

December 6, 1870.

CHARLES B. VIGNOLES, F.R.S., President,  
in the Chair.

The following Candidates were balloted for and duly elected :—  
CRAWFORD JAMES CAMPBELL, JOHN JAMES CAREY, WILLIAM BELLINGHAM CARTER, WALTON WHITE EVANS, ALEXANDER FRASER, WILLIAM FREDERICK MARCH PHILLIPPS, JOHN ARTHUR PHILLIPS, ARTHUR POTTS, JOSEPH QUICK, jun., WILLIAM ROBERT ROBINSON, and EDWARD WELSH, as Members; CHARLES AUGUSTUS ALBERGA, Stud. Inst. C.E., JOHN PHILIP CORTLANDT ANDERSON, THOMAS ASHTON, ROBERT WILLIAM PEREGRINE BIRCH, Stud. Inst. C.E., JAMES BISSET, JOSEPH BOURNE, JOHN CHARLES COODE, Stud. Inst. C.E., CHARLES COWAN, Capt. ARTHUR EDWARD DOWNING, FREDERICK DRESSER, FRANCIS FOX, THOMAS WILSON GRINDLE, JOHN FALSHAW HOBSON, ARTHUR LUCAS, JAMES CHATBURN MADELEY, WILLIAM MATTHEWS, GEORGE PALMER, ALEXANDER RHODES, WILLIAM GEORGE SCOTT, PETER SOAMES, HERBERT UNWIN, THOMAS FINSBURY SEPTIMUS WAKLEY, WILLIAM THOMAS WALKER, EDWARD ORANGE WILDMAN WHITEHOUSE, CHARLES HENRY WILKS, JOHN HATTON WILSON, as Associates.

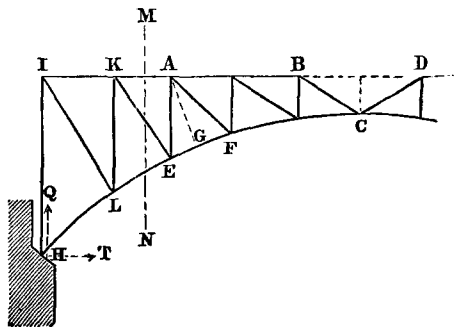
It was announced that the Council, acting under the provisions of Sect. III. Cl. VII. of the Bye-Laws, had transferred the following Associates to the class of Member :—HERBERT LOUIS AUGUSTUS DAVIS, THOMAS MANSON RYMER JONES, and HENRY SHIELD.

Also that the following Candidates, having been duly recommended, had been admitted by the Council, under the provisions of Sect. IV. of the Bye-Laws, as Students of the Institution :—ARTHUR TURNOUR ATCHISON, JAMES THOMAS ATCHISON, EDWARD KYNASTON BURSTAL, EDMUND EMSON, WALTER FREETH, CHARLES JOHN GOODMAN, ARTHUR GROSE, ARDEN HARDWICKE, FLETCHER JAMES IVENS, FREDERICK JACKSON, WALTER ROBERTS JONES, JOHN HERMAN MERIVALE, JOHN NOWLAN, WILLIAM PATTERSON ORCHARD, ERNEST EDWARD SAWYER, GILBERT STIFF, THOMAS SUGDEN, JOSEPH CALISTE GROSVENOR DU VALLON, and CHARLES EDWARD SABINE YOUNGHUSBAND.

No. 1,224.—“On the Theory and Details of Construction of Metal and Timber Arches.” By JULES GAUDARD, Civil Engineer, Lausanne. (Translated from the French by William Pole, F.R.S., M. Inst. C.E.)

1. ELASTIC arches supporting a road or railway are connected to the horizontal platform by pillars and filling pieces, occupying the spandrels. The office of these parts is to transmit the load of the platform to the resisting arch; no other function is usually attributed to them, and the arch is the important member on which the whole rests. The spandrel, however, forming always a rigid filling or system, contributes powerfully to increase the resistance of the structure, so that it would be justifiable to consider as the chief member, not the isolated arch, but the framework, more or less complex, constituted by the arch, the spandrel filling, and the longitudinal horizontal piece placed at the level of the platform. Under this point of view, the arch would no longer be restricted to a rigid whole; it might be reduced to a chain of articulated segments. Now in all bridges hitherto constructed, the arches are rigid at the same time that the spandrels present a certain resistance; there is a superabundance of organs, and consequently an uncertain distribution in the functions assigned to each of them. The calculation is only rendered determinate by means of an arbitrary hypothesis of this distribution. Usually the spandrels are neglected, and the arch is assumed to resist without their aid. But there is nothing to prevent the system being assimilated to an articulated form. Hence the two modes of calculation which will now be examined.

Fig. 1.



2. *Articulated System.*—Suppose the form represented in Fig. 1. It is necessary to choose a single triangulation, where every piece

is essential; for otherwise, if, for example, two diagonals were introduced in each bay, the calculation would become indeterminate. At the summit C the two half-spans are joined by a simple point of articulation, like those of the other summits of the triangulation. The piece BD is useless in theory; and in practice it would be desirable to provide it with a free sliding joint, in order that nothing might impede the expansion of the fixed portion.

Under these conditions, the elements of statics furnish easily the stresses on all the pieces. If, for example, it be asked what is the stress  $t$  of the segment EF of the arch? Let fall AG perpendicular to EF, then  $t \times AG$  is the *moment of resistance*, by virtue of which the piece EF prevents the pivoting round the point A. By making this equal to the *bending moment* relative to the same point, there will result an equation which gives the value of  $t$ . The bending moment in question comprehends the sum of the moments of the forces applied to the portion AEHI. If, for example, there is a weight = P applied to the point K, and if the reaction of the abutment H is decomposed into a vertical force = Q, and a horizontal thrust = T, the equation giving  $t$  will be

$$t \times \overline{AG} = Q \times \overline{AI} - T \times \overline{HI} - P \times \overline{AK}.$$

If the second number is positive,  $t$  is a tension; if it is negative the portion EF of the arch will be compressed.

The calculations are the same for the longitudinal IB. For example, the piece KA ought to prevent the solid EKIH from pivoting round E.

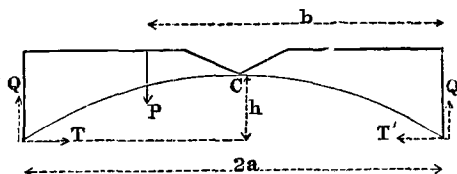
3. When the stresses in the various parts of the arch or of the longitudinal are known, the stresses in the spandrel bars can be deduced from them by simple graphic decompositions, based on the equilibrium of the summits of the system. For example, the point E is subject first to two known forces exerted by EL and EF; construct the resultant of these two forces, and then decompose it between the given directions EA and EK; there will thus be obtained lines equal and opposite to the forces which the pieces EA and EK ought to exert upon the point E, in order to maintain it at rest.

These forces may also be obtained directly by calculation. Taking a vertical section such as MN, note down the equilibrium of vertical translation of the portion of the solid situated to the left of MN, introducing there the reactions proceeding from the segment to the right; that is to say, the stresses of the severed bars AK, KE, EL (of which the first has no influence, AK being

horizontal). This amounts to saying that the vertical projection of the required stress of the bar K E, added to the projection of the known stress of E L, ought to equilibrate the *shearing force*  $Q - \Sigma P$  (understanding by  $\Sigma P$  the sum of the various weights P applied between I and the section M N).

4. Hitherto the reactions Q and T have been regarded as known. In fact their preliminary calculation is easy, according to the equilibrium of the entire system. Using the notations of Fig. 2, and

Fig. 2.



designating by  $\Sigma$  the sums extended to the whole span, then  $T = T'$ ;  $Q + Q' = \Sigma P$ ; and  $2aQ = \Sigma Pb$ . The fourth equation necessary will be furnished by the equilibrium of rotation of one half-span round the summit C, i.e.,  $Th = Qa - \Sigma' P(b - a)$ ; the accent here put on  $\Sigma'$  indicates that this sum only embraces the terms arising from the half-span considered.

5. If the mutual thrust be required of the two half-spans at C, it can be as readily ascertained  $N'$  and  $N''$  being the horizontal and vertical components, it suffices, Q and T being already calculated, to write down the equations of equilibrium of translation of the half-span, i.e.,  $N' = T$ , and  $N'' = Q - \Sigma' P$ . The vertical component  $N''$  will be null if there is complete symmetry of form and of load round the summit C.

The pieces requiring most strength are those near the summit. The Author would be inclined, in the execution of the framing before mentioned, to place the summit C notably lower than the line of the longitudinal, or to adopt a central portion of solid plate.

6. For a determined fixed load, there may be given to the arch the form of a *funicular polygon*, a figure of equilibrium such that the articulated chain may maintain itself in position without the intervention of the other pieces of the framework. In a bridge the load varies, but it is desirable that the figure of the arch should approach that of equilibrium corresponding to the complete load. This form would be the curve called the *catenary* for an arch of uniform section carrying only its own weight. It will be the *parabola* for an arch loaded uniformly per unit of length on a

horizontal line. This latter case is that of suspension bridges, and also of bridges with compressed metallic arches, for the weight of the arch and of the spandrels has but little influence, proportionally to that of the horizontal platform and its test load.

The theory of M. Yvon Villarceau furnishes differential equations for determining the figure of equilibrium, and the variable thickness of the joints, of an arch of equal resistance submitted to a fixed load. The reason why the circular shape is generally preferred to these theoretical forms is, in the first place, the simplification of the design and of the execution, and also the consideration that the theoretical figure only satisfies rigorously a single arrangement of the load, and not all those to which the arch may be exposed.

7. *Rigid Arches.*—The second mode of calculation referred to in Art. 1, consists in restricting the spandrels to simple supports for transmitting the load; the arch is then more strained, and ought to be rigid, for the funicular form only gives an instable equilibrium, corresponding to a particular state of the load. As soon as this state is changed, the arch no longer suffers simple compression, but is disposed to bend, *i.e.*, to change its figure.

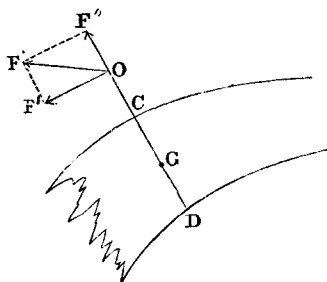
In an arch of masonry, the effect of the mortar is neglected, and the voussoirs are regarded as blocks placed in juxtaposition without adherence, having the power of pivoting one on the other, round the edge either of the intrados or of the extrados; that is what is called a *system of alternative articulation*. The *centre of pressures*, *i.e.*, the point of application of the resultant of the elementary reactions of the joint is considered for each joint. The locus of these centres, or the *curve of pressures*, ought not to pass outside the thickness of the arch, or the pivoting of certain of the voussoirs would take place; the curve ought even to keep within a zone more limited than that of the arch, for fear of endangering the crushing of the stones.

An elastic arch is subject to other conditions. If the connections are very good, as is (or ought to be) the case with plate iron, the arch forms an entire piece, suitable to resist both tension and compression. If it is treated as an arch by the curve of pressures, this curve will no longer be required to remain within the arch.

8. Let  $F$  be the total reaction on a joint  $CD$  (Fig. 3), and  $O$  its point of application, lying in the curve of pressures. Let  $OG = x$  the distance from this point  $O$  to the centre of gravity or of elasticity (mean fibre) of the section  $CD$ . The force  $F$  is decomposed into a normal pressure  $F'$  and a shearing stress  $F''$  which, acting in the direction  $CD$ , tends to shear the solid. The longitudinal force  $F'$  may be applied at the centre of elasticity  $G$ ,

provided that we combine with it a couple whose moment =  $F' x$ , which will be the bending moment. Then the fibre most strained,

Fig. 3.



*i. e.* C in the case of the figure, will be subjected to a pressure

$$R = \frac{F'}{v} + \frac{v F' x}{I}$$

per unit of surface:  $\omega$  is the area of section,  $I$  its moment of inertia, and  $v$  the distance  $CG$ . The formula supposes that, under the stress mentioned, the section  $CD$  has slightly pivoted while remaining plane.

9. When the arch rests upon the abutments by an extended sustaining surface, being keyed by a range of wedges, there is nothing to prevent certain of these wedges being driven tighter than others. This will then modify the point of concentration of the thrust upon the abutment, and consequently also all the other points of the curve of pressures. Hence arises the uncertainty which generally attends the method of the curve of pressures, these pressures only being determined by arbitrary data in regard to the original keying-up, or the yielding of the materials. An infinite number of curves are possible. It is not known which is effectively realized, but suppose the material system to seek, in the first place, its equilibrium at the cost of the *least resistance* (Moseley's principle); then certain edges are compressed and oblige the curve of pressures to modify itself continuously, until it arrives at a stable condition. If no stable condition is met with, the system soon exhausts the series of conditions possible, and ends at the extreme limit, which is then the curve of the greatest resistances, beyond which the failure of the structure takes place. According to that a curve of any pressures whatsoever, arbitrarily chosen among all those compatible with the equilibrium, will give a strong presumption of stability, if it only submits the edges to

pressures offering all practical security; for, in the absence of any other stable condition, the system ought to stop at that, if the velocity acquired is insensible. Thus is justified the method of Méry, notwithstanding that it appears strange at first sight to base confidence on the existence of a certain curve, at the same time that there is no assurance that this curve will be effectively realized.

By these considerations, the safety is established, but it may be so in different degrees; and there is another consideration to bring into the question—that of economy. Now these two conflicting objects, the economy of material and the stability, lead the play or field open to the virtual oscillations of the curve of pressures to be confined within just limits. The amplitude of this play, if it is great, will give a superabundance of safety, and if it is small, it will testify to great boldness of design. M. Durand Claye has ('Annales des Ponts et Chaussées, 1868') indicated a method of appreciating the degree of boldness—a method of which the following résumé will present the essential features for the case of elastic arches, not excluding the action of tension.

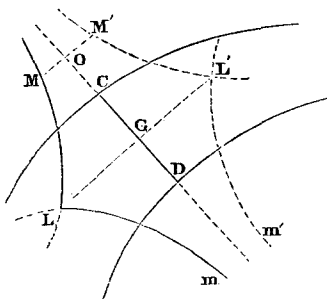
10. Two preliminary remarks may be offered:—

1st Remark.—In Art. 8, is stated the formula of resistance which gives the stress  $R$  on the fibre most strained in a joint  $C D$ . Suppose that this stress attains precisely the limit imposed by practical safety, it becomes a given datum, and the unknown quantity is the corresponding normal pressure  $F'$  in respect of abscissa  $GO = x$  of its point of application. Now

$F' = \frac{R I \omega}{I + \omega v x}$ , so that, considering the line representing the force

$F'$  as an ordinate,  $OM = y$  (Fig. 4), answering to the abscissa

Fig. 4.



$GO = x$ , the geometrical locus of the points  $M$  will be a branch of

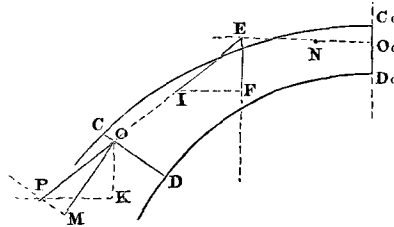
an equilateral hyperbola  $ML$ , of which  $DC$ , being prolonged, represents an asymptote. Beyond the point  $L$ , which corresponds to  $x = 0$ , the first hyperbola ceases to apply, and is replaced by another  $Lm$ ; seeing that, when the centre  $O$  of the pressures passes to the other side of the centre of elasticity  $G$ , it is no longer the fibre  $C$  which is the most strained, but the fibre  $D$ .

For an arch capable of extension  $R$  and  $F'$  may be admitted to be negative, which leads to a second hyperbolic contour  $M'L'm'$ ; but this case rarely demands attention.

These different curves define generally the exigencies of the section  $CD$ . If a curve of pressures brings on this joint a resulting normal reaction  $F'$ , exceeding the ordinate, such as  $OM$ , or in other words, whose summit passes outside the contour  $MLm \dots m'L'M'$ , this curve ought to be rejected, as involving a pressure,  $R$ , greater than the limit admitted.

2nd Remark.— In order to trace a curve of pressures, the voussoirs, such as  $C_0 D_0 DC$  (Fig. 5) are considered as departing from the ver-

Fig. 5.



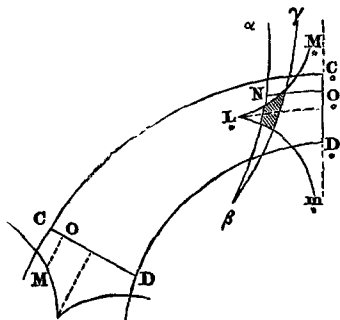
tical joint of the crown  $C_0 D_0$  and stopped successively at various joints  $CD$ . Let the case of an arch be considered which is symmetrical both as to figure and load, so that the thrust at the crown is horizontal. Take the centres of pressure  $O_0$  and  $O$  (arbitrary if they do not result from constructions already existing); through  $O_0$  draw the horizontal  $O_0 E$  which will cut in  $E$  the vertical from the centre of gravity of the voussoir  $C_0 D_0 DC$  and of its load; draw a length  $EF$  equal to the weight, join  $EO$ , and draw  $FI$  horizontal. The three forces which maintain the equilibrium are represented by the three sides of the triangle  $EFI$ ; the weight =  $EF$ , the thrust at the crown =  $FI$  (but applied at  $O_0$ ), and lastly the reaction of the joint  $CD$  equal to  $IE$ , and applied at  $O$ . This force may be transferred to  $OP = EI$ , and the normal component of it  $OM$  taken, which is that called above  $F'$ .

Inversely, this pressure  $OM$  may be assumed *à priori*, and its point of application  $O$ , in order to arrive therefrom at the thrust

on the crown. In effect, take  $OK$  vertical and equal to the weight  $EF$ ; draw  $KP$  horizontal, and  $MP$  parallel to  $DC$ ; then join the point of intersection  $P$  to  $O$ ; prolong  $PO$  to  $E$ , where it encounters the vertical of the centre of gravity, and draw  $EO_0$  horizontal.  $O_0$  is thus known, and the value of the thrust at this point is measured by  $KP$ . It may be transferred to  $O_0N = KP$ , and a point  $N$  will be derived from  $M$ . The constructions should be modified in the case of non-symmetry, for then the thrust at the crown is not horizontal.

