ON UNIFORM AND NON-UNIFORM CONVERGENCE AND DIVERGENCE OF A SERIES OF CONTINUOUS FUNCTIONS AND THE DISTINCTION OF RIGHT AND LEFT*

By W. H. YOUNG, Sc.D., F.R.S.

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1. Writers on uniform and non-uniform convergence have for the most part contented themselves hitherto with determining the mode of distribution of the points of uniform and non-uniform convergence, without occupying themselves with the various types of non-uniform convergence at a point that may occur, and how far such types are to be regarded as normal or exceptional. In particular, the question whether the character of the non-uniform convergence may be different on the left and on the right has been barely mooted, and the distribution of points at which this is the case has not been discussed at all. For some time the idea of a point of uniform convergence itself was only imperfectly grasped, uniform convergence being thought of as something pertaining to an interval alone; it was not realised that a series could be uniformly convergent at a point, without being uniformly convergent in any interval containing the point; in other words, that a point of uniform convergence may be a limiting point on both sides of points of non-uniform convergence, and this even though all the functions concerned are continuous. Still less was it realised that, when the functions whose sum is considered are discontinuous functions, a point of uniform convergence may be absolutely isolated.†

In the present paper I take as fundamental the definition‡ of uniform convergence at a point I have already employed in previous papers, one which is now coming into general use.§

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* A brief account of the results of this paper was communicated to the British Association at Leicester on Monday, August 5th.

† W. H. Young, "Points of Uniform Convergence . . .," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, pp. 358–360.

‡ W. H. Young, "On non-Uniform Convergence . . . ," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 90.

§ I am not sure who was the first to actually formulate the definition. In Osgood's original paper in the *American Journal*, Vol. XIX., and in Schoenflies's account of it in his *Bericht*, I can

Inevitable as this definition appears, it has several disadvantages:—

(1) It involves the remainder function $R_n(x)$, and therefore the possibly unknown sum, or limiting function $f(x)$, which may be discontinuous even when the functions $f_n(x)$ are all continuous.

(2) It is an "ϵ-definition."

(3) It affords us no means of *classifying* points of non-uniform convergence.

The close connection, or analogy, between the uniform convergence of a series and the continuity of a function is well known. As regards the continuity of a function we may avoid the ϵ-method by introducing* the *associated upper* and *lower right-hand and left-hand limiting*

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find no such formulation. In Townsend's dissertation, "Doppellimes," Göttingen, 1900, I find the following statements:—"Also gleichmässige Convergenz bezieht sich auf das ganze Intervall, dagegen hat der einfache Limes

$$\underset{y=y_0}{\mathrm{Lt}} f(x_0, y) = f(x_0, y_0),$$

nur mit einem einzelnen Punkte des Intervalls zu thun," (p. 29); again, on p. 65, "So weit ist diese Bedingung ganz dasselbe wie obige Bedingung für die Stetigkeit von $f(x)$. Sie unterscheidet sich aber davon, indem bei gleichmässiger Convergenz das für jedes n definierte Intervall eine untere Grenze grösser als Null haben muss, wenn n über alle Grenzen wächst. Dagegen braucht bei der Stetigkeitsbedingung . . . , dieses Intervall keine untere Grenze grösser als Null zu haben." These statements suggest that even in 1900 in Göttingen a precise formulation of uniform convergence at a point had not been made. Of course points of uniform convergence play an important part, none the less, in Townsend's dissertation, and Arzelà's paper, which preceded it ("Sulle Serie di Funzioni," Parte 1, *Mem. di Bologna*, Serie 5, Vol. VIII., pp. 131–186, 1899). The phraseology adopted for these points by Townsend is "points at which the series converges regularly," regular convergence being expressed in terms of, and thought of in connection with, the behaviour of the allied functions of two variables introduced by Du Bois Reymond ("Ueber die Integration der Reihen," *Sitzungsbericht d. Berliner Akademie*, 1886, pp. 359–371), and used also by Arzelà. This definition only applies in the case considered by Townsend, which is also that considered in the present paper, when the functions to be summed are continuous.

In Osgood's paper, although the statement is made that "Uniform and non-uniform convergence are conceptions that relate to the behaviour of the variable function throughout an interval," p. 166, the points in question are used, and called ζ-points; their definition as given by Osgood is clear and precise though somewhat complicated (pp. 163–165).

