An Extension of Vandermonde's Theorem. By F. H. JACKSON. Read March 14th, 1895. Received, in revised form, May 14th, 1895.

1. The function

$$\underset{\kappa=\infty}{L}\frac{(a-n+1)(a-n+2)\dots(a-n+\kappa)}{(a+1)(a+2)\dots(a+\kappa)}\kappa^n \equiv \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \quad \dots\dots(1).$$

If n be a positive integer,

$$\Gamma(a+1) = a(a-1)(a-2)\dots(a-n+1)\,\Gamma(a-n+1),$$

and

$$\frac{\Gamma(a+1)}{\Gamma(a-n+1)} = a(a-1)(a-2)\dots(a-n+1).$$

Similarly, if n be a negative integer $(=-m)$, function (1) reduces to

$$\frac{1}{(a+m)(a+m-1)\dots(a+1)}.$$

2. Let a_n denote the product of n related quantities,

$$a,\ a-1,\ a-2,\ \dots\ a-n+1,$$

then such expressions as $a_{\frac{1}{2}}$, a_{-n}, $a_{p/q}$ seem to be without meaning. Exactly the same might have been written concerning $a^{\frac{1}{2}}$, a^{-n}, $a^{p/q}$, so long as a^n was regarded as the product of n factors each equal to a. As soon as the general law

$$a^m \times a^n = a^n \times a^m = a^{m+n}$$

was assumed in the Theory of Indices, fractional and negative powers were interpreted, and the Binomial Theorem was shown to be true (with certain restrictions) for negative and fractional values of the index. Vandermonde's Theorem is a finite algebraical identity analogous to the Binomial Theorem for positive integral indices. We shall show that

$$(a+b)_n = a_n + na_{n-1}b_1 + \frac{n\,.\,n-1}{2!}a_{n-2}b_2 + \dots$$

$$+\frac{n\,.\,n-1\,\dots\,n-r+1}{r!}a_{n-r}b_r + \dots \quad \dots\dots(2),$$

where n is not restricted to being a positive integer, and a_n denotes the function (1).

3. Firstly, writing

$$a_n = a(a-1)(a-2)\dots(a-n+1),$$

$$a_m = a(a-1)(a-2)\dots(a-m+1),$$

(m and n being positive integers), we have

$$a_m \times (a-m)_n = a_n \times (a-n)_m = a_{m+n} \quad \dots\dots\dots\dots\dots (A).$$

Let us assume these to be general laws in a manner analogous to the assumption

$$a^m \times a^n = a^n \times a^m = a^{m+n}$$

in the Theory of Indices, then we must find in general functions a_m and a_n which satisfy the relation (A), m and n being unrestricted.

Function (1), namely $\frac{\Gamma(a+1)}{\Gamma(a-n+1)}$, is such a function, for, on writing

$$a_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)},$$

we get $$a_n \times (a-n)_m = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \frac{\Gamma(a-n+1)}{\Gamma(a-n-m+1)}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a-n-m+1)} = a_{m+n}.$$

In the same way $$a_m \times (a-m)_n = a_{m+n}.$$

Of course the relations (A) would be satisfied if we assumed

$$a_n = \frac{f(a)}{f(a-n)},$$

$f(a)$ denoting any function of a whatever, but the function a_n must be such as will reduce to

$$a(a-1)(a-2)\dots(a-n+1)$$

if n be a positive integer. We therefore take

$$a_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)},$$

which function, we know, reduces to

$$a(a-1)(a-2)\dots(a-n+1),$$

if n be a positive integer.

An extended form of the relation (A) is

$$a_p \times (a-p)_q \times (a-p-q)_r \times (a-p-q-r)_s \times \ldots = a_{p+q+r+s+\ldots}.$$

Let each of the m quantities $p, q, r, s, \ldots$ be equal to $\frac{n}{m}$, where n and m are both integers; then $(a)_{n/m}$ will be a function such that

$$(a)_{n/m} \times \left(a-\frac{n}{m}\right)_{n/m} \times \left(a-\frac{2n}{m}\right)_{n/m} \times \ldots \times \left(a-\frac{m-1\,.\,n}{m}\right)_{n/m}$$

$$= a_{n/m+n/m+\ldots \text{ to } m \text{ terms}}$$

$$= a_n.$$

The function (1) satisfies this relation.

Putting $n = 0$ in $$a_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)},$$

we get $$a_0 = 1.$$

4. Let $F_1(\alpha, \beta, \gamma)$ denote the hypergeometric series in which the element x is equal to unity; then

$$\frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} = F_1(\alpha, \beta, \gamma) \qquad \text{(B)},$$

where Π denotes Gauss's Π function. In Gamma Functions this may be written

$$\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} = F_1(\alpha, \beta, \gamma) \qquad \text{(C)},$$

For α substitute $-n$,

" β " $-b$,

" γ " $a-n+1$.