11. Now, let there be chosen successively for  $M$  all the points of the hyperbolic contour  $MLm$  of Fig. 4. The points  $N$  corresponding to them at the crown, will describe certain curves  $\alpha\beta\gamma$  (Fig. 6), which will represent at the crown the exigencies of any

Fig. 6.



joint  $CD$ . A thrust  $O_0N$  will only be admissible if its summit  $N$  falls upon the contour  $\alpha\beta\gamma$ , or in the interval between the branches  $\alpha\beta$  and  $\beta\gamma$ . But the joint  $C_0D_0$  possesses, by the same title as  $CD$ , its own limiting contour  $M_0L_0m_0$  formed of hyperbolic arcs. Severally, then, the summits  $N$  of the thrusts  $O_0N$  admissible at the crown ought to fall in the interior of the space cross-shaded in Fig. 6, in order to satisfy the simultaneous exigencies of the joints  $CD$  and  $C_0D_0$ .

Repeat the constructions for the joints other than  $CD$ , transferring everything to the crown joint  $C_0D_0$ . The various cross-hatched spaces will contract more or less among themselves, and their common part alone must be taken into consideration, in order to obtain the thrusts at the crown, or the curves of pressure, compatible with the limiting resistance  $R$  imposed simultaneously at all the various joints.

If it is now desired to compare the boldness of two works, it will suffice to seek for each of them, by trial and error, the limiting stress  $R$  for which the various curves of pressure realizable become reduced to one only, or the cross-shaded space becomes reduced to a single point. The value thus found will be a well-characterized definition of the boldness sought. Such is the method of M. Durand Claye.

12. *Bridges with three pivots.*—A metallic arch might be provided with three pivots or hinges, one at the summit, the other two at the supports. The axis of these pivots would determine necessarily the centres of pressure at the crown and at the abutments, and then nothing would be arbitrary in tracing the curve of pressures, or in the methods of calculation. The calculations of the reactions, and consequently those of the resistances of the arch, would be simple and certain; the expansion would have free play, without straining the metal. These three pivots have been proposed, but hitherto, as far as the Author is aware, no one has ventured to apply them to large works, for fear that the too great mobility should favour the dislocation in certain parts. There have only been employed two pivots at the supports, an innovation introduced by M. Manton in an iron bridge of the St. Denis Canal. Without doubt, for an isolated arch, a pivot at the crown would have the effect of a spherical knee, provoking the structure to turn over; but in a bridge sufficiently wide relatively to the opening, it would seem that the connection together of several arches might be sufficiently well arranged to realize in the whole a long joint, resisting effectually any tendency to bending, at least for an ordinary road, where the rolling weight is less than on a railway.

*Bridges with two pivots.*—With two pivots at the supports, the expansion is subject to some constraint, because it ought to augment a little the rise of the arch, on the hypothesis that the supports are rigidly fixed. This effect contributes in certain cases to increase the stress of the material; it is sought to lessen this by reducing the dimensions of the middle of the piece.

The reactions of the abutments can no longer be obtained by simple statics, as in Art. 4, for the last equation of this Article contains the rise  $h$ , which would be unknown, as it ought to be measured to the unknown centre of pressure of the middle joint. If no supplementary equation existed, the problem might be deemed indeterminate, and a choice be arbitrarily made of the centre of pressure of the summit, as, for example, the middle of the joint; this is the process of M. Méry.

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In reality, however, the equation does exist; it depends on the deformation or on the yielding of the material of the arch, for this deformation must conform to certain conditions, as, for example, with respect to the primitive invariable situation of the points of support.

Now, when treating of an apparatus not homogeneous, the expression of such a delicate condition must be abandoned, and reliance must be placed upon M. Méry's process, notwithstanding its vague nature. Such is the case of an arch of masonry, a heterogeneous agglomeration of stones and mortar; and further, as there can be no question here of pivots at the supports, the same uncertainty prevails as to the point of concentration of the thrust upon the abutments. Such would also be the case with an arch of carpentry, the wood yielding much, being very sensitive to atmospheric influence, having a texture full of knots and other irregularities, and finally only allowing imperfect jointing. Arches of cast iron, even, are not rigorously homogeneous; they consist of several voussoirs fitted more or less exactly one on the other, and the material assumes irregularities of texture in the casting. A certain indecision seems, therefore, legitimate for such works, even in the case of two pivots at the supports. Further, it cannot be believed that a metallic arch would be menaced with failure by the simple fault of its having been calculated by the process of M. Méry; the great guarantee in construction always being the wise precaution of keeping at a respectful distance from the stresses of rupture.

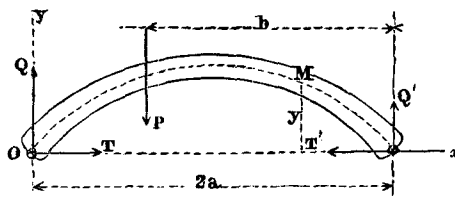
There only remain, therefore, arches in a single piece of homogeneous metal, to which the theoretical calculations of deformation may be applied with confidence. Arches in solid wrought-iron plate may be considered to belong to this category, the metal having been well worked, and the connections being as solid as the continuous parts. The theory of these arches has been the object of the researches of M. Bélanger, and subsequently of M. Bresse, who has entered into great details respecting them in his *Treatise on Applied Mechanics* (*Stabilité des Constructions*). His principal formulæ will be indicated.

13. The equilibrium of the entire system furnishes always, as in Art. 4, the three statical equations  $T = T'$ ,  $Q + Q' = \Sigma P$ , and  $2aQ = \Sigma Pb$ , the  $P$ 's expressing the weights or vertical forces (Fig. 7). The equation of deformation, which will express the invariability of the chord  $2a$ , is the equation following (see Art. 27 of the Author's *Memoir on the Strength of Materials*).<sup>1</sup>

<sup>1</sup> *Vide Minutes of Proceedings Inst. C.E., vol. xxviii., p. 561.*

$$2\tau a + \int_0^{2a} \left( \frac{\mu y}{EI} \frac{ds}{dx} + \frac{N}{E\omega} \right) dx = 0.$$

Fig. 7.



The co-ordinate axes are  $Ox, Oy$ , the origin being taken at one of the supports;  $s$  expresses a length of the curved mean fibre, and  $ds$  its differential. The elastic arch may be more or less strained by expansion or keying;  $\tau$  expresses the linear elongation resulting from these causes independent of the loads  $P$ . At the point  $M$  of the mean fibre, defined by the abscissa  $x$  and the ordinate  $y$ , let us imagine a transverse section:  $\omega$  is the area of this section;  $I$  its moment of inertia with respect to the horizontal axis through its centre of gravity;  $\mu$  is the bending moment, and  $N$  the total normal pressure, to which it is subject. The shearing stresses (parallel to the section), which have but little influence on the deformation, may be neglected. The letter  $E$  is the coefficient or modulus of elasticity; the section being supposed homogeneous,  $E\omega$  is what is called the longitudinal spring (*ressort longitudinal*), and  $EI$  the moment of flexibility.

The three statical equations will give  $T', Q$ , and  $Q'$ , then  $T$ , which alone remains unknown, ought to be derived from the last equation, where it enters implicitly. In order to show this,

put  $\mu = \mu_1 - Ty$ , and  $N = N_1 - T \frac{dx}{ds}$ ; then the moment  $\mu_1$  and

the longitudinal stress  $N_1$  are known, since they no longer take account of the unknown quantity  $T$  put aside; hence

$$T = \frac{2\tau a + \int_0^{2a} \frac{\mu_1 y}{EI} \frac{ds}{dx} dx + \int_0^{2a} \frac{N_1}{E\omega} dx}{\int_0^{2a} \frac{y^2}{EI} \frac{ds}{dx} dx + \int_0^{2a} \frac{1}{E\omega} \frac{dx}{ds} dx}$$

Such a formula is, indeed, alarming for practical application, especially when the calculation must be several times repeated for

different cases of loading. However, by reducing it to the most essential terms, these may be taken approximately

$$T = \frac{2\tau a + \int \frac{\mu_1 y}{E I} ds}{\int \frac{y^2}{E I} ds};$$

and further, the two integrals or quadratures alone retained, and taken between the limits 0 and S (equal to the total length of the arch), may each be calculated by the approximate method of Thomas Simpson, whatever be the figure of the given arch whose stability it is desired to verify. Dividing the arch into an equal number of equal parts, the values of  $\frac{\mu_1 y}{E I}$  and of  $\frac{y^2}{E I}$ , may be determined for each point of division, and Simpson's formula may be applied.

14. *For a symmetrical arch symmetrically loaded*, the axis of  $y$  may be transferred to the axis of symmetry passing through the summit of the arch, and then the general formula for  $T$  will be

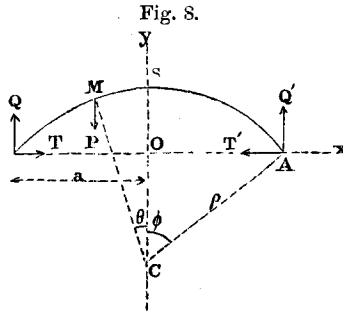
$$T = \frac{\tau a + \int_0^a \frac{\mu_1 y}{E I} \frac{ds}{dx} dx + \int_0^a \frac{N_1}{E \omega} dx}{\int_0^a \frac{y^2}{E I} \frac{ds}{dx} dx + \int_0^a \frac{1}{E \omega} \frac{dx}{ds} dx}.$$

If such an arch is rigidly fixed by building in (*encastré*) at the extremities, the abutment will exercise reactions  $Q$ ,  $T$ , which may always be regarded as applied to the point situated upon the mean fibre, provided there is combined with it a couple or moment of *encastrement*  $\sigma$ , destined to maintain invariable the angle between the crown joint and the joint at the origin. This couple will then appear in  $\mu$ , from which it is disengaged by making  $\mu = \mu_2 - T y + \sigma$ . Then the theory of M. Bresse furnishes these two equations for determining  $T$  and  $\sigma$ , the origin of the co-ordinates being in the middle of the span.

$$\begin{aligned} \sigma \int_0^a \frac{1}{E I} \frac{ds}{dx} dx - T \int_0^a \frac{y}{E I} \frac{ds}{dx} dx + \int_0^a \frac{\mu_2}{E I} \frac{ds}{dx} dx &= 0; \\ T \left( \int_0^a \frac{y^2}{E I} \frac{ds}{dx} dx + \int_0^a \frac{1}{E \omega} \frac{dx}{ds} dx \right) - \sigma \int_0^a \frac{y}{E I} \frac{ds}{dx} dx \\ &= \tau a + \int_0^a \frac{\mu_2 y}{E I} \frac{ds}{dx} dx + \int_0^a \frac{N_1}{E \omega} dx. \end{aligned}$$

15. *Circular Arches of uniform section.*—M. Bresse has further advanced the study of circular arches of uniform section, for which  $\omega$  and  $I$  are constant, and  $y$  = the ordinate of the arc of a circle. The following are the more important formulæ for the case of simple supports without *encastrement*.

Let  $\rho$  be the radius of the circle,  $a = \rho \sin \phi$ , the demi-chord,  $\phi$  the half angular amplitude  $S C A$ , between the crown and the origin  $A$ , Fig. 8.



Let there be considered first the isolated action of a single weight  $P$ , applied to a point  $M$  defined by the angle  $S C M = \theta$ . The integrals will be calculated after having expressed  $\mu_1$  and  $N_1$  which change on either side of  $M$ . The vertical components of the reactions of the abutments will be

$$Q = \frac{P}{2} \left( 1 + \frac{\sin \theta}{\sin \phi} \right), \quad Q' = \frac{P}{2} \left( 1 - \frac{\sin \theta}{\sin \phi} \right);$$

and the horizontal thrust will be

$$T = T' = P \cdot \frac{2a^2 A - r^2 \sin^2 \phi (\sin^2 \phi - \sin^2 \theta)}{2a^2 B + 2r^2 \sin^2 \phi (\phi + \sin \phi \cos \phi)};$$

$r$  designates the radius of gyration  $= \sqrt{\frac{I}{\omega}}$  of the uniform section.

Further, for the sake of abridgment, let  $A$  designate the quantity

$$\frac{\sin^2 \phi - \sin^2 \theta}{2} + \cos \phi (\cos \theta + \theta \sin \theta - \cos \phi - \phi \sin \phi),$$

and  $B$  the quantity,  $\phi + 2\phi \cos^2 \phi - 3 \sin \phi \cos \phi$ .

As a general form, the preceding thrust  $T$  may be expressed :—

$$T = P \frac{A}{B} \cdot \frac{1 - \lambda \frac{r^2}{a^2}}{1 + \lambda^1 \frac{r^2}{a^2}}.$$

Now, the principal coefficient  $\frac{A}{B}$  is furnished ready calculated by the Table I. of M. Bresse's 'Mecanique Appliquée' for different values of  $\frac{2\phi}{\pi}$  and of  $\frac{\theta}{\phi}$ . The Table IV. gives the coefficient of correction

$$\frac{1 - \lambda \frac{r^2}{a^2}}{1 + \lambda' \frac{r^2}{a^2}}.$$

for a series of values of  $\frac{2\phi}{\pi}$ , and of  $\frac{r^2}{a^2}$ . The factor  $\lambda$  is equal to  $\frac{\sin^2 \phi (\sin^2 \phi - \sin^2 \theta)}{2A}$ , but differs little from  $\frac{\pi^2 \sin^2 \phi}{4\phi^2}$ . And on the other hand,  $\lambda' = \frac{\sin^2 \phi (\phi + \sin \phi \cos \phi)}{B}$  may be replaced approximately by  $\frac{15 a^2}{f^2}$ ,  $f$  designating the rise or height of the arch.

16. Now, leave  $P$  aside, and only consider a simple dilatation  $\tau$  per running metre. This will produce no vertical reaction, but a simple horizontal thrust

$$T = \frac{2 a^2 E I \sin^3 \phi}{a^2 B + r^2 \sin^2 \phi (\phi + \sin \phi \cos \phi)} \cdot \tau.$$

Writing  $T = \frac{D}{1 + \lambda' \frac{r^2}{a^2}} \cdot \frac{E I \tau}{a^2}$ , there will be found the principal

coefficient  $D$  in the tables of M. Bresse. In the absence of tables, approximately  $T = \frac{15 E I \tau}{15 r^2 + 8 f^2}$ , if the arch is sufficiently flat.

17. For the third case, a *uniform load  $p$  per metre of length of the arch* (such as the weight of the arch itself). Supposing it applied to the entire arch, of length =  $S$ , the vertical reactions will be

$Q = Q' = \frac{pS}{2}$ , and the thrust ;

$T = p \rho \times$

$$\frac{a^2 \phi (\frac{1}{2} - 5 \cos^2 \phi) + \frac{3}{2} a^2 \sin \phi \cos \phi - r^2 \sin^2 \phi (\frac{1}{2} \phi + \frac{1}{2} \sin \phi \cos \phi - \phi \cos^2 \phi)}{a^2 B + r^2 \sin^2 \phi (\phi + \sin \phi \cos \phi)}.$$

This expression is of the form  $T = 2 p \rho \phi \cdot D' \cdot \frac{1 - \lambda \frac{r^2}{a^2}}{1 + \lambda' \frac{r^2}{a^2}}$ , and

the coefficient  $D'$  has also been calculated in the tables cited. In the absence of tables, the following approximate formula will suffice for a flat arch :

$$T = \frac{4 f p \rho \phi}{7 a} \left( \frac{7 a^2 - 3 f^2}{8 f^2 + 15 r^2} \right).$$

18. Finally, for a *uniform load  $p$  per unit of length on the horizontal line* (weight of the platform). As the moving load may only be partial, let  $\theta_1$  and  $\theta_2$  be the angles analogous to  $\theta$  (Fig. 8) which limit the portion of the arch loaded. Then  $T$  will be given by the following equation :

$$\begin{aligned} \frac{T}{p \rho} [2 a^2 B + 2 r^2 \sin^2 \phi (\phi + \sin \phi \cos \phi)] \\ = a^2 (\sin \theta_2 - \sin \theta_1) (3 \sin^2 \phi - 2 - 2 \phi \sin \phi \cos \phi) - \frac{a^2}{3} (\sin^3 \theta_2 - \sin^3 \theta_1) \\ + \frac{a^2 \cos \phi}{2} (\theta_2 - \theta_1 + 2 \theta_2 \sin^2 \theta_2 - 2 \theta_1 \sin^2 \theta_1 + 3 \sin \theta_2 \cos \theta_2 - 3 \sin \theta_1 \cos \theta_1) \\ - r^2 \sin^2 \phi [\sin^2 \phi (\sin \theta_2 - \sin \theta_1) - \frac{1}{3} (\sin^3 \theta_2 - \sin^3 \theta_1)]. \end{aligned}$$

When  $p$  extends to the entire length, make  $\theta_1 = -\phi$ ,  $\theta_2 = \phi$ , and consequently we have

$$T = p a \frac{\frac{1}{6} \sin^2 \phi (7 a^2 - 6 a^2 \phi \cot \phi - 4 r^2 \sin^2 \phi) + \frac{1}{2} a^2 (\phi \cot \phi - 1)}{a^2 B + r^2 \sin^2 \phi (\phi + \sin \phi \cos \phi)},$$

$B$  designating, as above, the quantity  $\phi + 2 \phi \cos^2 \phi - 3 \sin \phi \cos \phi$ .

This last expression for the thrust differs little from

$$T = \frac{4 f p}{7} \left( \frac{7 a^2 - f^2}{8 f^2 + 15 r^2} \right).$$

But it may also, without altering its value, be written under the

$$\text{form } T = 2 p a D'' \frac{1 - \lambda \frac{r^2}{a^2}}{1 + \lambda' \frac{r^2}{a^2}}, \text{ and } D'' \text{ sought in the tables of M.}$$

Bresse. The vertical reactions are  $Q = Q' = p a$  (in the case of the complete load). If the parabolic form for the arch had been adopted, the horizontal thrust would be, as with suspension bridges,

$$T_1 = \frac{p a^2}{2 f}.$$

19. When once the determination of the reactions of the abutments has been arrived at, there is no further difficulty in applying the formulæ of resistance to any section whatsoever. The complete bending moment  $\mu$  and the longitudinal force  $N$  are calculated from the known forces, then the formula  $R = \frac{N}{\omega} + \frac{v\mu}{I}$  gives the stress  $R$  per superficial unit on the most strained fibre in the section ( $v$  = distance of this fibre from the neutral axis,  $\omega$  = area of section, and  $I = \omega r^2$  its moment of inertia). Considering successively various sections, the section will be found where the stress  $R$  attains the highest value. This *maximum maximorum* of stresses is obtained rapidly in the case of a load  $p$  per running horizontal metre on the entire length, by multiplying  $\frac{pa}{\omega}$  by a coefficient  $\beta$ , which M. Bresse's Table V. gives ready calculated for a series of values of  $\frac{r^2}{a^2}$ , of  $\frac{2\phi}{\pi}$ , and of  $\frac{h}{a}$  (or  $\frac{2v}{a}$ ). This applies always to a circular arch of uniform section, this section being also symmetrical relatively to the neutral axis, which occupies the middle of its height  $h$ .