In Hobson's paper ("On Non-Uniform Convergence . . . ," *Proc. London Math. Soc.*, Vol. XXXIV., pp. 254 *et seq.*, Jan., 1902), uniform convergence is defined only for an interval. In his recent book *Functions of a Real Variable* (1907), the definition for a point is implicitly given.

Van Vleck, in a recent paper, "A Proof of some Theorems on Pointwise Discontinuous Functions," *Trans. Amer. Math. Soc.*, Vol. VIII., April, 1907, p. 204, footnote, gives the definition in the form adopted by myself; and I understand that Hilbert has now done the same for some time in his lectures.

* W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," Aug., 1907, *Quarterly Journal of Math.*, Vol. XXXIX., pp. 67–83.

functions ϕ_R, ϕ_L, ψ_R, ψ_L. We are thus naturally led to devise similar functions in the case of convergence of series.

This idea is in embryo in Osgood's paper, above cited, in which he introduces the word "peak" and the term "indices at a point." Osgood deals, however, only with those series of continuous functions whose sum is a continuous function, and his indices have relation to the remainder function; in the case when the sum is zero, his indices are closely connected with the χ and π of the present paper. The case which usually arises in practice is that where the functions of which the sum is considered are continuous, but it is an unnecessary and an undesirable restriction to suppose that their sum is continuous. In what follows I begin by defining four functions π_L, π_R, χ_L, χ_R, which I call *peak* and *chasm functions*, and which are strictly analogous to the associated functions ϕ_L, ϕ_R, ψ_L, ψ_R, above referred to, and I shew that, in the case considered, *the equality of these functions at a point is the necessary and sufficient condition for uniform convergence at the point*, so that we may, if we please, give this equality as a new definition of uniform convergence at a point. This definition is precisely analogous to that referred to of continuity at a point; it is not, however, intended to replace the other, which is indeed fundamental in character, and cannot easily be dispensed with when the functions to be summed are not continuous. It has, however, the advantage of being free from the objections pointed out as inherent in the other. I take occasion to shew how this second definition may be used directly to obtain the well known distribution of the points of non-uniform convergence, viz., that they form an ordinary outer limiting set of the first category.*

The main use, however, that I make of the new definition is to examine the character of the types of non-uniform convergence that may arise, more especially with respect to the distinction of right and left. I shew that *only at a countable number of points, which may, however, be dense everywhere, can the behaviour of a series, as regards non-uniform convergence, be different on the right and on the left of a point.* In particular the points at which the series is uniformly convergent on one side and non-uniformly convergent on the other side of a point, can be, at most, countably infinite.

This discussion would appear to complete the qualitative study of

* Proofs of this result have been given by Townsend and Hobson, *loc. cit.* Both of these authors deduce it from considerations connected with the theory of functions of two variables. I myself have given a proof of a more general result, including this as a special case, by a method more on the lines of Osgood's classical paper. The proof in the text thus constitutes a fourth proof of the result.

points of non-uniform convergence, in the case when the functions to be summed are all continuous. It should be noted, however, that though many of the theorems obtained apply as they stand, or with slight modifications, to the more general case, the information afforded by the investigations of the paper is not adequate for a complete analysis of the facts of non-uniform convergence except when the functions to be summed are continuous.

It remains to be added that throughout the paper all that has been required from the series of functions is that it should have a definite sum. The possibility of that sum having the value ∞ ($+\infty$, or $-\infty$, as the case may be), is not excluded. In other words, divergence is allowed, provided it be not an oscillatory divergence. Moreover the functions whose sum, or limit, is considered, are not necessarily bounded functions; in other words, their continuity is of the generalised character, in which infinite values are allowed. Thus the new proof, above referred to, relating to the distribution of points of non-uniform convergence, is really wider in scope than any of its predecessors, and the result obtained is of a more general character, having reference moreover to points of "uniform divergence," as we may conveniently call them, as well as to points of uniform convergence.

2. It will be convenient to repeat here the definitions, already referred to, of the associated left- and right-hand upper and lower limiting functions ϕ_L, ϕ_R, ψ_L, ψ_R of a discontinuous function.

