2}})^{-1}\,\psi_0^2\,\Omega^{\frac{1}{2}} = (\Omega^{\frac{1}{2}})^{-1}\,\psi\Omega^{\frac{1}{2}} = \phi.$$
Conversely, if
$$\phi_0^2 = \phi,$$
and
$$\breve{\phi}_0\,\Omega\,\phi_0 = \Omega,$$
then, if
$$\psi_0 = \Omega^{\frac{1}{2}}\,\phi_0\,(\Omega^{\frac{1}{2}})^{-1},$$
we have
$$\breve{\psi}_0\psi_0 = \delta,$$
and
$$\psi_0^2 = \psi.$$

As stated above, the orthogonal substitutions of the second kind are all imaginary. But the linear substitutions of the second kind which transform automorphically certain real symmetric bilinear forms are not all imaginary. Thus the bilinear form

$$\mathfrak{F} \equiv ax_1y_1 + 2bx_3y_3 - 2b\,(x_2y_4 + x_4y_2) + b\,(x_3y_4 + x_4y_3),$$

is transformed automorphically, if we put

$$x_1 = -\xi_1, \quad x_2 = -\xi_2 + \xi_3, \quad x_3 = -\xi_3 + \xi_4, \quad x_4 = -\xi_4,$$
$$y_1 = -\eta_1, \quad y_2 = -\eta_2 + \eta_3, \quad y_3 = -\eta_3 + \eta_4, \quad y_4 = -\eta_4;$$

and this substitution, which is real, is of the second kind.

If a and b are both positive, three of the roots of the equation

$$\Gamma(z) \equiv \begin{vmatrix} a-z, & 0, & 0, & 0 \\ 0, & -z, & 0, & -2b \\ 0, & 0, & 2b-z, & b \\ 0, & -2b, & b, & -z \end{vmatrix} = 0$$

are positive and one negative. If a and b are of different sign, two of the roots of this equation are positive and two negative. If both a and b are negative, all but one of the roots of this equation are negative. But any real symmetric bilinear form

$$(\Omega \between x_1, x_2, x_3, x_4 \between y_1, y_2, y_3, y_4)$$

with cogredient variables can be transformed into the form $\mathfrak{J}$ by a real linear substitution ϖ, if the number of positive roots of the equation

$$| \Omega - z\delta | = 0$$

is equal to the number of positive roots of the equation

$$\Gamma(z) = 0.$$

If this condition is satisfied, and if ϕ denotes the linear substitution given above, the real linear substitution $\varpi\phi\varpi^{-1}$ transforms

$$(\Omega \between x_1, x_2, x_3, x_4 \between y_1, y_2, y_3, y_4)$$

automorphically, and is of the second kind. Whence it follows that any real symmetric bilinear form

$$(\Omega \between x_1, x_2, x_3, x_4 \between y_1, y_2, y_3, y_4),$$

with two sets of four cogredient variables the roots of whose characteristic equation

$$| \Omega - z\delta | = 0$$

are not all of the same sign, is transformed automorphically by a real inear substitution of the second kind.

Thursday, June 13th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Gilbert Thomas Walker, M.A., Fellow of Trinity College, Cambridge, was elected a member.

Mr. Bryan communicated a note "On an Extension of Boltzmann's Minimum Theorem," by Mr. S. H. Burbury, F.R.S.

Dr. Larmor gave a sketch of a paper by Mr. J. Brill, entitled "On the Form of the Energy Integral in the Varying Motion of a Viscous Incompressible Fluid for the case in which the Motion is Two-Dimensional, and the case in which the Motion is Symmetrical about an Axis."

A paper by Dr. Routh, "On an Expansion of the Potential Function $1/R^{\kappa-1}$ in Legendre's Functions," was taken as read.

Mr. Macaulay read a paper entitled "Groups of Points on Curves treated by the Method of Residuation."

The President informed the meeting of the death of Prof. A. M. Nash, of the Presidency College, Calcutta, which took place on the voyage home, for a two years' furlough, after twenty years' service in India.

The following presents were made to the Library :—

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Mantel, W.—"Gewone Lineaire Differentiaalvergelijkongen," pamphlet, 8vo, 1894.

D'Ocagne, M.—"Sur la Composition des Lois de Probabilité des Erreurs de Situation d'un Point sur un Plan," pamphlet.

D'Ocagne, M.—"Abaque en Points Isoplèthes de l'Equation de Képler," pamphlet.

"Proceedings of the Physical Society of London," Vol. XIII., Pt. 7, No. 57 ; June, 1895.

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Schouten, Dr. G.—"Theorie der Functies naar Weierstrass," pamphlet, 8vo; 1895.

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"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 1895, 1-25.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IV., Fasc. 8, 9, 10; Roma, 1895.

"Annali di Matematica," Ser. 2, Tomo XXIII., Fasc. 2; Milano, 1895.

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Weierstrass, K.—"Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen," herausgegeben von H. A. Schwartz, Zweite Ausgabe, Abt. 1, 8vo; Berlin, 1893.

"Annales de la Faculté des Sciences de Toulouse," Tome IX., Fasc. 2; Paris, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXV., Heft 1; Berlin, 1895.

"Indian Engineering," Vol. XVII., Nos. 16-20.

"Application de la Géométrie à la Résolution d'une Classe de Problèmes relatifs au Calcul des Probabilités," by Rev. T. C. Simmons. (Offprint of the Association Française, Congrès de Caen, 1894.) From the Author.