20. The best ratio  $\frac{f}{2a}$  to adopt between the rise  $f$  and the opening  $2a$  depends on  $\frac{r^2}{a^2}$ ; it would be, for example—

For $\frac{r^2}{a^2} =$	0.0001	0.0002	0.0003	0.0005	0.0008	0.0010	0.0015
$\frac{f}{2a} =$	0.124	0.150	0.158	0.176	0.198	0.212	0.221

Good sections to adopt for arches are those which make  $\frac{r}{h}$  large, i.e., which have a large radius of gyration or moment of inertia, without too great height.

21. The tables prepared by M. Bresse simplify considerably the calculation of circular arches with a uniform section. The circular form is, indeed, that which is ordinarily chosen in practice; but the restriction to uniform section is more troublesome, for there is a double motive to vary the section. By diminishing the middle of the arch the play of the expansion is facilitated (in the absence

of a pivot at the summit), and by enlarging the section near the abutments the strength is increased where the pressures become greater. It is better to give to a structure the form which is practically most advantageous rather than that which most facilitates the theoretical calculations. If, then, in the case of a variable arch, either from want of time or of skilful aid, the general formula for T (Art. 13) be thought too complicated, the method of M. Méry may be adopted, by choosing arbitrarily the centre of pressure at the crown; but in this case it will be advisable to make say two calculations instead of one, with situations notably different from the arbitrary point. If these operations indicate advantageous results for the two hypotheses (repeated in the various cases of loading presumed to be dangerous), doubtless the safety of the work may be considered as sufficiently established.

If the joints of attachment at the abutments are keyed at several points, instead of being pivoted, it would be illusive, as in a masonry arch, to pretend to make a rigorous theoretical calculation; for it could not be ascertained that the keys were all driven equally in the first instance, or that in touching them afterwards, in case of derangement, the resistance of the structure had not been altered. Further, the wedging-up by numerous keys should be considered objectionable. If the centre of pressure of the extreme section, which is not adherent to the support, passes certain limits (*noyau central*) a key of the extrados or the intrados will become loosened, while the opposite one will suffer excessive strain. To avoid these evils it would be necessary to have recourse to bolting or *encastrement*, often difficult to apply, or of little utility.

#### ARCHES OF DIFFERENT MATERIALS. DETAILS OF CONSTRUCTION.

22. Structures in timber are the most economical in many countries, but their durability is limited. This may be prolonged by successive renewals of the parts deteriorated; but such reparation is costly, and it tends to augment the general deformation or giving way, because every new piece introduced in a space shortened by previous alterations shortens it further when it takes its share of the work.

Metals, cast iron, wrought iron, or steel, promise generally a greater durability.

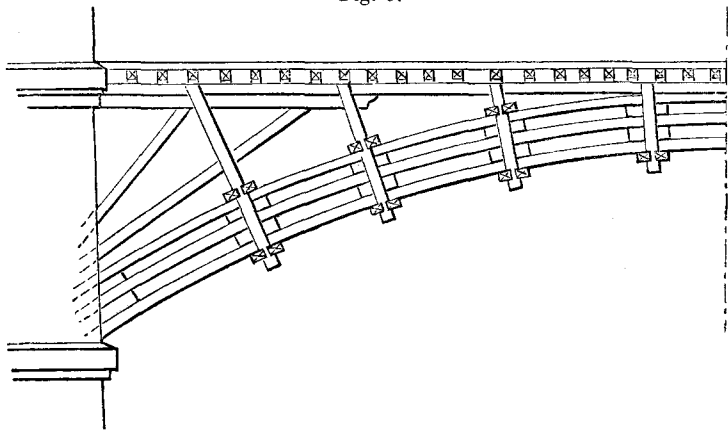
*Arches in Timber* do not form entire structures so well connected together as to admit of the application of M. Bresse's formulæ, nor of the adoption of pivots. It is, therefore, necessary to proceed nearly as for a vault of stone, by the curve of pressures. The

difference only lies in the form of section of the arch, and in a different mode of distribution of the loads

As the wood alters and twists by hygrometrical variations, it is generally desirable to give a slight rise (camber) to the platform. Doubtless, at first, if the bridge has many spans, this camber may be slightly visible to the eye, and may appear as a series of convex curves breaking the continuity of the horizontal line of the platform; but this appearance is certainly less disagreeable than the hollows, signs of sinking, which, even while the solidity is undoubted, impress the public with an idea of danger.

23. The principal type of timber arches appears to be that inaugurated at Yvry (Seine) by M. Emery, namely, that where the arches are composed of strong pieces of carpentry superposed to the number of three, for example. But it is preferred in many cases to leave open the intervals between the several pieces in order to allow better ventilation, and to prevent them from heating; and further, because this plan gives an increase of depth to the arch, and consequently a more ample field to the oscillations of the curve of pressures. The sketch (Fig. 9) will recall the type referred to.

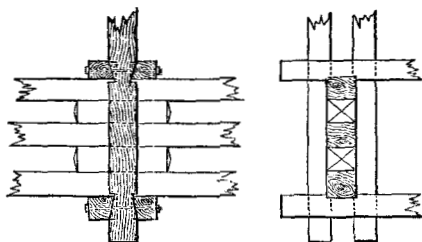
Fig. 9.



The horizontal and the hanging members are connected together (Fig. 9a) by obliquely-cut joints, forming sliding planes, so that the bolts produce a reciprocal tightening of all the pieces brought together. With all surfaces in contact, moisture and heating are to be feared; but it is advisable to apply layers of coal-tar here as well as in other internal surfaces where the parts are fitted together. The iron-work should be as much as possible in the

form of straps and stirrups, clasping the wood round without piercing it. In the butt-joints of the segments of the arch it is useful to interpose plates of copper, or of very hard wood, in order

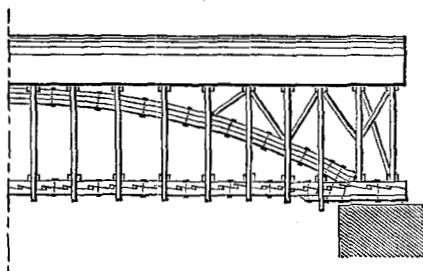
Fig. 9a.



to prevent the mutual penetration of the fibres. If it is absolutely necessary to use wood imperfectly seasoned, it ought not to be painted till a year or two after the construction.

Certain timber bridges present a compressed arch, a tensile tie-bar, and vertical connecting-rods (Fig. 10). This system is allied to the bowstring form, which will be referred to hereafter.

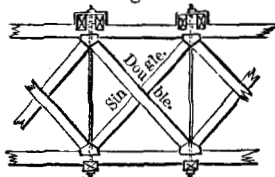
Fig. 10.



The flat arches of Wiebeking (Bavaria) and those in thin layers of planks superposed (Emy) would appear only suitable for roofing purposes, being too subject to deformation for bridges.

Among the American forms of trellis framework for straight beams, capable of competing with arched openings, the system of Howe, having iron vertical tie-rods, deserves special mention (Fig. 11). Two observations may be made, by way of digression on this system.

Fig. 11.



A. It permits, within certain limits, of the correction of too

much deflection by simply tightening the screws of the ties. In effect this operation shortens the rods which the flexure has elongated. Further, the tightening compresses the cross pieces in the diagonal which has distended, and may even, if pushed further, compress this simple diagonal more than the other, of which the section is double; these effects are contrary to those of the load, which has deformed the beam. In practice this resource is but limited, especially in bridges of several spans with continuous girders. A bridge on this system, constructed on the Rhone, near St. Maurice, in Switzerland, has suffered considerable deformation, resulting, at length, in requiring the application of thick packing pieces to level the rails; this work, built with green timber, and painted prematurely, has required almost complete reconstruction within three or four years after its first erection; the successive replacement of the pieces of timber, during the working of the railway, has not been possible without aggravating the deformation already produced by the load.

B. The association of two different materials, wood and iron, in one and the same resisting system, is not free from inconvenience. The tightened joints of the cross pieces occasion shocks under the vibrations of the trains; besides, it has been sometimes remarked, that the lowering of the temperature in winter may unduly contract and fracture the iron rods.

24. *Constructions in Metal*.—Steel is the material of the future for large bridges; for it sustains twice as much as iron, and the dead weight is diminished. It is, moreover, a material homogeneous and elastic, and little affected by vibrations. England has already for some time inaugurated the use of steel in bridges.<sup>1</sup>

*Cast iron* resists compression well, but tension badly. It makes good arches in the cases where the curve of pressures does not pass out of a certain central zone, and where the flexure is nothing or insensible; and on the condition that there are no powerful vibrations, *i. e.*, that the dead weight is large compared to the moving load. This last consideration leads, in the case of railways, to the spreading of a thick layer of ballast on the platform, in spite of the increase of load resulting therefrom. This has been done especially on the bridge of Tarascon, over the Rhone, a great work of seven arches of 62 mètres (203 feet) each. Cast iron is less costly than wrought iron, for equal weights, and it may easily be used, under compression, up to a practical coefficient of 6 kilogrammes per square millimètre (3·8 tons per square inch) and

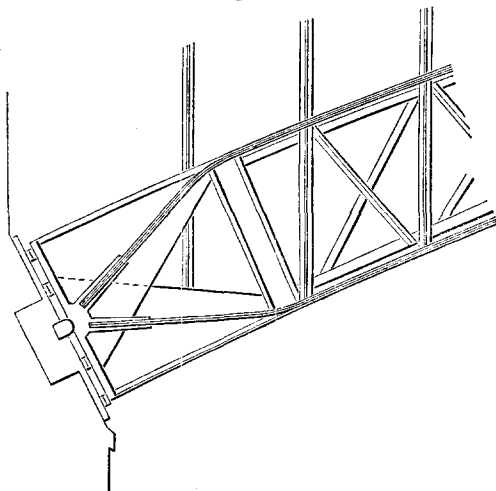
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<sup>1</sup> This paragraph ought, I think, to be taken with some reserve.—Tr.

upwards. The stress on the mean fibre in works constructed, generally approaches 3 kilogrammes (1·9 ton), the voussoirs are applied one on the other by flanches rebated and bolted together. It is desirable to avoid thicknesses less than 2 centimètres (0·8 inch nearly), and also too abrupt variations in the thickness of the same piece, to avoid inequalities in cooling, and differences of texture.

*Wrought iron* is preferable to cast iron, in spite of its higher price, whenever there is reason to fear the effects of flexure in certain parts, or when it is desired to make a light structure, without loading of ballast. The rivetted joints render the piece as strong as if it were rolled entire;<sup>1</sup> there is the opportunity of adopting pivoted supports, and of calculating the thrust according to the theoretical deformation. However, this conclusion is not absolutely true for trellised arches, which it is necessary to use in the case of very large spans. Such, for example, is the bridge at Coblenz over the Rhine, consisting of three arches, each 96·70 mètres (317 feet) span (Fig. 12). In this bridge the pivot applied to the abutments is interposed between two cast-iron plates, one

Fig. 12.



bedded with cement on the masonry, the other provided with boxes or jaws to receive the feet of the arch, which are bent to converge

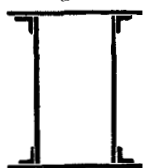
<sup>1</sup> Not always; rivetted joints are often badly designed, and worse made.  
—Tr.

to this point. The arch, however, has not been reduced in depth, the triangular parts have been preserved, and sustained by ordinary keying, in order to diminish the mobility. This may be justified for a work of this kind, with a trellised arch, but in the case of a solid plate web, the pivot alone without keys would appear better.

In general, a solid plate web is preferable to trellis-work for arches of moderate dimensions. In effect, the flexure is small, and the longitudinal pressure much predominates; the solid plate acts, consequently, more usefully than in a straight beam, which presents neutral fibres. Moreover, it will only be necessary to employ rivets at considerable intervals in the parts outside the joints.

To satisfy the advantageous condition named in Art. 20, *i. e.*, a large moment of inertia, without too great depth, it is desirable to adopt for the section a double T with large wings. For large openings, however, a box section (Fig. 13) appears to be preferred.

Fig. 13.



The lateral stiffness will depend essentially on the cross framing which connects together the different arch ribs of the same span. Where this resource is wanting, *i. e.*, where an isolated arch rib must maintain itself alone, the oval section, analogous to that of the arches of the gigantic bridge at Saltash, presents itself as one of the most favourable.

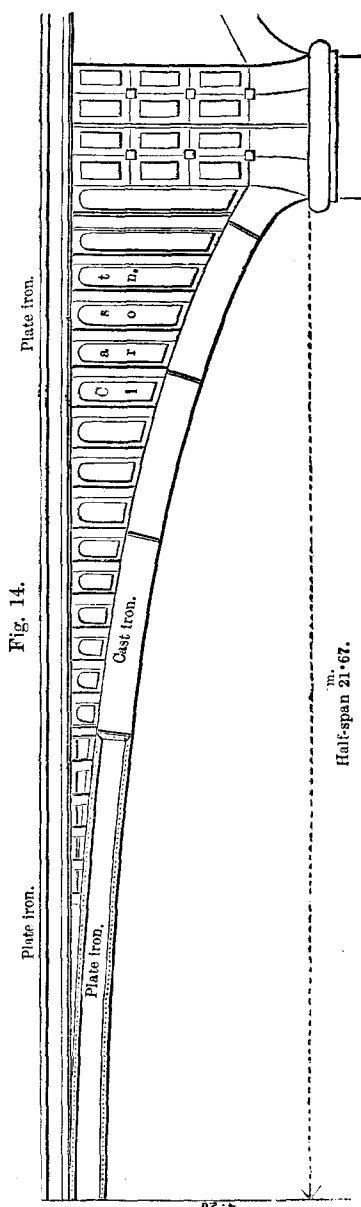
25. Wrought and cast iron may be associated in the same work. This combination has been adopted, for example, in a bridge of three arches constructed in the Park at Neuilly, near Paris. Fig. 14 indicates the general aspect of a semi-span of this work, many details of which have been suggested by the new Westminster Bridge. All the pieces essentially under compression, *i. e.*, the spandrels and the portions of the arch near the abutments, have, for the sake of economy, been made of cast iron, while the central portion of the arches and the longitudinal bearers of the platform are of plate iron. The summit of the arch is very flat, which exposes it to sensible flexure; at certain parts of the crown the stress of tension may attain 4 kilogrammes per square millimètre. The longitudinal bearer is, moreover, destined to act as an anchoring tie, in order to unite the three parts into a solid whole, securely fastened to the masonry; and this is extended to the abutments on either side. The resistance of such a structure is undoubtedly considerable, but it is difficult to estimate, with any certainty, the internal strains which may arise from expansion.

Arched bridges are, on principle, often more economical than those formed with girders; they admit better of an augmentation of the dead weight, for the purpose of deadening the vibrations and increasing the probable durability. But it is necessary that sufficient height should be available, and that the abutments should be able to resist the thrusts. When the work is low and the ground solid, it is easy to give the abutments the necessary stability, without too much expense. The compression, tending to close the molecules of the metal, appears to promise to such works a longer duration than would be due to pieces in tension, which threaten, after a long period, to be subject to enervation. These considerations in favour of arched bridges are, it is true, somewhat counterbalanced by the greater complication of the forms.

#### BOW AND STRING BRIDGES.

26. The girders called by this name are, as the words imply, arches provided with a tie-rod which receives the thrust upon the extremities, without the intervention of supports; so that the abutments are freed from the effect of these thrusts, and only exert reactions in a vertical direction.

The most remarkable example is the bridge at Saltash, of two great spans of 139 mètres each.



The arch, being single, had to be kept at the two ends at the height above the roadway necessary to leave a free passage for the trains; and this led the eminent designer, Mr. Brunel, to adopt a curved tie, and to suspend the platform at a lower level.

When there are two arches, one on each side of the road, there is nothing to prevent their extremities descending to the level of the platform; the tie is then straight, and being strengthened in order to serve as a longitudinal bearer, it may support the roadway. An example of this disposition is the bridge of Audenarde, on the Scheldt. Between the arch and the tie, there is fixed a framework composed of vertical bars more or less strong, and of lighter diagonals. The office of these pieces is analogous to that of the spandrels in ordinary arched bridges, with this difference, that the tensions predominate, or are the only forces present. The attachment of the tie to the ends of the arch ought to be very strong.

When the bow or arch of such a structure is made very stiff, the vertical bars may be considered as simple suspending-rods, and the diagonals as stiffening bars bearing scarcely any strain. If, on the other hand, the arch requires to be stayed at intervals, then, for the sake of precaution, the trellis may be calculated as in the case where the arch might be made of articulated bars. This hypothesis does not entail so much expense as might be thought at first sight, inasmuch as certain practical exigencies otherwise lead already to the vertical rods being given a somewhat stronger section. As to the diagonals, they may remain slender, on condition that they are calculated for stresses of tension, *i.e.*, in the case where rods exist following the two diagonals of each interval, duty must only be expected from that which is brought into tension by the load considered. If the arch is parabolic, it maintains its figure of equilibrium under a uniform load per running mètre of the platform, extended to its whole length; then the diagonals only work under partial loads, extending over only a portion of the roadway. The vertical bars, like portions of the trellis opposed to the deformations of the arch, fulfil the office of compressed members, but at the same time they act as suspending rods under tension; it may be conceived, therefore, that the resulting stress on them will always remain as a stress of tension, and that in every case their accidental compression, if such occur, will be but small. Also, the stiffening of these pieces at the price of an increased section, would be determined, not so much by the chance of compression, as by the need of seeking in them fixed

points of attachment for the stiffening or gusset-pieces necessary to prevent turning over.

In a work published in 1865 (*Étude comparative sur les Ponts en Fer*; published by Lacroix), the Author of the Paper has given formulæ (Chap. XII.) expressing the stresses on the various parts of bow and string girders, as well as the comparison between this and other systems of construction.

APPENDIX.<sup>1</sup>I. *Application to a Design for an Iron Arch Bridge over the Rhone, near St. Maurice, for the Western Railway of Switzerland. (Plate 5<sup>A</sup>.)*

1. The Author had the honour a short time ago to submit to the Institution a memoir on Arch-Bridges in Metal and Timber. It may be objected to that essay that it is confined to the vague terms of general theory; and for this reason, the Author, having received a commission from the West of Switzerland Railway Company to design an iron arch-bridge, to be substituted for the temporary wooden structure at St. Maurice, believes the Institution would be interested to receive a *résumé* of the calculations he has made for this design. A particular application, in spite of its imperfections, or even by reason of its imperfections, may sometimes better show the difficulties of the problem, and better illustrate certain facts connected with it, than pure general investigations.