Let P be any internal point of a segment throughout which a function $f(x)$ is defined. Take any interval with P as right-hand end-point, then $f(x)$ has, for the points *internal* to this interval, an upper limit; as this interval diminishes, this upper limit cannot increase, and therefore has a limit, which is, at the same time, its lower limit; denote this limit by $\phi_L(P)$.

We thus get for every point P of the segment a function $\phi_L(x)$, or shortly ϕ_L, which may be called *the upper left-hand limiting function of* $f(x)$.

Similarly, changing left into right, we define a function ϕ_R, *the upper right-hand limiting function of* f. Further, interchanging the words "upper" and "lower," "increase" and "decrease," in the definition, we define corresponding *lower limiting functions* which we shall denote by ψ_L and ψ_R.

If at each point P we choose that one of the two upper limiting functions which is not less than the other, we get a new function, which may be called the (modified) *upper limiting function*, and be denoted by ϕ. Similarly we define the (modified) *lower limiting function* ψ, by taking that one of the two lower limiting functions which is not less than the other.

3. Let $f_1, f_2, \ldots$ be a series of functions, having a definite limiting function f; in other words, at any point P, we have

$$\underset{n=\infty}{\mathrm{Lt}}\, f_n(P) = f(P). \tag{1}$$

We now define auxiliary numbers at P precisely analogous to the upper and lower left- and right-hand limiting functions of a discontinuous function; these we shall call *the left- and right-hand peak and chasm functions*, and denote them by π_L, π_R, χ_L, χ_R.

We take an interval PQ with P as right-hand end-point, and denote the upper limit of f_n for points x inside this open interval by $M_{n,Q}$. Then for all such points x,

$$f_n(x) \leqslant M_{n,Q}, \tag{2}$$

while either there is such a point x at which

$$f_n(x) = M_{n,Q},$$

or else there is at least a sequence of points passing along which $f_n(x)$ has the limit $M_{n,Q}$.

These numbers $M_{n,Q}$ for the successive integers n, may be conveniently plotted off on the axis of y; they form a countably infinite set* which has therefore at least one limiting point, and, in any case will have a first derived set which may be countable or of potency c. Let the highest of these derived points, or corresponding numbers, be denoted by M_Q, and, though this will be less used in the sequel, the lowest of these derived points, or numbers, be denoted by M'_Q.

Now, if Q_1 and Q_2 are two positions of Q of which Q_2 lies between P and Q_1, it follows from the definitions that

$$M_{n,Q_2} \leqslant M_{n,Q_1}.$$

Hence any limiting point of the points M_{n,Q_2}, for successive values of n, will determine one or more limiting points of the points M_{n,Q_1}, none of which will lie below the former limiting point. It follows that

$$M_{Q_2} \leqslant M_{Q_1},$$

and also

$$M'_{Q_2} \leqslant M'_{Q_1}.$$

Therefore, if we make the point Q approach P as limit, moving along a

* Counting two of the points as distinct whether or no they coincide in position. If there is only a finite number of positions, the highest derived point is, of course, the highest of the points which is repeated an infinite number of times.

sequence, or continuously, or in any manner, the quantities M_Q will have a definite limit, which will be at the same time their lower limit, and which will be denoted by $\pi_L(P)$, and, for all positions of P, be called the *left-hand peak function.* Similarly, the quantities M'_Q have a definite limit, which is their lower limit, and may be denoted by $\pi'_L(P)$.

Here, of course, $$\pi'_L \leqslant \pi_L.$$

Working in like manner on the right of P we obtain the corresponding right-hand quantities, denoted by the subscript R instead of L. Again, interchanging "upper" and "lower," we get the *left- and right-hand chasm functions* χ_L and χ_R, as well as the quantities χ'_L and χ'_R, where

$$\chi'_L \geqslant \chi_L \quad \text{and} \quad \chi'_R \geqslant \chi_R.$$

4. THEOREM 1.—

$$\chi_L(P) \leqslant \chi'_L(P) \leqslant \psi_L(P) \leqslant \phi_L(P) \leqslant \pi'_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, for the right-hand functions.)