Then the equation (C) becomes

$$\frac{\Gamma(a-n+1)\,\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(a+b-n+1)} = F_1(-n, -b, a-n+1)$$

$$= 1 + \frac{(-n)(-b)}{1!\,(a-n+1)} + \frac{(-n)(-n+1)(-b)(-b+1)}{2!\,(a-n+1)(a-n+2)} + \ldots$$

$$+ \frac{(-n)(-n+1)\ldots(-n+r-1)(-b)\ldots(-b+r-1)}{r!\,(a-n+1)(a-n+2)\ldots(a-n+r)} + \ldots$$

$$= 1 + \frac{n_1 b_1}{1!\,(a-n+1)_1} + \frac{n_2 b_2}{2!\,(a-n+2)_2} + \ldots + \frac{n_r b_r}{r!\,(a-n+r)_r} + \ldots$$

$$\qquad \text{(D)}.$$

Now $\dfrac{a_{n-1}}{a_n} = \dfrac{\Gamma(a+1)}{\Gamma(a-n+2)} \, \dfrac{\Gamma(a-n+1)}{\Gamma(a+1)} = \dfrac{\Gamma(a-n+1)}{\Gamma(a-n+2)} = \dfrac{1}{(a-n+1)}$,

$$\frac{a_{n-2}}{a_n} = \frac{1}{(a-n+2)_2},$$

$$\frac{a_{n-r}}{a_n} = \frac{1}{(a-n+r)_r},$$

and
$$\frac{\Gamma(a-n+1)}{\Gamma(a+1)} \, \frac{\Gamma(a+b+1)}{\Gamma(a+b-n+1)} = \frac{(a+b)_n}{a_n};$$

therefore we have

$$\frac{(a+b)_n}{a_n} = \frac{a_n}{a_n} + \frac{n_1}{1!}\,\frac{a_{n-1}b_1}{a_n} + \frac{n_2}{2!}\,\frac{a_{n-2}b_2}{a_n} + \ldots + \frac{n_r}{r!}\,\frac{a_{n-r}b_r}{a_n} + \ldots.$$

Multiplying both sides by a_n, we have

$$(a+b)_n = a_n + n_1 a_{n-1} b_1 + \frac{n.n-1}{2!} a_{n-2} b_2 + \ldots + \frac{n.n-1\ldots n-r+1}{r!} a_{n-r} b_r$$
$$+ \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(E)},$$

subject to the convergence of the infinite series on the right side of the above equation.

5. Denoting the general term of the series (E) by u_r, the ratio

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r} \, \frac{b-r+1}{a-n+r},$$

which approaches unity when r increases without limit.

Using the general test

$$\operatorname*{Lim}_{r\to\infty} \left[\left\{ r \left(\frac{u_r}{u_{r+1}} - 1 \right) - 1 \right\} \log r \right] > 1 \text{ (convergent series)},$$
$$< 1 \text{ (divergent series)},$$

we find the condition of convergence is

$$a+b+1 > 0.$$

Thursday, April 4th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

The Rev. T. C. Simmons read a paper on "A New Theorem in Probability." Messrs. Bryan, Cunningham, the President, and Dr. C. V. Burton (a visitor) joined in a discussion on the paper.

The President (Mr. Kempe, Vice-President, in the Chair) communicated a Note on "The Linear Equations that present themselves in the Method of Least Squares."

The President then read the title of a paper by the Rev. W. R. W. Roberts, viz., "On the Abelian System of Differential Equations, and their Rational and Integral Algebraic Integrals, with a discussion of the Periodicity of Abelian Functions."

The following presents were received:—

Miller, W. J. C.—"Mathematical Questions and Solutions," Vol. LXII., 8vo; London, 1895.

"Smithsonian Report, 1893," 8vo; Washington, 1894.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 3; Leipzig, 1895.

"Mittheilungen der Mathematischen Gesellschaft im Hamburg," Bd. III., Heft 5, 1895.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXIV., Heft 1; Jahrgang 1892; Berlin, 1895.

"Archives Néerlandaises," Tome XXVIII., Livr. 5; Harlem, 1895.

"The Silver Question: Injury to British Trade and Manufactures," papers by G. Jamieson, T. H. Box, and D. O. Croal, 8vo; London, 1895.

"Bulletin de la Société Mathématique de France," Tome XXIII., No. 1; Paris, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie III., Vol. I., Fasc. 1, 2; 1895.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 1; 1895.

Braune, W., and O. Fischer.—"Der Gang des Menschen," Th. 1, royal 8vo; Leipzig, 1895.

Bruns, H.—"Das Eikonal," R. 8vo; Leipzig, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1., Vol. IV., Fasc. 5; Roma, 1895.

"Educational Times," April, 1895.

"Acta Mathematica," XIX., 1; Stockholm, 1895.

"Journal für die reine und angewandte Mathematik," Bd. cxiv., Heft 4; Berlin, 1895.
"Annals of Mathematics," Vol. ix., No. 2, January, 1895; University of Virginia.
"Indian Engineering," Vol. xvii., Nos. 8, 9, 10.

A New Theorem in Probability. By Rev. T. C. Simmons, M.A.

Read April 4th, 1895. Received, in revised form, June 6th, 1895.

1. "*If an event happen on the average once in* m *times*, m *being greater than unity, then it is more likely to happen less than once in* m *times than it is to happen more than once in* m *times.*" In the present paper I undertake to prove this novel proposition, which may be enunciated more explicitly thus:—"If an event may happen in b ways and fail in a ways, a being greater than b, and all these ways are equally likely to occur, then, μ trials being made, where μ is any multiple of $a+b$, large or small, or any *random* number, the event is more likely to happen less than $\frac{\mu b}{a+b}$ times than it is to happen more than $\frac{\mu b}{a+b}$ times." Moreover, if the ratio of a to b be greater than 4, I shall venture to assert and prove a wider proposition, viz., that the event is *more likely than not* to happen less than $\frac{\mu b}{a+b}$ times. This amounts to saying that if a die, for instance, be thrown any number of times, large or small, chosen at random, the number of appearances of the ace is more likely than not to be less than $\frac{1}{6}$ of the number of throws. For reasons which will be stated in Art. 32, I am compelled *at present* to qualify the foregoing statements by the limitation that $b = 1$.

2. The first suggestion of such a proposition arose in this way. At the beginning of the present year I was engaged, for a purpose to be elsewhere recorded, in the collection and examination of upwards of 40,000 random digits; and was considerably surprised to find that, aggregating the results, each digit presented itself, with unexpected