The accompanying plate gives the principal features of the proposed arch. The skew-opening<sup>2</sup> is 68·19 metres between the abutments; the chord of the arch is 69 metres, measured between the extremities of the mean fibre, these extremities being keyed on a nucleus of steel, shaped as a segment of a circle, to serve as a pivot, as shown in detail No. 6. By virtue of these pivots the bending moment will be always = 0 at the supports or extremities of the mean fibre.

The two arch-ribs are constructed as for a square bridge, and abut on normal supporting faces; but they are placed one in advance of the other, to provide for the skew. The cross-girders and vertical supports are normal to the axis of the work (see transverse section, Fig. 5), to avoid the complication of oblique joints.

2. The load on each of the two arch-ribs of the bridge, per running metre on the horizontal, is estimated as follows:—

	Kil.
Permanent dead load . . . . .	$p = 2,500$
Live, or moving load . . . . .	$p' = 2,000$
Maximum total load . . . . .	$p + p' = 4,500$

<sup>1</sup> This Appendix was received at different dates, after the original Memoir, and had not been translated or circulated previous to the discussion.

<sup>2</sup> The dimensions have been retained in their original form, as the metric system is now thoroughly understood and much used by English Engineers.—Tr

The arch receives the load in a discontinuous manner by the vertical struts; but these being tolerably numerous, it is permissible to regard the distribution of weight as sensibly uniform. In principle, the only difference between a continuous load and a load applied at points some distance apart is, that in the latter case the locus of the bending moments is polygonal, instead of being curvilinear.

The mean fibre is an arc of a circle (Fig. 7) presenting a chord of 69 metres =  $2a$ , with a rise  $f = 7.575$  metres. Consequently, its radius  $r = \frac{a^2 + f^2}{2f} = 82.352$  metres.

The vertical component  $Q$  of the reaction of the abutment, in the case of the complete load, is  $Q = 4500 \times \frac{69}{2} = 155,250$  kil.

To obtain the horizontal thrust  $T$ , which depends on the deformation of the arch, recourse must be had to the following complicated formula, due to Messrs. Belanger and Bresse:

$$T = \frac{2\tau a E + \int \frac{\mu_1 y}{I} ds + \int \frac{N_1}{\omega} dx}{\int \frac{y^3}{I} ds + \int \frac{1}{\omega} \frac{dx}{ds} dx},$$

and in which the integrals are defined for the whole length of the arc or of the chord.

$E = 14,000,000,000$  is the modulus of elasticity of the metal, assumed homogeneous.

$\tau$  = elongation, per unit of length, due to causes independent of the load, such as expansion by heat, or an artificial wedging-up.

$\omega$  = area, and  $I$  = moment of inertia of the normal section of the arch at the point  $(x, y)$  of the mean fibre.

$ds$  = elementary arc;  $\frac{dx}{ds}$  is the cosine of the angle  $\alpha$  of the normal with the vertical.

$\mu_1$  = partial bending moment exerted round the point  $(x, y)$  by the weight and the reaction  $Q$  considered alone; or, in other words, the moment determined solely according to the vertical forces, abstraction being made of  $T$ , so that the complete moment  $\mu$  should be  $= \mu_1 - Ty$ . The moment is regarded as positive when it tends to turn from  $Ox$  to  $Oy$  (i.e., in a direction contrary to that of the hands of a watch), the portion of the solid situated to the right of the section under consideration.

$N_1$  = longitudinal force which stretches the section considered, always excluding the action of  $T$ . Thus the true total normal

tension will be  $N = N_1 - T \cos \alpha$ ; that is, the sum of the components, normal to the section, of the forces acting between this section and one extremity of the solid. The term 'tension' is purposely employed here, for the formula is written under this hypothesis. It is therefore necessary to remark that in the bridge arches, where  $N$  acts by compression, this force must be given a negative value. This will be seen, indeed, by inspection of the formula; for if  $N_1$  shortens the arch, it ought to diminish the thrust  $T$ . Similarly,  $\tau$  would be negative if it were a question of contraction by cold, tending to loosen the arch from its abutments.

3. The expression for  $T$  comprises quadratures which will be approximately effectuated here by dividing, with this view, the arch into an arbitrary number of finite parts, not necessarily corresponding to the vertical struts, as the load is assumed continuous. If, for example, the semi-arch is subdivided into twelve parts of equal amplitude  $= 2^\circ 3' 50''$ , the abscissæ and ordinates will have the value given in the annexed table (p. 101). For an angle  $\alpha$  with the vertical, the abscissa, measured from the middle, is  $x' = r \sin \alpha$ ; and, reckoned from the extremity 0 (Fig. 7), it becomes  $x = a \pm r \sin \alpha$ . The ordinate will be  $r(1 - \cos \alpha)$  starting from the tangent at the summit; it becomes  $y = f - r + r \cos \alpha$ , measured above the chord,  $f$  being the rise or versed sine. The length  $\Delta s$  of the equal divisions taken on the arc is  $= \pi r \frac{2^\circ 3' 50''}{180^\circ} = 2.9664$  metres.

In this table will be found two kinds of abscissæ, those  $x'$  measured from the middle of the opening, and those  $x$  reckoned from the left extremity 0 (Fig. 7). In order to designate any particular point, the value corresponding to  $x'$  may be used, if treating of a symmetrical load; but in the case of a load unequal on the two halves of the arch, recourse must be had to the abscissæ  $x$  to specify, without the embarrassment of sign, which of the two symmetrical points is intended. The table establishes clearly the correspondence of  $x$ ,  $x'$ ,  $y$ , and  $\alpha$ .

The table also contains other columns, the mode of calculating which is as follows:

The bending moments  $\mu_1$  being estimated solely according to the vertical forces, neglecting the horizontal thrust  $T$ , are independent of the ordinate  $y$ ; they have the same values as they would have for a straight beam. They may be obtained graphically, for whatever load, knowing that their geometric locus is the funicular polygon. For the present case of uniform and complete load, they

Angle $\alpha$ of the Section with the Axis.	Abscissae.		Ordinate on the Chord $y$ .	Moment $\mu_1$ .	Force $N_1$ .	Half-thickness of the Arch $\frac{e}{2}$ .	Area of the Section $\omega$ .	Moment of Inertia $I$ .	$\frac{\mu_1 y}{1000 I}$ .	$\frac{N_1 \cos \alpha}{\omega}$ .	$\frac{y^2}{I}$ .	$\frac{\cos^2 \alpha}{\omega}$ .
	From the middle $x'$ .	From the Extremity $x$ .										
0 0	m. 0	m. 34.500 { 37.466 and 31.534 40.428 and 28.572 { 43.982 25.618 { 46.325 29.675 { 49.252 19.748 { 52.160 13.954 { 55.046 11.095 { 57.905 8.267 { 60.733 5.472 { 63.528 2.716 { 66.284 0	m. 7.575	2678060	k. 0	m. 0.350	m <sup>2</sup> 0.07282	0.0078728	2576800	0	7288	13.7
2 3 50	2.966		7.522	2658260	— 480	0.352	0.07290	0.0079405	2518200	— 6580	7126	13.7
4 7 40	5.928		7.361	2599000	— 1920	0.359	0.07318	0.0082693	2313500	— 26170	6552	13.6
6 11 30	8.882		7.095	2500560	— 4310	0.369	0.07358	0.0087515	2027200	— 58248	5752	13.4
8 15 20	11.825		6.722	2363440	— 7640	0.383	0.07414	0.0094013	1689900	— 101980	4806	13.2
10 19 10	14.752		6.243	2188390	— 11892	0.401	0.07486	0.0103938	1314400	— 156290	3750	12.9
12 23 1	17.660		5.659	1976260	— 17042	0.423	0.07574	0.0116117	963150	— 219770	2758	12.6
14 26 51	20.546		4.971	1728270	— 23067	0.449	0.07678	0.0131456	653540	— 290930	1880	12.2
16 30 41	23.405		4.179	1445580	— 29933	0.479	0.07798	0.0150446	401545	— 368030	1160	11.8
18 34 31	26.233		3.285	1129700	— 37604	0.514	0.07938	0.0174376	212820	— 449040	619	11.3
20 38 21	29.028		2.290	782210	— 46014	0.551	0.08086	0.0201790	88768	— 532880	260	10.8
22 42 12	31.784		1.194	405050	— 55202	0.592	0.08250	0.0236421	20456	— 617270	60	10.3
24 46 2	34.500		0	0	— 65039	0.636	..	..	0	— 700880	0	9.8

are represented by the ordinate of a parabola, or by

$$\mu_1 = \frac{p+p'}{2} (a^2 - x'^2) = 2250 (1190.25 - x'^2), \text{ the formula from}$$

which they have been calculated.

The longitudinal forces  $N_1$  are calculated by  $(p+p')(a-x)\sin\alpha$ , which will always give negative values, regarding the angle  $\alpha$  as negative when taken to the left of the vertical of the summit, in which case  $x < a$ . This expression may also be written

$$N_1 = -(p+p') \frac{x'^2}{r} = -\frac{4500}{82.352} x'^2,$$

$$\text{or simply } -(p+p')r \sin^2\alpha.$$

Passing on to the calculation of the areas  $\omega$ , and of the moments of inertia  $I$  of the sections: at present the arch has been only introduced by the figure of its mean fibre, and by a rough estimate of its weight, which enters into the valuation of the dead load  $p$ . But it is now desirable to give entirely, *à priori*, the dimensions of this member; the calculations only take the form of a verification *à posteriori*, not of a direct solution of the problem.

As a basis for the preliminary hypothesis, it may be considered in the first place what section would be necessary if the flexure were = 0 in the case of the complete load, *i.e.*, if the arch had the parabolic form. The thrust  $T$  would then be expressed by the simple formula of the suspension bridge—

$$\frac{p+p'}{2f} a^2 = \frac{4500 \times 34.50^2}{2 \times 7.575} = 354 \text{ tons, and the total oblique thrust}$$

against the abutment would become  $\sqrt{354^2 + (4.5 \times 34.50)^2} = 386$  tons. These stresses would require a section at the summit = 0.0506 metres, and at the springing = 0.055, making the metal do a duty of 7 kilogrammes per square millimètre.

It will be necessary to adopt stronger sections, possessing a moment of inertia sufficient to resist the flexures due to the non-parabolic form, and particularly to the displacements of the load, and to the expansion by heat. It might be desirable perhaps to double the preceding sections, if the object were to realize a constant thickness for the webs or sole-plates. But it will be preferable only to add slightly to these dimensions; the calculations of verification may then probably discover some points which are too weak, and it will thus suffice to apply strengthening plates solely to the defective parts, in imitation of what is done with straight beams.

4. Let the section at the summit, drawn in Fig. 8, and whose area is = 0.0728, be tried. It is for convenience of execution that

this somewhat irregular form has been adopted; it will, however, be remarked that the mass of matter is distributed symmetrically above and below the neutral axis, in order that this axis may occupy the middle of the height. The part in surplus of the lower member is compensated for by an inequality of the horizontal plates, in such wise that the section may theoretically be assimilated to Fig. 9, which is perfectly symmetrical. Its moment of inertia will be

$$I = \frac{0.9 \times \overline{0.748^3} - 0.58 \times \overline{0.7^3} - 0.258 \times \overline{0.672^3} - 0.042 \times \overline{0.5^3}}{12} \\ = 0.0078728.$$

Departing from this section at the key, the mean fibre is traced with the radius  $r = 82.352$  metres, and the curve of the intrados with a radius  $r' = 79$  metres only. These two curves are not concentric; their distance apart  $\frac{e}{2}$ , measured on the normal to the mean fibre, depends on the distance apart  $b$  of the centres of the two circles and on the variable angle  $a$  of the normal under consideration. It is calculated by the formula

$$\frac{e}{2} = r - b \cos a - \sqrt{r'^2 - b^2 \sin^2 a} = 82.352 - 3.002 \cos a - \\ \sqrt{6241 - (3.002 \sin a)^2}.$$

Where  $b = 82.352 - 79 = 3.350 = 3.002$  metres, on the understanding that the curve of the intrados of 79 metres radius is taken on the layer separating the angle-irons and sole-plates, so that the half-depth at the summit is  $\frac{e}{2} = 0.350$  metres (Fig. 8 or 9). The radical term differs but very little from 79; the greatest difference is at the extreme joint ( $a = 24^\circ 46' 2''$ ), which gives 78.990. By laying down the variable half-depths  $\frac{e}{2}$  above the mean fibre, the curve of the extrados is obtained.

The section from the summit to the springings only varies by the fact of the increase of  $e$ . The area  $\omega$  only increases by virtue of the increase of height of the vertical plates. The successive moments of inertia  $I$  are calculated as above, by the aid of a table of cubes. It would be useful to have tables still more convenient, in order to obtain quickly the values of moments of inertia, the laborious calculation of which is inherent in all problems of resistance.

5. The elements are now prepared for the calculation of  $T$ . Having taken an even number of uniform divisions  $\Delta s = 2.9664$

metres, measured on the arch, with a view of applying Simpson's approximate formula of quadrature, all may be expressed in  $ds$  in the formula of T; that is to say, that it will be written thus, substituting  $ds \cos \alpha$  for  $d\alpha$ :

$$T = \frac{2 \tau a E + \int \frac{\mu_1 y}{I} ds + \int \frac{N_1 \cos \alpha}{\omega} ds}{\int \frac{y^2}{I} ds + \int \frac{\cos^2 \alpha}{\omega} ds}.$$

Table No. 2 contains the successive values of the quantities

$$\frac{\mu_1 y}{I}, \quad \frac{N_1 \cos \alpha}{\omega}, \quad \frac{y^2}{I} \text{ and } \frac{\cos^2 \alpha}{\omega},$$

easy to calculate with the elements furnished by the preceding columns.

The arch being symmetrical and symmetrically loaded, it suffices to consider here one-half of it. According to Simpson's parabolic method of quadratures, every integral of the form  $\int z ds$  will be the product of the third part of the equidistance (for example  $\frac{\Delta s}{3}$ ) by a sum compounded—

Of the sum of the two extreme ordinates ( $z_0 + z_n$ );

Of the quadruple sum of the ordinates with an odd index  
 $4(z_1 + \dots + z_{n-1})$ ;

And of the double sum of the intermediate ordinates with an even index  $2(z_2 + \dots + z_{n-2})$ .

Thus

$$\int \frac{\mu_1 y}{I} ds = 40476990000 \cdot \frac{\Delta s}{3}, \quad \int \frac{N_1 \cos \alpha}{\omega} ds = -9511976 \cdot \frac{\Delta s}{3},$$

$$\int \frac{y^2}{I} ds = 115109 \cdot \frac{\Delta s}{3}, \text{ and } \int \frac{\cos^2 \alpha}{\omega} ds = 443 \cdot \frac{\Delta s}{3}.$$

Omitting, at first, the dilatation  $\tau$ , due to the temperature or to the initial wedging-up,

$$T_1 = \frac{40467480024}{115552} = 350210 \text{ kil.}$$

The terms in  $\cos \alpha$  and  $\omega$  might perhaps have been neglected without great error; for, without taking account of them, there would be

$$\frac{40476990000}{115109} = 351640.$$

The difference between this value and the former one is less than the unknown errors that might be produced by the expansion and

the wedging. At any rate, the terms in  $\frac{N_1 \cos \alpha}{\omega}$ , which arises from the shortening of the mean fibre, has but small influence; without it, there would follow that

$$\frac{40476990000}{115552} = 350293.$$

It is to be remarked that the circular form has not much modified the thrust 354 tons, given by the parabolic curve, in Article 3.

6. Now let  $\tau = 0.0004$ ; this will be the linear variation due to an increase of temperature of  $33^\circ$  centigrade, or to a less increase, accompanied by a certain amount of wedging-up. This effect gives rise, in the numerator of the formula of T, to a supplementary term

$$2a\tau E \frac{3}{\Delta s} = 69 \times 0.0004 \times 14000000000 \times \frac{3}{2.9664} = 390777000;$$

it produces, consequently an increase of thrust

$$= \frac{390777000}{115552} = 3380 \text{ kil.}$$

The initial wedging-up is unknown, or arbitrary; the temperature varies. These effects, also, may very well be neutralized by an opposite cause; such as the settlement of an abutment, which may allow the arch to expand slightly while the dilatation recompresses it. Now write separately the bending moments  $\mu = \mu_1 - T_1 y$  and the forces  $N = N_1 - T_1 \cos \alpha$  due to the loading only, then the augmentations  $-3380 y$  and  $-3380 \cos \alpha$  produced by the supplement of thrust 3380 kil., due to the admitted dilatation. These augmentations will then only enter into account for the sections whose conditions of resistance are aggravated by them.

Abcisse, starting from the Middle $x'$	Effect of the Complete Load.		Supplements due to the Expansion T.	
	$\mu$	N	For $\mu$	For N
0	+25220	-350210	-25600	-3380
2.966	+23982	-350460	-25425	-3378
5.928	+21104	-351220	-24880	-3370
8.882	+15820	-352480	-23980	-3360
11.825	+ 9328	-354220	-22720	-3345
14.752	+ 2030	-356440	-21100	-3325
17.660	- 5578	-359100	-19130	-3300
20.546	-12622	-362200	-16800	-3273
23.405	-17947	-365700	-14125	-3240
26.233	-20740	-369570	-11103	-3204
29.028	-19770	-373780	- 7740	-3163
31.784	-13100	-378280	- 4036	-3118
34.500	0	-383040	0	-3070

7. In any section whatever, submitted to a moment  $\mu$ , and to a longitudinal tension  $N$ , the work  $R$ , per unit of surface, on the fibres at the distance  $v$  from the neutral axis, is expressed by

$R = \frac{v\mu}{1} - \frac{N}{\omega}$ . By the convention, our values of  $N$  being compressions, are negative. For the maximum of the stress  $R$ , the fibres must be considered as the farthest removed from the neutral axis,

for which  $v = \frac{e}{2} + 0.024$  metre (Fig. 9); and we may take  $v$  positive for the upper fibre, or that of the extrados, and negative for the intrados. The fibre which the flexure tends to break by extension is aided by the pressure  $-N$ ; on the opposite fibre, however, the two stresses  $\mu$  and  $N$  act together to produce crushing; this dangerous point of the section is found at the extrados when  $\mu$  is positive, and at the intrados if  $\mu$  is negative. The value of  $\mu$  is found to be positive in the central region, and negative in the external portions; the arch tends slightly to sink in the middle and to rise at the haunches, when not acted on by expansion.

The expansion, or a wedging-up of the arc, produces negative moments. Taking the intensity  $\tau = 0.0004$ , these moments overcome those of the load, so that the central sinking disappears, and the arch rises throughout.