For, if x be any point inside the interval PQ, we had as equation (2) of § 3,

$$f_n(x) \leqslant M_{n,Q}. \tag{2}$$

Making n increase indefinitely, $f_n(x)$ has the single limit $f(x)$, which, therefore, cannot lie above any limit of the quantities $M_{n,Q}$; therefore

$$f(x) \leqslant M'_Q.$$

Now letting x describe a suitable sequence with P as limit, we obtain for $f(x)$ the limit $\phi_L(P)$, therefore, Q being still fixed,

$$\phi_L(P) \leqslant M'_Q.$$

Since this is true for all positions of Q,

$$\phi_L(P) \leqslant \pi'_L(P).$$

Similarly $$\psi_L(P) \geqslant \chi'_L(P),$$

which proves the theorem.

THEOREM 2.—*If the functions f_n are continuous at P,**

$$\chi_L(P) \leqslant \chi'_L(P) \leqslant f(P) \leqslant \pi'_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, since $f_n(x)$ is continuous at P, it has the definite limit $f_n(P)$, so

* It follows from Theorem 1, using the results of my paper quoted on p. 30 that this inequality holds whatever the f's are, except at most at a countable set of points.

that, by equation (2) of § 3 or § 4,

$$f_n(P) \leqslant M_{n,Q}.$$

Since this is true for all values of n, the single limit $f(P)$ approached by the left-hand side of the inequality cannot be higher than any limit approached by the right-hand side; therefore

$$f_n(P) \leqslant M'_Q.$$

Since this holds for all positions of Q,

$$f(P) \leqslant \pi'_L(P).$$

Similarly $$f(P) \geqslant \chi'_L(P),$$

which proves the theorem.

5. From Theorem 1 it follows that, if the peak and chasm functions are equal at P, they are equal to the upper and lower associated limiting functions, and therefore, with the possible exception of a countable set of points,* they are all equal to $f(P)$; if, however, the f_n's are continuous, there are no such exceptional points, by Theorem 2.

In other words, *at a point where*

$$\chi(P) = \pi(P) = f(P),$$

f is continuous; if the f_n's are continuous, we may drop the $f(P)$ from this equation.

We see, however, from the enunciations of Theorems 1 and 2 that, though this condition is sufficient for continuity it is not necessary; it is still sufficient if

$$\chi'(P) = \pi'(P) = f(P),$$

but this condition also is not necessary; anyone familiar with examples of non-uniform convergence will recognise that this is so, and that thereby hangs a tale.

6. The definition of uniform convergence at a point P is as follows:—

* *Loc. cit.*, p. 30.

"The series of functions $f_1, f_2, \ldots$ is said to 'converge uniformly to the function f at the point P,' if, given any positive quantity ϵ, however small, an interval d can be described, having P as internal point, so that, for all points x within the interval d,

$$|f(x)-f_n(x)| < \epsilon$$

for all values of $n \geqslant m$, where m is an integer, independent of x, which can always be determined.

"Similarly we may define the expressions right-handed and left-handed uniform convergence* at P; in this case the interval d will have P as end-point."

This definition may easily be adapted so as to give what we may call "uniform divergence" at P, when the value $f(P)$ is infinite with determinate sign; we merely have instead of the above inequality,

$$f_n(x) > A \quad \text{or} \quad f_n(x) < -A,$$

according as the sign of f is $+$ or $-$, A being, like ϵ, preassigned, and being, of course, not "however small" but "however large."

7. The connection of the peak and chasm functions with these definitions is determined by the following theorems.

THEOREM 3.—*If the f_n's are continuous functions, and P a point at which*

$$\chi_L(P) = \pi_L(P),$$

the series $f_1, f_2, \ldots$ is uniformly convergent or divergent, at P on the left.

CASE 1.—Let $\pi_L(P)$ be finite, then, since π_L is the limit of the quantities M_Q, we can choose Q_1 so that

$$M_{Q_1} \leqslant \pi_L(P)+\epsilon.$$

Therefore, since M_{Q_1} is the highest possible limit approached by M_{n, Q_1}, we can determine k_1, so that, for all values of $n \geqslant k_1$,

$$M_{n, Q_1} \leqslant M_{Q_1}+e \leqslant \pi_L(P)+2e.$$

Since M_{n, Q_1} is the upper limit of $f_n(x)$ in the interval PQ_1, it follows that, for all points x of this interval and all values of $n \geqslant k_1$,

$$f_n(x) \leqslant \pi_L(P)+2e.$$

Similarly, working with χ_L instead of π_L, we can find a point Q_2 inside

* It should be noted that in the definitions of right-handed and left-handed uniform convergence or divergence of a series of continuous functions it is immaterial whether the interval with P as end-point be supposed to include P or not, provided we know that the series is actually convergent, or divergent, at P as the case may be.