The section at the summit is strained at the extrados, when  $\tau$  is nul; the maximum work, at this point, is then:

$$R = \frac{0.374 \times 25220}{0.0078728} + \frac{350210}{0.07282} = 1198100 + 4809200 \\ = 6007300 \text{ kil.}$$

per square metre, or 6 kilogrammes per square millimetre. The expansion above specified almost destroys at this place the effect of flexure, bringing back the centre of pressure upon the mean fibre. The special curve of pressure which would give the expansion considered alone, would be directed according to the chord of the arch.

When the expansion acts the greatest absolute value of  $\mu$  is found at the abscissa  $x' = 23.405$ , where we have  $\mu = -17947 - 14125 = -32072$ , with  $N = -365700 - 3240 = -368940$ . The maximum stress per unit of surface is, in this section, at the fibre of the intrados:

$$R = \frac{0.503 \times 32072}{0.0150446} + \frac{368940}{0.07798} = 1042500 + 4731200 = 5773700.$$

At the preceding joint  $x' = 20.546$ , the actions are a little less, and the section is a little weaker also.

In conclusion, the case of the complete loading of the bridge does not give any strains reaching the limit imposed of 7 kilogrammes per square millimetre.

But this is not the most dangerous test that the bridge has to submit to; it is necessary to consider the arch as loaded on one half only, and unloaded on the other half. The foregoing calculations are, however, not useless; for, according to a known theorem, the thrust estimated for the case of symmetrical loading will furnish, in a very simple manner, the new thrust when the load is unsymmetrically disposed.

CASE WHERE THE LOAD ( $p' = 2000$  Kil.) ONLY COVERS HALF THE ARCH.

8. The dilatation  $\tau$ , which will exert precisely the same effects as above, since it is independent of the load, may be neglected.

The arch supports a dead load  $p = 2500$  kilogrammes per horizontal running metre, on the left half, *i.e.*, between the abscissæ  $x = 0$  and  $x = a = 34.500$  metres (measured from the left abutment); then it carries  $p + p' = 4,500$  kilogrammes on the right half, from  $x = 34.5$  to  $x = 69.0$ .

If the symmetry were re-established by loading the left semi-span, it has been seen that the thrust would be  $T_1 = 350,210$  kilogrammes.

If, on the contrary, the symmetry be re-established by unloading the portion to the right, so that the whole bridge be reduced to its dead weight  $p$ , the thrust  $T_0$  would be obtained by reducing

$$T_1 \text{ in the ratio of } p \text{ to } p + p', \text{ i.e., } T_0 = 350210 \times \frac{2500}{4500} = 194560.$$

To return to the actual case of unsymmetrical load, it suffices to take the arithmetical mean of  $T_1$  and  $T_0$ , that is, the horizontal thrust will be  $T = 272385$ .

The vertical component  $Q$ , of the reaction of the right abutment will be  $Q = pa + \frac{3}{4}p'a = 138000$  kilogrammes, and that of the left abutment  $pa + \frac{p'a}{4} = 103500$ . Between  $x = 0$  and  $x = a = 34.500$  metres, the bending moment will be expressed by  $\mu = px\left(a - \frac{x}{2}\right) + \frac{p'ax}{4} - Ty$ ; then, beyond by

$$\mu = px\left(a - \frac{x}{2}\right) + \frac{p'ax}{4} - \frac{p'}{2}(x - a)^2 - Ty.$$

On the left half, the longitudinal force is  $N = \left[ p(a-x) + \frac{p'a}{4} \right] \sin \alpha - T \cos \alpha$ , and on the other half ( $x > a$ ) it becomes

$$N = \left[ (p + p')(a-x) + \frac{p'a}{4} \right] \sin \alpha - T \cos \alpha.$$

This force  $N$  is always negative (compression), taking  $\alpha$  negative for  $x < a$ .

The terms in  $p$  may be deduced from the values of  $\mu_1$  and  $N_1$  in the preceding calculations, reduced in the ratio  $\frac{p}{p+p'} = \frac{5}{9}$ . Or the operations may be abridged, as is done for straight beams, by the aid of diagrams or graphic delineations; after having calculated a limited number of moments  $\mu$  and of forces  $N$ , they may be drawn as ordinates of a geometrical locus, the figure of which may be finished by hand, and which will serve afterwards for graphic interpolation.

Neglecting the term  $-Ty$ , the expressions of  $\mu$  are parabolic functions, of which the second differences are constant. This property may be made use of to obtain rapidly a large number of successive values corresponding to the values of  $x$ , increasing by equal steps.

More conveniently still, by cutting out a parabolic curved template in cardboard, having  $\frac{1}{p}$  for semi-parameter, this curve will serve to draw, as at AOB (Fig. 10), the representative locus of the  $\mu$  due to the dead load  $p$ . Afterwards, the additional values of  $\mu$  due to  $p'$  will be represented in ODB by a rectilinear portion OD, followed by an arc tangent DUB of a parabola; this latter may be traced with a second template curve having a semi-parameter  $= \frac{1}{p'}$ , the axis of which is kept perpendicular to the abscissæ  $x$ .

Then, for any abscissa OM, the moment  $\mu_1$ , due to the vertical forces alone, is represented by the accumulated ordinate VU. It remains to deduct from it  $Ty$ , a value proportional to the ordinate  $y$  of the mean fibre above the chord. This  $Ty$  may itself be obtained practically by reducing, in the proper ratio, by a reducing-compass, the ordinate of the arch on a large-scale drawing.

9. The values of  $\mu$  and  $N$ , given below, have, however, been calculated directly. The last columns are obtained by adding algebraically to the former ones the supplements already known (No. 6) produced by the expansion.

	Abcissæ starting from the Left Abut- ment $x$	Without Expansion.		With Expansion.	
		Moment $\mu$	Force N	$\mu$	N
Left abutment	0	0	-290690	0	-293760
	2·716	- 53350	-288600	- 57390	-291720
	5·472	- 94810	-286560	-102550	-289720
	8·267	-124570	-284580	-135670	-287780
	11·095	-143810	-282680	-157935	-285920
	13·954	-153170	-280890	-169970	-284160
Unloaded side	16·840	-153015	-279220	-172145	-282520
	19·748	-144075	-277680	-165175	-281000
	22·675	-126810	-276280	-149530	-279625
	25·618	-101460	-275050	-125440	-278410
	28·572	- 68270	-273990	- 93150	-277360
	31·534	- 28110	-273100	- 53535	-276480
Summit	34·500	+ 19620	-272385	- 5980	-275765
	37·466	+ 65420	-272350	+ 40000	-275730
	40·428	+101100	-272360	+ 76220	-275730
	43·382	+126080	-273250	+102100	-276610
	46·325	+141330	-274725	+118610	-278070
	49·252	+147240	-276780	+126140	-280100
Loaded side	52·160	+144350	-279390	+123220	-282690
	55·046	+133530	-282530	+116730	-285800
	57·905	+115890	-286180	+101765	-289420
	60·733	+ 92310	-290300	+ 81210	-293500
	63·528	+ 64060	-294870	+ 56320	-298030
	66·284	+ 32960	-299820	+ 28924	-302940
	69·000	0	-305140	0	-308210

This table brings into view the dangerous points. Without expansion, the loaded side would be the most strained as far as the abscissa  $x = 49\cdot252$ , and this upon the extrados, as it is sinking (positive moment). Farther on, the danger is transferred to the intrados of the unloaded half span which swells out. Thus the point  $x = 16\cdot840$  is exposed to the moment  $-153015$ , while the symmetrical point  $x = 52\cdot160$  is only exposed to the moment  $+144350$ , and the quantities N are nearly equal. The calorific expansion, or an artificial wedging-up of the arch between its abutments, ameliorates the condition of the loaded portion, but aggravates that of the unloaded half.

The greatest moment is realized at the section  $x = 16\cdot840$ . The stress is

$$R = \frac{0\cdot447 \times 172145}{0\cdot0116117} + \frac{282520}{0\cdot07574} = 6626800 + 3730100 = 10356900,$$

or more than 10 kilogrammes per square millimètre, on the fibre of the intrados. The extrados supports a tension  $= 6626800 - 3730100 = 2896700$ , nearly 3 kilogrammes per square millimètre.

At  $x = 19.748$  the forces are less, but the section is less also. The maximum stress is

$$R = \frac{0.425 \times 165175}{0.0103938} + \frac{281000}{0.07486} = 6754000 + 3753700 = 10507700.$$

These amounts being too high, it is necessary to strengthen the arch. It suffices to apply a supplementary plate from the abscissa 5.472; for in this section, in the actual state of things, the stress, including expansion, presents itself under the admissible value

$$R = \frac{0.575 \times 102550}{0.020179} + \frac{289720}{0.08086} = 6505200.$$

Similarly, the reinforcement may be stopped at 3 metres distance from the summit; for, at the abscissa  $x = 37.466$  metres, we find

$$R = \frac{0.376 \times 65420}{0.0079405} + \frac{272350}{0.0729} = 6833800;$$

and the symmetrical point  $x = 31.534$  is less fatigued. It is clear, besides, that two symmetrical sections of the arch are exposed reciprocally to the same accidents when the load is moved.

The deficit of strength being considerable, the arch will not only be strengthened at the haunches, but its thickness at the key will be augmented 5 centimètres, in spite of longer calculations being entailed.

#### NEW CALCULATIONS, WITH STRENGTHENING PLATES AT THE HAUNCHES.

10. The old theoretical section (Fig. 9) is now replaced by that of figure 13, where the height between the sole-plates is increased to 0.750 metres, and the thickness of the soles to 26 millimètres, instead of 24. In addition to this first general reinforcement, supplementary plates of 12 millimètres will be applied on the haunches, between the abscissæ 7.50 metres and 30 metres, and then between 39 metres and 61.50 metres (Fig. 11); so that in these parts the sole-plates may attain the thickness of 38 millimètres.

The depth  $e$  will vary according to the radius of the intrados, which we will take now = 78.50 metres, the radius of the mean fibre remaining always = 82.352 metres.

At the joint  $x = 19.748$ , the half-depth  $\frac{e}{2}$  will be = 0.433 metres, and the distance of the extreme fibre at the neutral axis will become

$v = 0.433 + 0.038 = 0.471$ ; the area becomes  $\omega = 0.10134$ , and the moment of inertia  $I = 0.017615$ . If, then, the moment  $\mu$ , and the reaction  $N$ , preserve values of the preceding case, the greatest stress would be reduced to

$$\frac{0.471 \times 165175}{0.017615} + \frac{281000}{0.10134} = 7189400.$$

At this point the researches might be stopped, and the reinforcement made a little thicker, since this last figure slightly exceeds the limit adopted, of 7 kilogrammes per square millimetre. But the series of calculations will be taken up anew under the modified conditions indicated, reserving the intention of adding hereafter other smaller and shorter reinforcements at the points which betray weakness.

The strengthening mentioned above produces some increase in the dead load. Put it at  $p = 2650$  kilometres per metre: the total load will be  $p + p' = 4650$ , or  $\frac{1}{30}$  greater than that of the former calculations; so that  $\mu_1$  and  $N_1$  themselves will have, in the case of complete loading, the values of the Table No. 2, augmented by  $\frac{1}{30}$ .

In general, a second calculation is always more rapid than a first trial, not only because the process becomes more certain, but because a great number of results already obtained are utilized again. With this view it is important to preserve the sketches and memoranda of the arithmetical operations, the logarithms of  $\cos \alpha$ , of  $y$ , etc., arranged in order. The student should not be too much afraid of extended calculations, made methodically; the labour becomes almost mechanical, and consequently rapid; and the accidental errors almost always betray themselves, on a glance of revision of the series of figures; or, if necessary, at their first differences.

In the present case, there are the sudden variations of section at the points where the additional plates begin and end. The theory assumes a gradual variation; it is not, therefore, absolutely correct to employ the same formula for the calculation of the thrust; but that approximation will doubtless suffice.

11. The following table is obtained for the complete load:

Abcissæ starting from the Summit $x'$ .	Half Depth of the Arch between the Sole Plates $\frac{e}{2}$	Area $\omega$ .	Moment of Inertia I.	$\mu_1$ .	$N_1$ .	$\frac{\mu_1 y}{1000 I}$	$\frac{N_1 \cos \alpha}{\omega}$	$\frac{y^2}{I}$	$\frac{\cos^2 \alpha}{\omega}$
m.	m.								
0 (Summit)	0.375	0.07742	0.009623	2767330	0	2178380	0	5963	12.9
2.966	0.377	0.07750	0.009730	2746870	— 497	2123530	— 6409	5815	12.9
5.928	0.384	0.09938	0.013843	2685630	— 1984	1539280	— 19912	4219	10.0
8.882	0.396	0.09986	0.014722	2583910	— 4455	1245270	— 44352	3419	9.9
11.825	0.412	0.10050	0.015940	2442220	— 7894	1029900	— 77733	2834	9.7
14.752	0.433	0.10134	0.017615	2261340	— 12290	801450	— 119313	2213	9.6
17.660	0.460	0.10242	0.019902	2042135	— 17610	580667	— 167940	1609	9.3
20.546	0.490	0.10362	0.022617	1785880	— 23836	392520	— 222760	1092	9.0
23.405	0.524	0.10498	0.025920	1493770	— 30930	240836	— 282480	638	8.8
26.233	0.564	0.10658	0.030117	1167360	— 38857	127329	— 345590	358	8.4
29.028	0.608	0.10834	0.026279	808280	— 47580	70435	— 410990	200	8.1
31.784	0.657	0.11030	0.030947	418550	— 57040	16148	— 477070	46	7.7
34.500 (Naissance)	0.709	0.11238	..	0	— 67210	0	— 543050	0	7.3

The use of Simpson's formula leads to

$$\int \frac{\mu_1 y}{I} ds = 27925604000 \frac{\Delta s}{3},$$

$$\int \frac{N_1 \cos \alpha}{\omega} ds = 7325136 \frac{\Delta s}{3}, \int \frac{y^2}{I} ds = 76735 \frac{\Delta s}{3} \text{ and}$$

$$\int \frac{\cos^2 \alpha}{\omega} ds = 342 \frac{\Delta s}{3}.$$

Consequently, the thrust due to the loads will be

$$T = \frac{27932929136}{77077} = 362410 \text{ kilogrammes.}$$

Now take the unfavourable case, where the moveable load of 2000 kilogrammes per metre is only applied on one half of the arch. The thrust will be a mean between the above thrust of complete load, and that of the arch unloaded, which is

$$= \frac{2650}{4650} \times 362410 = 206530.$$

It will therefore be = 284470 kilogrammes.

An expansion  $\tau = 0.0004$  will give rise to an increase of thrust

$$= \frac{390777000}{77077} = 5070 \text{ kilogrammes.}$$

12. The following table is calculated as in Article 9 :

	Abcissa starting from the Left Abutment $x$ .	Without Expansion.		With Expansion.	
		Moment $\mu$ .	Force N.	$\mu$ .	N.
Left abutment .	0	0	-303720	0	-308320
	2·716	- 54280	-301600	- 60330	-306280
	5·472	- 96400	-299410	-108010	-304150
	8·267	-126610	-297290	-143260	-302100
	11·095	-146120	-295270	-167300	-300130
	13·954	-155630	-293360	-180830	-298270
Unloaded side .	16·840	-155530	-291590	-184220	-296540
	19·748	-146580	-289960	-178230	-294950
	22·675	-129250	-288500	-163330	-293520
	25·618	-103850	-287210	-139820	-292250
	28·572	- 70610	-286100	-107930	-291160
	31·534	- 30400	-285190	- 68530	-290260
Summit. . .	34·500	+ 17350	-284470	- 21050	-289540
	37·466	+ 63130	-284410		
	40·428	+ 98760	-284470		
	43·382	+123690	-285400		
	46·325	+138880	-286940		
	49·252	+144740	-283070		
Loaded side .	52·160	+141880	-291760		
	55·046	+131070	-295000		
	57·905	+113580	-298770		
	60·733	+ 91270	-302940		
	63·528	+ 62460	-307710		
	66·284	+ 32040	-312820		
	69·000	0	-318280		

The last columns have not been finished, because the expansion is only prejudicial to the unloaded side.

On the loaded side the greatest stress will be found at the extrados, at the abscissa  $x = 46·325$ ; its value is

$$R = \frac{0·450 \times 133380}{0·015940} + \frac{286940}{0·1005} = 3920700 + 2855120 = 6775820.$$

The preceding point  $x = 49·252$  gives  $R = 6722640$ .

On the unloaded side, the greatest stress, with expansion, is produced near  $x = 19·748$ , where

$$R = \frac{0·471 \times 178230}{0·017615} + \frac{294950}{0·10134} = 4765600 + 2910500 = 7676100$$

kilogrammes per square metre for the compression at the intrados; and  $R = 4765600 - 2910500 = 1855100$  for the tension on the extrados.

The preceding point  $x = 16·840$  gives

$$R = \frac{0·498 \times 184220}{0·019902} + \frac{296540}{0·10242} = 4609600 + 2895300 = 7504900.$$

[1870-71. N.S.]

At the point  $x = 31.534$  metres, deprived of the strengthening-plate,

$$R = \frac{0.403 \times 68530}{0.00973} + \frac{290260}{0.0775} = 2838400 + 3745300 = 6583700.$$

13. It will be seen that it is necessary again slightly to strengthen the weak part of the haunches in order to limit the stress to 7000000, at least if we wish to be independent of the aid of the cross ties, which in reality maintain the arch against the flexure, and diminish consequently the calculated stresses.

An interesting question presents itself here: since it is the inequality of the load on the symmetrical sides which most threatens the stability, would it not be possible to ameliorate the conditions of the work by an addition of dead weight at suitable points? This is the case in some measure, as the following calculation will show.

Suppose two weights  $P$ , equal and symmetric, placed at the points  $x = 19.748$  and  $x = 49.252$  (i.e.,  $x' = \pm 14.752$ ), then let it be enquired what is the increase of thrust due to these weights?

For this special load it is necessary to prepare a table analogous to that of Article 11. The quantities  $I$  and  $\omega$  will remain the same;  $\mu_1$  will here be  $= P(a - x')$  between  $x' = a = 34.50$  metres and  $x' = 14.752$ ; then near the middle (for  $x' < 14.752$ ) they retain the constant value  $19.748 P$ . As to the longitudinal forces  $N_1$  developed by the two weights  $P$ , setting aside the thrust sought, they will be null in the middle part, and equal to  $P \sin \alpha$  (with a negative sign) between  $x' = 14.752$  and  $x' = 34.500$ . They may be omitted in the calculation of the thrust, which they influence very little. The result obtained is:

Abcisse $x'$ (from the middle) } =	0	2.966	5.928	8.882	11.825	14.752	17.660	20.546	23.405	26.233	29.028	31.784	34.500
$\frac{\mu_1}{P}$	= 19.748	19.748	19.748	19.748	19.748	19.748	16.840	13.954	11.095	8.267	5.472	2.716	0
$\frac{\mu_1 y}{IP}$	= 15545	15267	11319	9517	8328	6999	4788	3067	1789	902	477	105	0

Simpson's formula gives

$$\int \frac{\mu_1 y}{I} ds = 212375 \cdot \frac{\Delta s}{3} \cdot P.$$

Consequently the increase of thrust

$$= \frac{212375 P}{77077} = 2.755 P.$$

At the joint  $x = 19.748$  (where  $x' = 14.752$ ) found in the preceding article to be dangerous, the auxiliary load of the two symmetric weights  $P$  gives an additional positive moment  $= 19.748 P - 2.755 P \times 6.243 = 2.553 P$ , which will reduce the principal negative moment. This reduction, however, should be limited by the condition of not carrying the danger of rupture upon the symmetric section  $x = 49.252$ , in the case of non-expansion. Leaving aside the quantities  $N$ , which differ little, the equality of the two dangers, on the two symmetrical points, will take place for  $178230 - 2.553 P = 144740 + 2.553 P$ ; this equation gives approximately  $P = 6560$  kilogrammes. Then the total moment, at the point  $x = 19.748$ , becomes  $-178230 + 2.553 \times 6560 = -161480$ ; the force  $N$  becomes  $-294950 - 2.755 \times 6560 \cos \alpha = 312730$ , and the maximum stress

$$R = \frac{0.471 \times 161480}{0.017615} + \frac{312730}{0.10134} = 4317700 + 3086000 = 7403700,$$

instead of the value 7676100.

This advantage would be realized by lowering the roadway-plates near the portions ST of the longitudinal section (Fig. 3), in order to increase the thickness of the layers of ballast. But it is evident that the advantage would be very small; it is based on pure hypotheses of expansion, and would change to a disadvantage if it should happen, on the other hand, that a settlement of the abutment caused the arch slightly to expand. And, finally, the bracing, though light, would probably have more efficacy in diminishing the theoretical flexures.

14. If, therefore, the intervention of the auxiliary weights  $P$  be given up, and if no account be taken of the aid of the bracing, it only remains to determine the new strengthening plates required.

Now, by Article 12, the sections  $x = 13.954$  and  $x = 25.618$  are found to have a stress on them, the first of

$$R = \frac{0.528 \times 180830}{0.022617} + \frac{298270}{0.10362} = 7100000,$$

and the second of

$$R = \frac{0.434 \times 139820}{0.014722} + \frac{292250}{0.09986} = 7048500.$$

The joint  $x = 11.095$  gives  $R = 6486400$ . It will thus suffice to strengthen the arch between the abscissæ 13 metres and 26 metres, by the aid of additional plates of 5 millimètres, a thickness which ought to suffice, as may be verified by an approxi-

mate calculation. This thickness, however, would be weak when compared with that, 12 millimètres, of the first strengthening plate; and we will prefer to reduce the latter to 10 millimètres, and lengthen the 4 metres of the second portion to 7 millimètres thickness. New calculations of verification may be dispensed with, as the new addition further contributes slightly to cause the part to offer more resistance to expansion.

The weight of the arch is estimated at 210 tons, of which 130 are for the two arch ribs, and 80 for the other portions.

15. If it is desired to trace the curve of pressure, the operation will be easy when the calculations have been made of the moments  $\mu$  and the longitudinal forces  $N$ ; for the quotient  $\frac{\mu}{N}$  gives the ordinate or distance of the centre of pressure from the mean fibre; above it if  $\mu$  is positive, and below it towards the intrados if  $\mu$  is negative. For example, with the values of the table, Art. 12, without expansion, the curve of pressures dotted on Fig. 11 is obtained. This curve emerges slightly from the arch at M and M'. For example, at M, the distance of the centre of pressure is

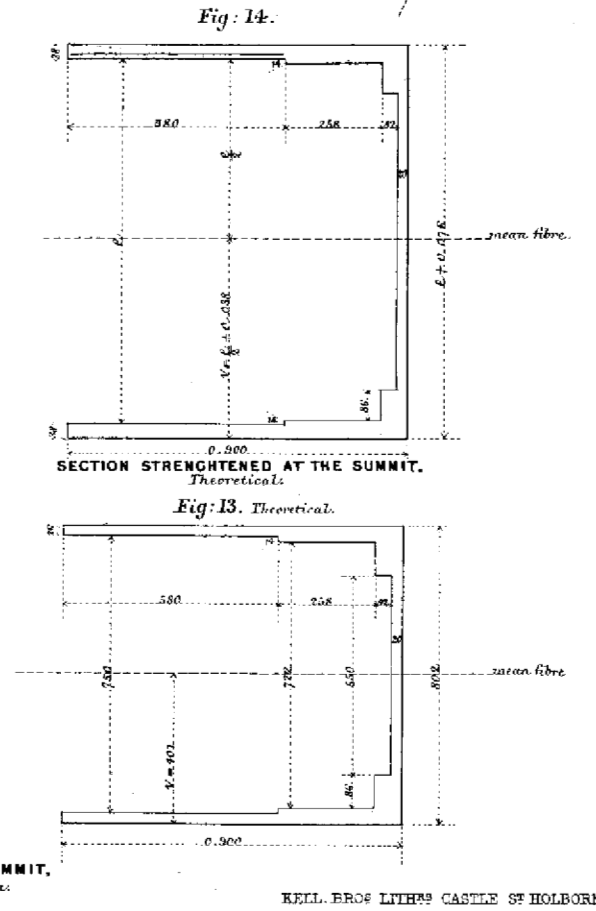
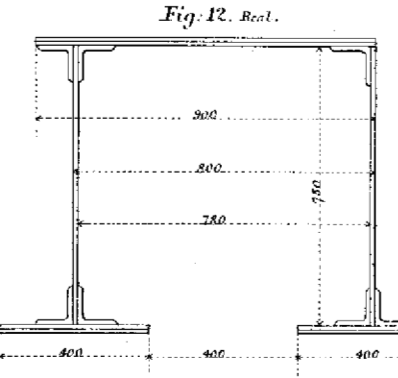
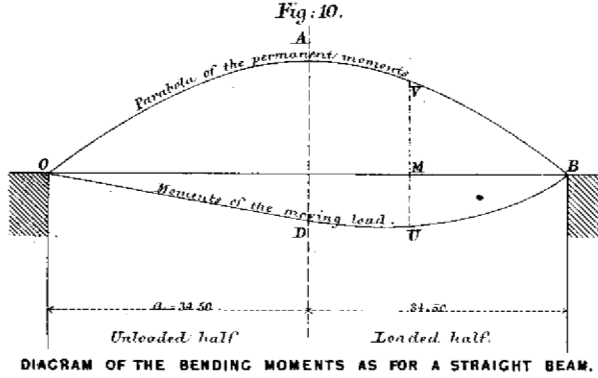
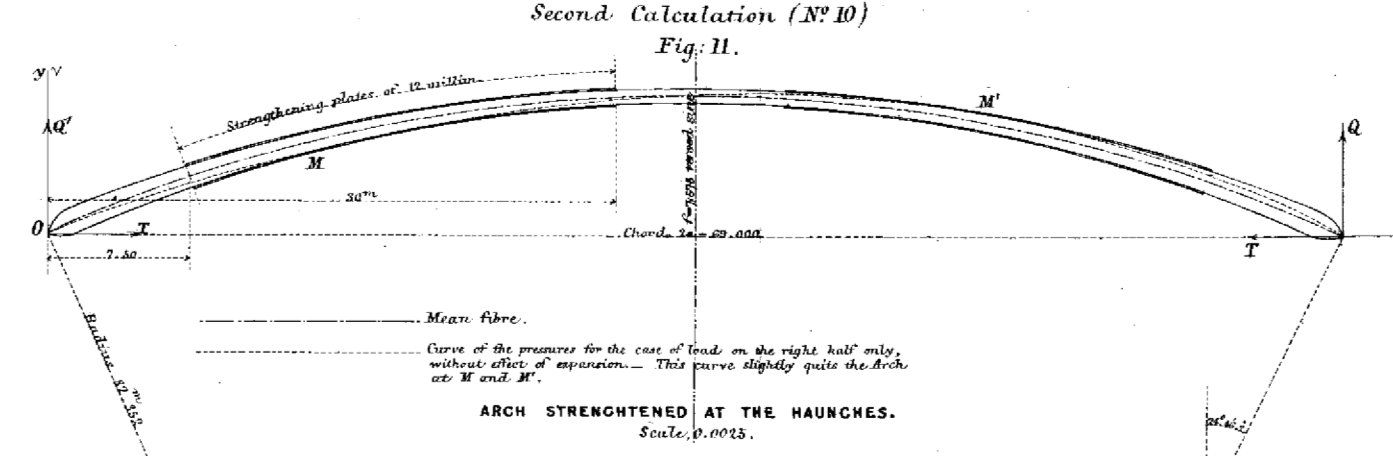
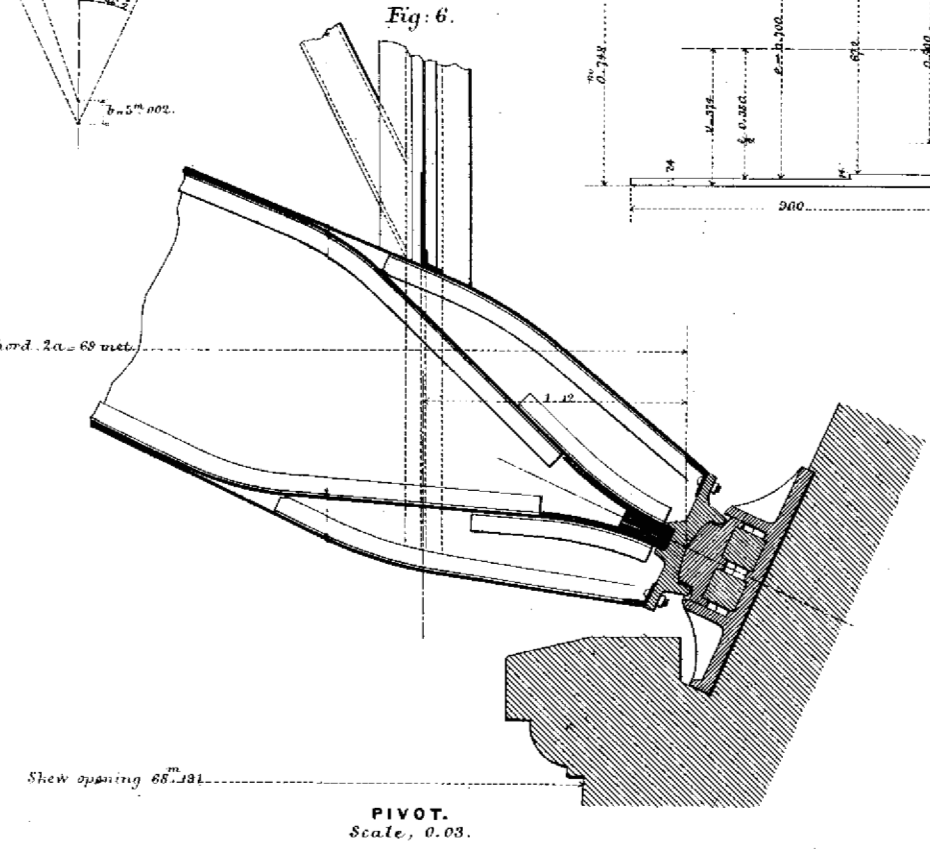
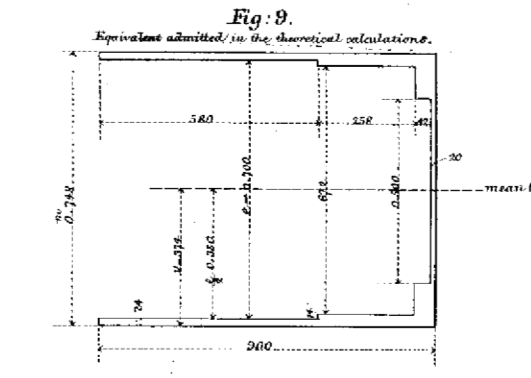
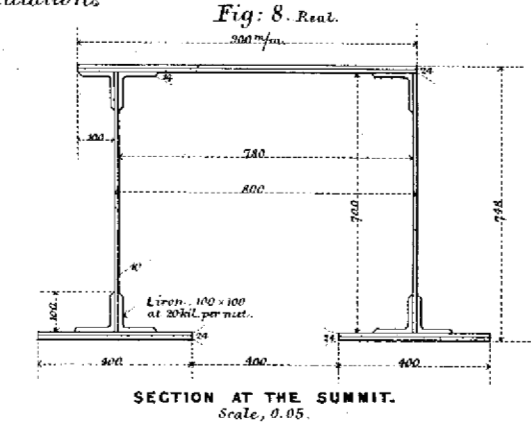
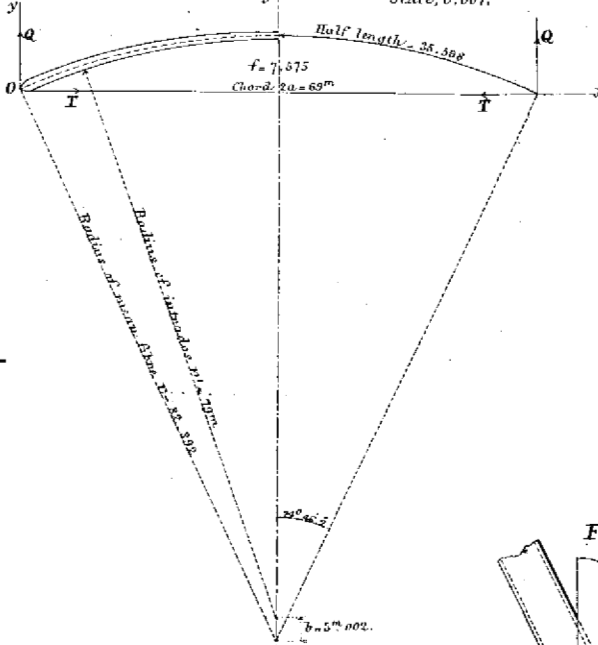
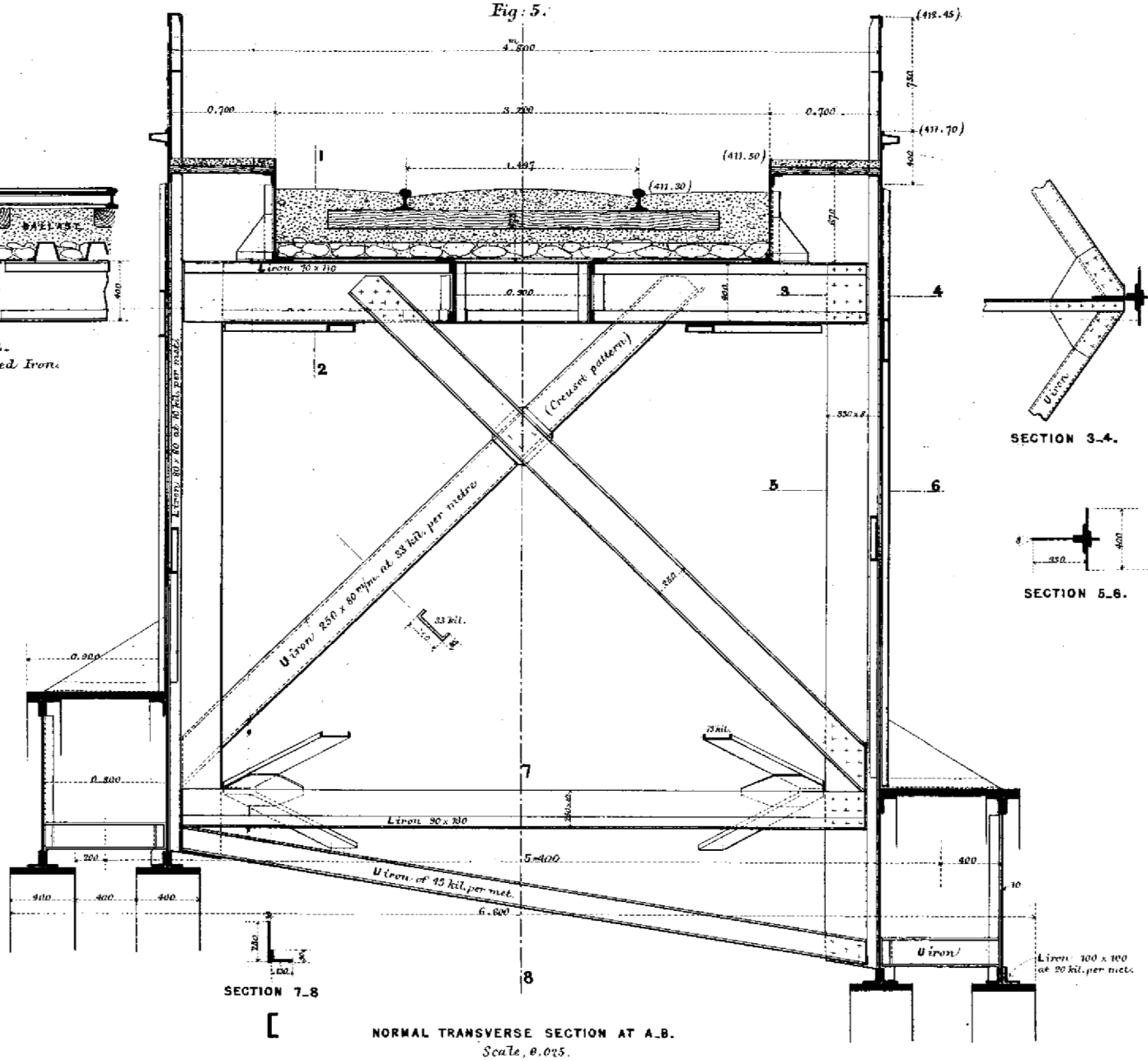
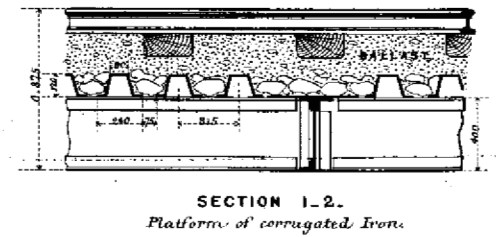
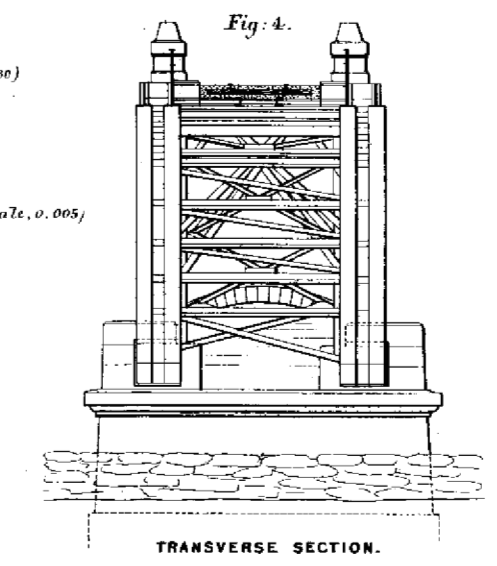
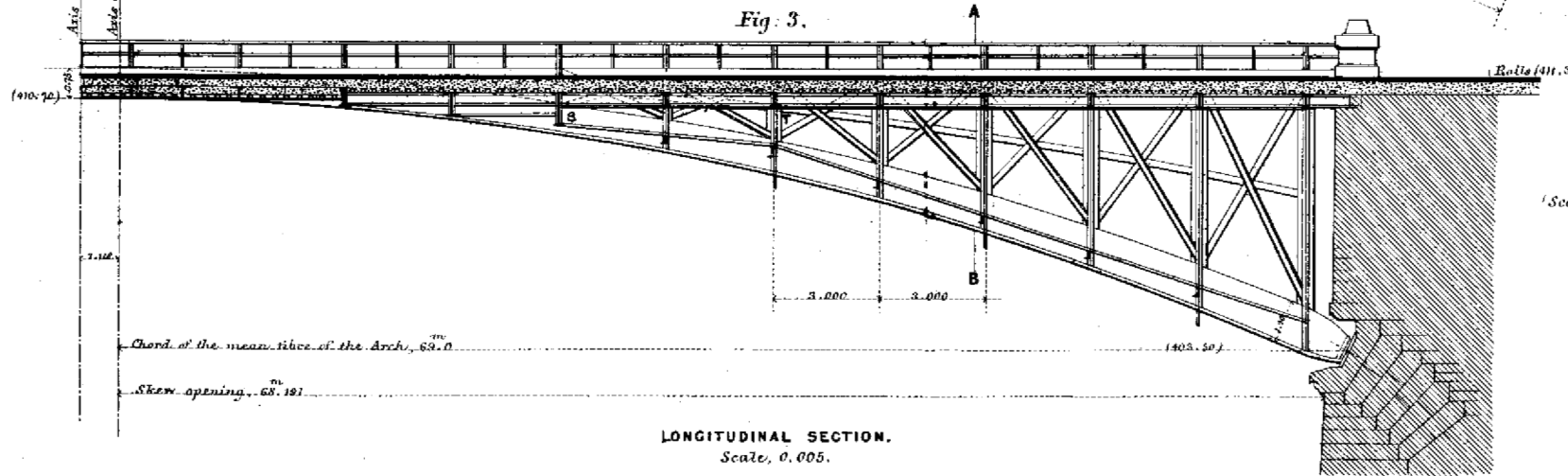
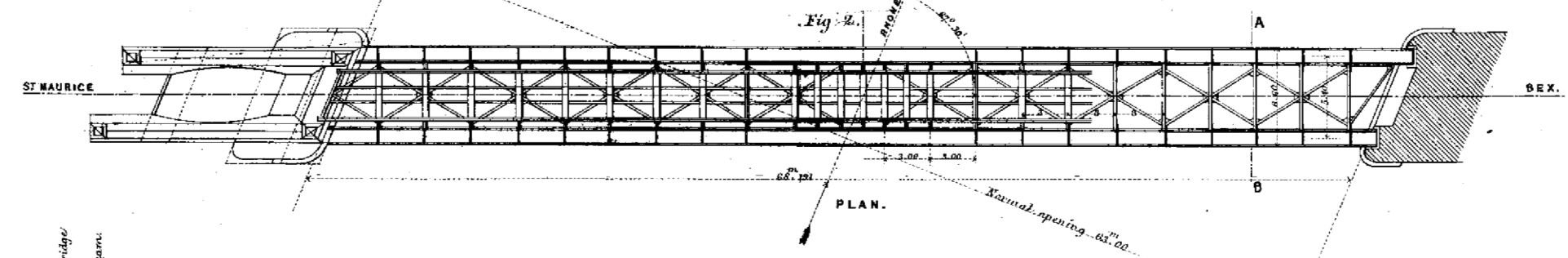
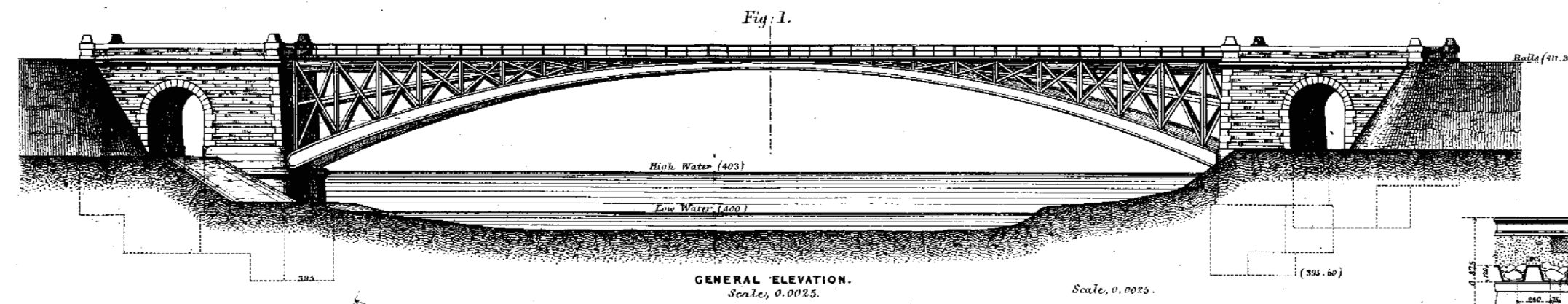
$$= \frac{146580}{289960} = 0.505, \text{ while the extreme fibre is only } 0.433 + 0.038$$