the interval PQ, and an integer $k_2 \geqslant k_1$, such that for all points x inside the interval PQ_2 and all integers $n \geqslant k_2$,

$$\chi_L(P) - 2e \leqslant f_n(x),$$

for these points, and for these values of n, the relation then holds,

$$\chi_L(P) - 2e \leqslant f_n(x) \leqslant \pi_L(P) + 2e.$$

But, by hypothesis, $\chi_L(P) = \pi_L(P)$;

therefore $\pi_L(P) - 2e \leqslant f_n(x) \leqslant \pi_L(P) + 2e.$

Proceeding to the limit with n,

$$\pi_L(P) - 2e \leqslant f(x) \leqslant \pi_L(P) + 2e.$$

From the last two relations it follows that in PQ_2,

$$|f(x) - f_n(x)| \leqslant 4e \quad (n \geqslant k_2),$$

which is the condition for uniform convergence, $f(P)$ being, by Theorem 2, certainly finite.

CASE 2.—Again, if the peak and chasm functions are not finite at P take the case when they are both $= -\infty$. Then we can, in like manner, choose Q so that

$$M_Q < -A,$$

A being any preassigned quantity, and then determine k_1 so that, for all values of $n \geqslant k_1$,

$$M_{n,Q} < -A;$$

and therefore, for all points x in (P, Q),

$$f_n(x) < -A,$$

which is the condition for uniform divergence, when the value of f is, as by Theorem 2 it must be here, $-\infty$.

Similarly, when the value of χ and π is $+\infty$, we get the condition for uniform divergence.

Thus the theorem is true in all cases.

COR.—*When the f_n's are not continuous, the same reasoning shews that, with at most a countably infinite set of exceptions,* the series is uniformly convergent or divergent at every point where the peak and chasm functions are equal.*

Where $f = \chi = \pi$, the series is certainly uniformly convergent or divergent.

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* Theorem 1 being now used instead of Theorem 2, so that in a countably infinite set of cases f may differ from the peak and chasm functions.

THEOREM 4.—*If $f_1, f_2, \ldots$ be continuous functions, and the series is uniformly convergent at P on the left* (*or uniformly divergent*), *then*

$$\chi_L(P) = \pi_L(P) = f(P).$$

CASE 1.—If the series is uniformly convergent at P on the left, assigning e, we can find an interval PQ, with P as right-hand end-point, and an integer k, such that, for all points x in PQ, and all values of $n \geqslant k$,

$$|f(x)-f_n(x)| < \epsilon. \tag{1}$$

Now, by the definition of $M_{n,Q}$ (§ 1), we can find at least one point, say x_n, inside the open interval PQ, such that

$$0 \leqslant M_{n,Q}-f_n(x_n) < e. \tag{2}$$

Therefore
$$|f(x_n)-M_{n,Q}| < 2e. \tag{3}$$

Now, if we let n describe a suitable sequence of constantly increasing integers, $n_1, n_2, \ldots, M_{n,Q}$ will have the limit M_Q.

The countably infinite set of points $x_{n_1}, x_{n_2}, \ldots$ has at least one limiting point, and may have more. Let x_Q be one of these limiting points.

Then we can pick out a subsequence of the set

$$n_1, n_2, \ldots,$$

say
$$n_1', n_2', \ldots,$$

such that
$$x_{n'_1}, x_{n'_2}, \ldots$$

is a sequence having x_Q as limit. In this case, by the definitions of ϕ and ψ, any limit approached by $f(x)$ will lie between

$$\psi(x_Q) \quad \text{and} \quad \phi(x_Q),$$

both inclusive.

But, by (3), any such limit differs from M_Q by at most $2e$, hence

$$\psi(x_Q)-2e \leqslant M_Q \leqslant \phi(x_Q)+2e. \tag{4}$$

Now, let us make e describe a sequence with zero as limit, and at the same time, as we may, let us take each interval PQ less than half the length of the preceding one; then Q will describe a sequence having P as limit, and the same will be true of x_Q, which always lies in the interval PQ.