$= 0.471$  metres from the mean fibre.

It may have been remarked, in certain parts of the application which forms the object of this Memoir, that some uncertainty has been expressed as to the unknown effects of the wedging-up, the expansion, or the settlement, which enter into the calculation of the strains in metallic arches. The adoption of a pivot, or free articulation at the summit, would eliminate these unknown quantities. The reason which has prevented the Author from proposing this in the present design is the smallness of the breadth in proportion to the length or span, a circumstance which might lead to a fear of lateral derangements.

The inconvenience, not ascribable to theory, just pointed out in regard to rigid arches, may be compared to that which is found in continuous straight beams of several spans, in which an unequal settlement of the various piers may seriously interfere with the conditions of resistance. The great wrought-iron viaduct of the Pandèze, near Lausanne, may be cited as an example. One of the abutments, situated near some subterranean excavations, underwent a settlement of more than 0.10 metre, causing a considerable curvature of the beams over the adjoining pier. It is clear that in such a case the iron must be subject to much higher

DESIGN FOR AN IRON ARCH BRIDGE OVER THE RHONE.  
NEAR ST MAURICE, WEST SWITZERLAND RAILWAY.  
(Not executed.)



stress than was intended in the calculations of the design, and this trial was continued for a certain time before measures were taken for remedying the evil.

The calculations described in the foregoing Paper do not appear of an unreasonable length when large works are in question. For designs of less importance the labour may be much simplified by making the arch of uniform section. In this case, in fact, the tables of M. Bresse (*"Mécanique Appliquée,"* 1st Part, *"Sur la Résistance des Matériaux et Stabilité des Constructions"*) furnish immediately the value of the thrust, so that it only remains to calculate the bending moments and the longitudinal forces at the sections whose resistance has to be verified. If the verification is not satisfactory, either another and stronger arch, still of uniform section, may be tried, or reinforcements may be added to the haunches, dispensing with a second calculation, for which the tables referred to would not suffice.

In the use of the formula for  $T$  in Art. 5, equal divisions on the arch have been taken. If it is preferred to take equal distances on the horizontal, with the view of giving values of  $x$  in round numbers, so as to simplify somewhat the calculations, the formula, for a circular arc, will be written in this form, in a function of the variable  $x$ :

$$T = \frac{2 a \tau E + \int \frac{\mu_1 y}{I \cos \alpha} dx + \int \frac{N_1}{\omega} dx}{\int \frac{y^2}{I \cos \alpha} dx + \int \frac{\cos \alpha}{\omega} dx}.$$

The term  $\int \frac{N_1}{\omega} dx$  may be neglected without material inconvenience, as explained in Art. 5.

## II. *Further Remarks and Illustrations.*

1. This note forms a second Appendix to the Memoir which the Author has had the honour to address to the Institution on the subject of Arched Bridges. The first Appendix consisted of a detailed numerical application of the theory of rigid arches to a design for an iron bridge over the Rhone, near St. Maurice on the Western railway of Switzerland. Since that was written, the Author has prepared another design, with straight girders, for the same site. This new design has been found somewhat cheaper than the former one; it is estimated at the sum of 170,000 francs, while the arch bridge had been estimated at 220,000 francs. For this reason the girder-bridge has been adopted by the company, and is now in course of execution. The principal features of this bridge are represented in Plate 5<sup>B</sup>, which will perhaps offer some interest as a comparative study. The arch design given in the first Appendix, although not carried out, does not on that account lose the utility it was intended to have as an example of calculation and as furnishing the opportunity of adding some general observations on the resistance of curved arches.

The economy found in favour of the new design does not imply, as a general rule, that the arch is more costly than the straight girder, for it will be remarked on comparing the two designs—

1st. That the normal water-way, allowed for the river stream, has been reduced in the later design to 60 metres, while in the earlier one it was 63 metres.

2nd. That the small shore openings have been made by light iron platforms, instead of by masonry arches, which would have been better architecturally, but more costly.

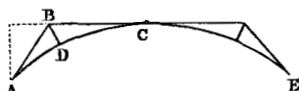
3rd. That the heavy loading of ballast, indicated in the first design with the object of lessening the vibrations, has disappeared in the final plan.

These various modifications might have been effected without abandoning the arch form, which, by its elegance and by the height of the free passage which it offered at the centre, would have been doubtless preferable for a navigable river, in the interior of a large city. But the Rhone, in this region, is not navigable; and, as at this remote locality questions of architecture were of very little weight, the economy of construction became of the first importance. The expense was found to be about the same in the two systems, when brought to parallel conditions; and hence it was considered preferable to adopt the plan which was most simple, in which the

calculations were most certain, and the execution the most convenient, and where the expansion might take place with full liberty.

2. If the new design had been made according to the same type as the first, the Author would have sought to sustain the haunches of the arch by an armature of pieces  $AB$ ,  $BC$ ,  $BD$  (Fig. 15) entering into the system of the spandrel filling. Such an armature would resist with advantage the settlement of the rising of the haunches, either loaded or unloaded.

Fig. 15.



It is true the rigorous calculation would become more complicated and more doubtful than for the simple arch, but also the structure, being less left to itself, and being thus only able to take an imperceptible play, would probably not depart materially from the conditions of uniform pressure on each section, which would admit the use of simple approximate calculation.

For example, the point  $D$  might be assimilated to a simple articulation, as is done, with no more exactness, in the lattices of straight beams. Or, in the first instance, the arch  $A D C E$  might be studied alone, without taking account of the armature, and making it sufficiently light to present a certain deficiency of resistance. Then, instead of covering this deficiency by strengthening plates added to the soles of the arch at the abutments (as proposed in the first Appendix), it should be done by the addition of the armature pieces. If, for example, the point  $D$  of the arch was found subject to a bending moment  $\mu$ , for which the moment of resistance of the section was insufficient,  $BC$  might be given such a resistance that the moment of this with respect to the point  $D$  would be capable alone of equilibrating  $\mu$ .

3. In the example treated in the first Appendix the load was regarded as transmitted from the platform to the arch by the intervention of the vertical risers, so that the weights retained their primitive direction. If it were a question of a bridge with radiating spandrel bars, analogous to the Victoria bridge, Pimlico, the arch would receive no more than the normal components of the load, these being decomposed between the direction of the radiating rods and the longitudinal bar of the roadway, which is subject to compression. If the load were further so arranged as to be distributed uniformly over the circumference of the arch, the strict figure of equilibrium would become circular, while it is parabolic in the case of vertical risers.

4. Some engineers propose to anchor the longitudinals to the

abutments, with the object of being able to consider the arch as composed of two isolated halves, disjoined at the summit, after the fashion of huge brackets or corbels, fixed (*encastrées*) in the masonry. This point of view, if it were followed out by really leaving a void space in the middle, would do away with the inconvenience of the expansion of the arch; but it would lead to a wasteful expenditure of metal, as the longitudinal would become as strong as the arch itself. Indeed, this is nearly the disposition adopted in certain swing-bridges in two halves, equilibrated at the back ends, but it is the dead weight alone which is thus balanced. When the bridge is shut, the two halves are rejoined together, in order to give the passing loads the benefit of the central butting joint. The abutment, therefore, cannot be dispensed with, except on the system of moveable pivots, if it is desired to render the expansion entirely free. Perhaps, in this case, there would be some advantage in substituting for the curved arch two triangular brackets (Fig. 16), whose points would be less slender. This is

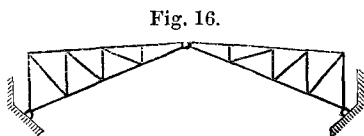


Fig. 16.

nearly the form of the swing-bridge at Brest. The calculations would be simple, and would follow the considerations of Arts. 4 and 5 of the original memoir.

Suspension-bridges, instead of presenting a multitude of joints, may be reduced to two links, or three joints to each span. The rigidity so often desired will then be obtained, still leaving liberty for expansion. It is stated that this is what has been done in the suspension-bridge at Frankfort. M. Bridel, Engineer-in-Chief of the Regulation of the Streams in the Jura (Switzerland), also proposes to adopt this disposition in a work on his district. It would seem that this is the true solution of the problem of rigid suspension-bridges; for each half-span may be latticed without inconvenience to the expansion, provided that a little play be left at A, Fig. 17,

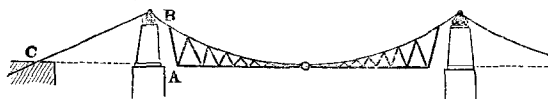


Fig. 17.

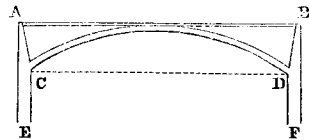
between the platform and the pier, in order to allow the expanded part to lower itself by moving on the central hinge. Rolling carriages B, at the summit of the piers, will keep the anchoring tie B C, stretched at all temperatures. In the case of several spans, the

oscillation of these moveable saddles might be limited by the addition of bracing ties.

5. *Moveable abutments; Bridges of several arches.*—It may be desirable to consider a peculiar cause of perturbation in the state of equilibrium of bridge arches. The formula of thrust which has been given and applied to an example is based on the hypothesis of immoveable supports. This hypothesis being admitted, there is nothing special to be said on bridges of several arches, every one of these being similar to an independent and isolated span. But this invariability of the separating piers is not rigorously true, as is proved by the fact, determined experimentally, that the load of one span may produce an elevation of a neighbouring arch.<sup>1</sup>

The abutments even of a single-arch bridge may yield slightly, at least in certain particular cases. For example, the ceiling arches of the Palais de l'Exposition de 1867, at Paris, had for abutments only light metal pillars, A E and B F, Fig. 18, maintained against upsetting by an upper tie rod A B.

Fig. 18.



In this combination, the chord C D does not remain invariable; it lengthens either by the stretching of the rod A B, which displaces the summits A and B, or by the flexure of the pillars, pressed laterally at c and d, and sustained only at their extremities. An analogous disposition occurs in the bridge of Szégédin, on the Theiss (Hungary), where the longitudinal member is utilized to hold together, by their summits, the metallic tubular piles.

6. The question of the mobility of the piles, under the influence of unequal loading of the contiguous arches, has been raised, and illustrated by a wood model presented to the Institution by Mr. E. A. Cowper.<sup>2</sup> It will be desirable to add here some theoretical considerations on the subject.

It is necessary to know, or at least to assume, by some hypothesis more or less plausible, the lateral resistance at the summit of the piers, *i.e.*, the quantity by which they are displaced under various intensities of resulting horizontal thrust. For piers which have no lateral resistance, as would be the case if the arches abutted on rolling saddles, the thrust of an arch would only be resisted by that of the adjoining arch; the point of support would be dis-

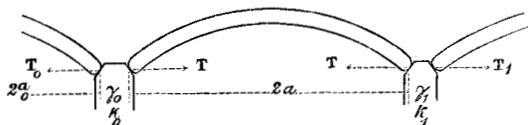
<sup>1</sup> *Vide* Minutes of Proceedings Inst. C.E., vol. xxvii., pp. 66 and 74.

<sup>2</sup> *Vide* *Ibid.*, p. 101.

placed until the opposing thrusts were equal, one arch resisting flattening, the other resisting a tendency to rise. This case would render inadmissible pivots at the summits, for then the arch that is crushed would no longer react with an increasing resistance; it would even tend to lose more and more a portion of its primitive resisting power, so that the other arch would thrust to falling. In the ordinary case, where the pier offers a certain degree of resistance, it will cease to move so soon as this resistance, added to that of the unloaded arch, has acquired a value equal to the thrust of the preponderating arch. This resists flattening under less favourable conditions than if the support was immovable, and the contiguous span resists the former one by its resistance to rising, if it has no central pivot.

Consider a loaded arch (Fig. 19), of which the primitive

Fig. 19.



chord  $2a$  has augmented by  $\gamma_0 + \gamma_1$ , by virtue of the accumulated displacement of the two piers. The left pier having moved  $\gamma_0$ , exerts, in consequence, a horizontal reaction  $k_0$ , connected with  $\gamma_0$  by a relation which is either given, or may be estimated as best we may; for example, by assimilating the pier to an upright cantilever, which bends a quantity  $\gamma_0$  under the lateral stress  $k_0$ . Similarly, the pile on the right side displaced  $\gamma_1$ , reacts by a resistance  $k_1$ , which co-operates with the thrust  $T_1$  of the following arch. The equilibrium of the piers furnishes the conditions:

$$T = T_0 + k_0 = T_1 + k_1 \dots [1].$$

Further, the theory of the deformation of the elastic curved piece, instead of being based on the supposed invariability of the chord, will express on the contrary that this has increased by  $\gamma_0 + \gamma_1$ , a quantity which is a function of known form,  $f(k_0, k_1)$ , of the developed lateral reactions  $k_0, k_1$ . Thus, with the same notations as already employed in Arts. 2, 5, and 15 of the first Appendix:—

$$\left. \begin{aligned}
 T &= \frac{2\tau a - f(k_0, k_1) + \int_0^{2a} \frac{\mu_1 y}{EI} \frac{ds}{dx} dx + \int_0^{2a} \frac{N_1}{E\omega} \frac{dx}{\omega}}{\int_0^{2a} \frac{y^2}{EI} \frac{ds}{dx} dx + \int_0^{2a} \frac{1}{E\omega} \frac{dx}{ds} dx}, \\
 \text{or,} \\
 T &= \frac{2\tau a E - Ef(k_0, k_1) + \int \frac{\mu_1 y}{I} ds + \int \frac{N_1 \cos \alpha}{\omega} ds}{\int \frac{y^2}{I} ds + \int \frac{\cos^2 \alpha}{\omega} ds}, \quad [2]. \\
 \text{or,} \\
 T &= \frac{2\tau a E - Ef(k_0, k_1) + \int \frac{\mu_1 y}{I \cos \alpha} dx + \int \frac{N_1}{\omega} dx}{\int \frac{y^2}{I \cos \alpha} dx + \int \frac{\cos \alpha}{\omega} dx}.
 \end{aligned} \right\}$$

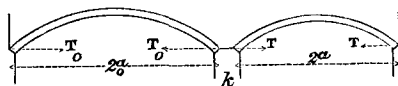
Any one of these equivalent formulæ [2], added to the double equation [1], connects the five unknown quantities  $T_0$ ,  $T$ ,  $T_1$ ,  $k_0$ ,  $k_1$ ; and by writing out the analogous equations for the different arches of the bridge, the determination of the problem may be arrived at.

Three particular cases will be considered.

**First Case.** *A single Arch, bearing against moveable Abutments.*—The arch rests, acquiring the increase of the chord which develops in the abutments the required resistance.  $T_0$  and  $T_1$  are made null, so that there only remain the three unknown quantities  $T$ ,  $k_0$  and  $k_1$ , determined by one equation [2], and the two equations [1]. It results from this that  $T = k_0 = k_1$ , so that  $T$  is obtained immediately by the formula [2], by substituting in it  $2\phi T$  for  $f(k_0 k_1)$ ; the function  $\phi T$  expressing the recoil of one abutment under a stress  $T$ .

**Second Case.** *Bridge of two Arches, with immovable Abutments, and one moveable Pier.*—With equal spans equally loaded, the pier does not move, each arch preserving its chord unaltered. But suppose the thrust  $T_0$  and  $T$  unequal (Fig. 20), then each span

Fig. 20.



gives rise to an application of the formula [2], and the displacement of the pier causes to one of the chords  $2a$  the loss of the

elongation gained by the other chord  $2a_0$ . Thus, by writing the abridged notations  $X, Y$ , instead of the sums of the integrals, the unknown thrusts  $T_0, T$ , and the supplementary reaction  $k$ , which is developed by the movement of the pier, will be given by the equations—

$$T_0 = \frac{2\tau a_0 - f(k) + X}{Y} \text{ and } T = \frac{2\tau a + f(k) + X'}{Y'},$$

joined to the equation of equilibrium of the pier  $T_0 = k + T$ . If  $k$  remained constantly null, *i.e.*, if the pier did not present any resistance to displacement, this displacement  $f(k)$  would only be arrested at the value resulting from the condition  $T_0 = T$ . In this case the notation  $f(k)$  would no longer be correct, and should be replaced by a single letter  $\gamma$ .

If the abutments themselves were moveable, they would introduce supplementary unknown quantities analogous to  $k$ ; but their conditions of equilibrium would also furnish two more equations.

*Third Case. Three Spans; Abutments immovable; two flexible Piers.*—The five unknown quantities,  $T_0, T, T_1, k_0, k_1$  (Fig. 19), will be determined by the equilibrium of the two piers,  $T_0 = \mp k_0 + T$ ,  $T = k_1 + T_1$ , and the deformations of the three arches:

$$T_0 = \frac{2\tau a_0 \mp f(k_0) + X}{Y}, \quad T = \frac{2\tau a \pm f(k_0) - F(k_1) + X'}{Y'},$$

$$\text{and } T_1 = \frac{2\tau a_1 + F(k_1) + X''}{Y''}.$$

The upper signs of the ambiguous terms suppose the left span driving the middle one, and this one driving the third; the lower signs suppose the intermediate arch alone to be loaded, and to thrust the two others. If the two piers are in similar conditions of construction, the functions  $f$  and  $F$  will have the same form.

If  $k_0$  and  $k_1$  remained null for every displacement, the real movements  $f(k_0), F(k_1)$ , (called in preference  $\gamma_0$  and  $\gamma_1$ ), would render equal the three thrusts  $T_0, T, T_1$ , and would be calculated by this condition.

Analogous considerations would apply to any number of spans.

*7. Numerical Example.*—Let us take the example of a bridge with three equal spans  $2a = 69$  metres, perfectly similar to that of the design treated of in the first Appendix; and let us first consider the case where the arch of the left bank is the only one loaded. With the sections of the second trial, the calculations of

Art. 11 in the said paper will furnish, immediately, without expansion—

$$\text{1st arch, loaded with 4650 kil. per metre, } T_0 = \frac{-E f(k_0) + 27932929136}{77077},$$

$$\text{2nd " " 2650 " " } T = \frac{E f(k_0) - E F(k_1) + 15918766067}{77077},$$

$$\text{3rd " " 3 " " } T_1 = \frac{E F(k_1) + 15918766067}{77077};$$

$$\text{also } T_0 = k_0 + T, \quad T = k_1 + T_1.$$

Admitting the same form of the functions  $f$  and  $F$ , it remains to assign this form. This is the delicate point, the uncertain element which renders the calculation but imperfectly applicable. The fault lies not in the theory, but in the variable, doubtful, or heterogeneous conditions of the foundation, and even of the substance of the piers. To complete the calculation, let us assume, for example, that the displacement of the pier, at the origin of the arches, provided that it remains very small, shall be proportional to the excess  $k$  of thrust, and equal to 1 millimètre per 5000 kilograms, for the portion of the pier acted on by the arch. This condition is expressed by replacing  $f(k_0)$  and  $F(k_1)$  respectively by 0.0000002  $k_0$  and 0.0000002  $k_1$ . This being done, let  $k_0$  and  $k_1$  be eliminated, and take  $E = 14,000,000,000$ , then these three equations are obtained :

$$79877 T_0 - 2800 T = 27932929136,$$

$$82677 T - 2800 (T_0 + T_1) = 15918766067,$$

$$79877 T_1 - 2800 T = 15918766067;$$

from which we deduce  $T_0 = 357120$  kil.,  $T = 211640$   $T_1 = 206,710$ .

Thus, in consequence of the compressibility of the right abutment, the arch of the left bank, loaded alone, only exerts a thrust of 357 metric tons, instead of 362, due to the hypothesis of immoveable supports. As to the two following arches, unloaded, the last, which is slightly compressed, gives a thrust of 206.7 tons, instead of 206.5, and the middle one 211.6, instead of 206.5. The second pier moves scarcely a millimètre, while the first one moves 0.0000002 ( $T_0 - T$ ) = 0.029 metre.

8. Now suppose the same bridge, loaded only on the second half of the central arch. The thrust of this arch half loaded is the arithmetical mean between the thrusts unloaded and completely loaded, for the same state of the chord. Then—

$$T_0 = \frac{2800 k_0 + 15918766067}{77077},$$

$$T = \frac{21925847600 - 2800 (k_0 + k_1)}{77077},$$

$$T_1 = \frac{2800 k_1 + 15918766067}{77077},$$

and  $T = T_0 + k_0 = T_1 + k_1.$

From this is found—

$$k_0 = k_1; \text{ then } T_0 = T_1 = 209083 \text{ kil., and } T = 279360.$$

Thus the lateral opposing arches acquire an energy of thrust equal to 209 tons, instead of  $206\frac{1}{2}$  tons, which they exert on the immovable piers; and the thrust of the central arch, half-loaded, is restricted to 279 tons, instead of 284 tons. The movement of the supports of this intermediate span favours evidently the unloaded side, diminishing its upheaval, while it aggravates the sinking of the loaded half. If it is required to know to what extent this takes place, it will suffice to repeat the calculations of resistance similar to those of Art. 12 in the first Appendix.

For the point of the abscissa  $x = 46.325$  metres, this No. 12 gives a pressure of 6.78 kilograms per square millimetre. In order to know what the actual case would give, it is necessary first to calculate, at the point under consideration, the bending moment  $\mu$ , and the longitudinal force  $N$ , according to the formulæ No. 8 (first Appendix). It will be found that  $\mu = 173230$ ,  $N = 281880$ , and consequently the molecular stress

$$R = \frac{0.450 \times 173230}{15940} + \frac{281880}{100500} = 7.70 \text{ kil. per square millimetre.}$$

It would seem then that the extension of the theory of elastic arches to the case of abutments susceptible of yielding under the load is not difficult. The uncertainty which exists as to the mode of resistance is attributable, not so much to the processes of the theoretical solution, as to the doubtful nature of the data of the problem; to the initial compression of the arches, and to the degree of mobility of the piers.

9. In the example which has thus been treated, the displacement which well-constructed piers may suffer has probably been exaggerated. Such at least would appear from the following considerations on the deflections by the effect of the load.

The sinking at the summit of a flat circular arch, with a uniform

section, by virtue of a load  $p'$  per horizontal running-metre, and under a linear expansion  $\tau$ , is valued approximately by

$$\frac{25 p' r^2 f}{2 E (8 \omega f^2 + 15 I)} \left( 1 + 0.012 \frac{\omega f^4}{I a^2} \right) \pm 1.56 \tau r,$$

$r$  being the radius of the arch,  $f$  its rise or versed sine,  $\omega$  the area of the section, and  $I$  its moment of inertia.

An arch entirely free, submitted to heat, would increase its chord by  $2 \tau a$  and its rise by  $\tau f$ ; but the chord being retained at its initial value  $2 a$ , the rise increases considerably, and M. Bresso estimates it at  $1.56 \tau r$  for an arch sufficiently flat. Now, by analogy, it may be said that if the chord sustains an increase  $2 \tau a$  by the fact of the recession of the piers, the summit of the arc will be depressed by  $1.56 \tau r$ .

In the example already considered,  $r = 82.352$  metres,  $f = 7.575$  metres, and  $a = 34.50$  metres. If  $\omega$  and  $I$  preserved at all points the minimum values at the summit  $\omega = 0.07742$  and  $I = 0.009623$ , the depression would be  $0.0000016 p' \pm 128 \tau$ . If, on the contrary, the section were retained constant with the maximum values immediately above the springing, i.e.,  $\omega = 0.1103$  and  $I = 0.030947$ , the depression would be reduced to  $0.000001 p' \pm 128 \tau$ . And for the variable section of our example, it may be assumed that the result will be intermediate between these two values. Thus, in the case of Art. 7 above, the first span would be depressed by a quantity comprised between  $0.0032$  and  $0.0020$ , by virtue of the load  $p' = 2000$  kilograms, and of another quantity  $128 \frac{0.029}{69} = 0.053$  metre, by the fact of the elongation of the chord. The second span, not loaded, but having the chord shortened by  $0.028$  by the mobility of the piers, would undergo an elevation at the centre of  $0.052$  metre.

Now, an elevation so considerable, by the sole effect of the load, has never, it is believed, been observed. On the Victoria bridge, Pimlico, the tests have only shown an elevation of  $4$  millimètres.<sup>1</sup>

<sup>1</sup> *Vide* Minutes of Proceedings Inst. C.E., vol. xxvii., p. 66. The bridge of Szegédin, with iron piers, did not give greater elevations than  $5$  or  $6$  millimètres according to the following information supplied by the "Nouveau Portefeuille de l'Ingénieur des Chemins de Fer," par A. Perdonnet et C. Polonceau. Texte, p. 366, 8vo., Paris, 1866.

"Lors des épreuves du pont, on a réalisé le cas le plus défavorable à la stabilité des piles, en chargeant chaque travée de  $8000$  kilog. par mètre courant, toutes les autres travées étant libres, et l'on a remarqué les lois suivantes.

"Toutes les piles fléchissent à la hauteur des naissances, en s'écartant de la travée chargée ;

It is therefore probable that the movements of the piers are, in general, insignificant, and need hardly be considered in the face of the much more important effects of expansion. It is seen, really,<sup>1</sup> that the expansion by heat has produced an alteration of level amounting to as much as  $1\frac{1}{2}$  inch, or 0.038 metre, twelve times as much as the elevation caused simply by the load on an adjoining span. It may be added, however, that the unknown part played by the spandrels and the longitudinals forbids us from affecting much precision in considerations of this delicate nature.

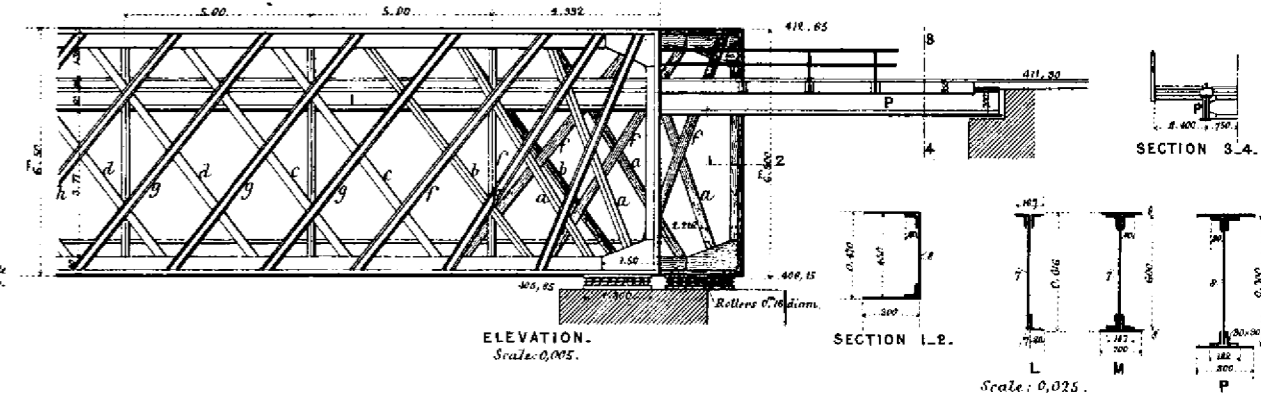
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“Les deux piles adjacentes à la travée chargée fléchissent, en moyenne, de 4 millimètres; les deux piles situées à la distance d’une travée fléchissent de  $1\frac{1}{2}$  millimètres: ces flèches diminuent rapidement quand on s’éloigne de la travée chargée; elles sont sensibles encore, quoiqu’on ne puisse les mesurer d’une extrémité à l’autre du pont.

“Les dépressions au sommet des travées, qui n’étaient que de 12 millimètres lorsque tout le pont était chargé, atteignaient 30 millimètres pour la travée chargée isolément, ce qui s’explique par l’augmentation de la corde; les deux travées adjacentes se relevaient de 5 à 6 millimètres, les suivantes de 2 millimètres au plus; plus loin, aucun mouvement n’a été observé.”

<sup>1</sup> *Vide Minutes of Proceedings Inst. C.E., vol. xxvii., p. 74.*

*Fig: 4.*



ELEVATION.  
Scale: 0,005.

Scale: 0,025.

Fig. 5.

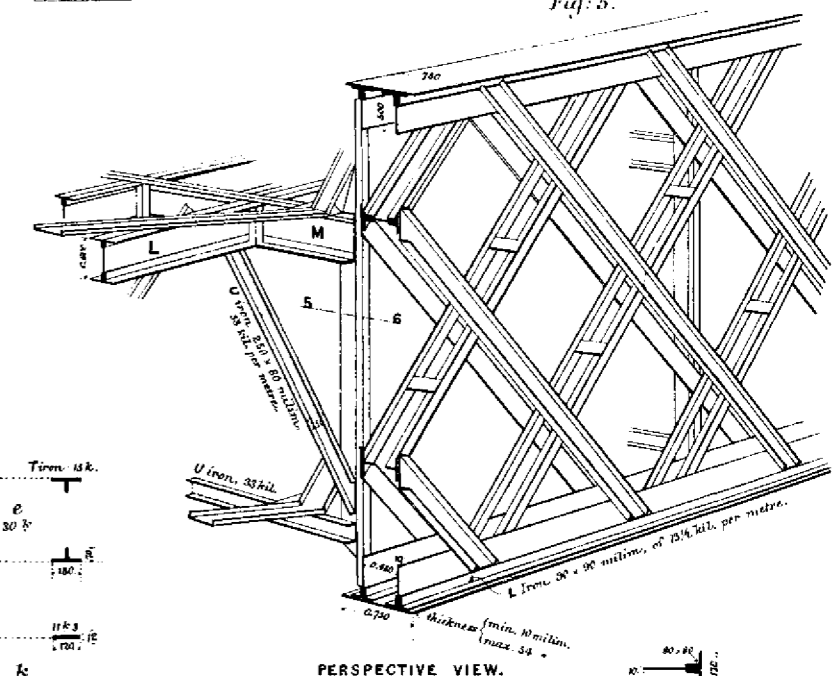
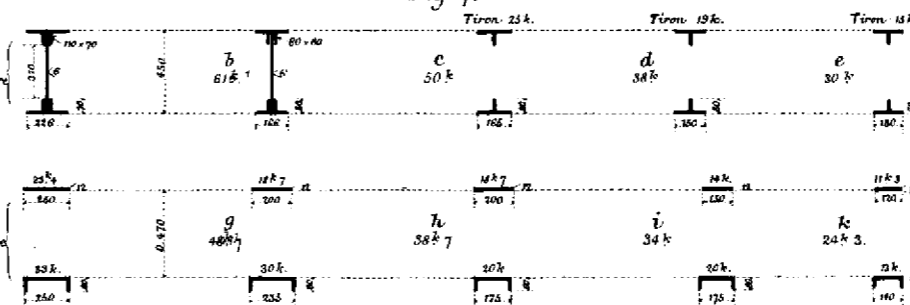


Fig:7.

SECTIONS OF THE BARS  
Scale : 0,0255

Compression: 6  
Tension: 5



COMPOSITION OF THE TOP AND BOTTOM MEMBERS OF THE GIRDER

PERSPECTIVE VIEW.

SECTION 5-6.