M_Q has then, as we saw in § 1, the definite limit $\pi_L(P)$, while $\phi(x_Q)+2e$ may, or may not, have a definite limit, but any limit assumed by it is $\leqslant \phi_L(P)$, by Theorem 1 of my paper quoted on p. 30 of the present memoir. Hence

$$\pi_L(P) \leqslant \phi_L(P).$$

Similarly any limit assumed by $\psi(x_Q)-2e \geqslant \psi_L(P)$, so that

$$\psi_L(P) \leqslant \pi_L(P).$$

Hence
$$\psi_L(P) \leqslant \pi_L(P) \leqslant \phi_L(P). \tag{5}$$

But, the f_n's being continuous and the series uniformly convergent at P, f is continuous at P, so that

$$\psi_L(P) = \phi_L(P) = f(P);$$

therefore, by (5),
$$\pi_L(P) = f(P).$$

Similarly
$$\chi_L(P) = f(P),$$

which proves the theorem in this case.

CASE 2.—Next, let the series be uniformly divergent on the left at P, and first let

$$f(P) = -\infty.$$

Then, by the condition for uniform divergence (§ 6), there is an interval (P, Q) and an integer k, such that, for all points x in that interval, and all integers $n > k$,

$$f_n(x) < -A;$$

and therefore
$$M_{n,Q} \leqslant -A;$$

and therefore
$$M_Q \leqslant -A.$$

Hence, $\pi(P)$ being the lower limit of the quantities M_Q,

$$\pi(P) \leqslant -A.$$

Since this is true for all values of A,

$$\pi(P) = -\infty;$$

and therefore also
$$\chi(P) = -\infty,$$

which proves the theorem in this case.

Similarly, if $f(P) = +\infty$, the result follows, using $\chi(P)$ instead of $\pi(P)$, $>$ for $<$, and replacing M_Q, $M_{n,Q}$ by the corresponding quantities connected with χ.

THEOREM 5.—*If the f_n's are not continuous throughout the interval having P as right-hand end-point, but the series is uniformly convergent (or divergent) at P on the left,*

$$\phi_L(P) = \pi_L(P) \quad and \quad \psi_L(P) = \chi_L(P).$$

CASE 1.—First, let the series be uniformly convergent at P on the left. The proof then proceeds as in Case 1 of the preceding theorem down to equation (5).

But, by Theorem 1, $\pi_L(P) \geqslant \phi_L(P)$;

therefore, by (5), $\pi_L(P) = \phi_L(P)$.

Similarly, $\chi_L(P) = \psi_L(P)$,

which proves the theorem in this case.

CASE 2.—Secondly, let the series be uniformly divergent at P on the left. It will be found on examination that the proof given of this case in Theorem 4 holds without modification.

COR.—*At a point of uniform divergence, or at a point of uniform convergence which is also a point of continuity of $f(x)$ on the left,*

$$\chi_L(P) = \pi_L(P) = f(P).$$

8. Making then our usual assumption that the f_n's are all continuous functions, it follows, by the results of the preceding article, that we may take as the definition of uniform convergence at the point P the equality of all the four peak and chasm functions.

Similarly, we can define uniform convergence on the right or left alone by the equality of the corresponding one-sided peak and chasm functions. It should be noticed that from this point of view uniform divergence is merely a special case of uniform convergence.

9. When the series is non-uniformly convergent at the point P, there are several special cases of interest.

(1) Let f be not less than π at P; then the function f *is upper semi-continuous at P.*

For, by Theorem 1, in this case,

$$\phi(P) \leqslant f(P).$$

(2) Similarly, if f be not greater than χ at P, the function f *is lower semi-continuous at P.*

These conditions are again, as in the case of continuity, sufficient but not necessary; in particular, Theorem 1 shews that it is still sufficient for upper semi-continuity if $f(P) \geqslant \pi'(P)$, and for lower semi-continuity if $f(P) \leqslant \chi'(P)$.

(3) Let $$\chi'(P) = \pi'(P),$$

then the function f is continuous at P, by Theorems 1 and 2.

[It is clear from Theorem 1 that $\chi'(P) \not> \pi'(P)$.]

This condition is sufficient but not necessary.

10. In Case 3, an argument precisely the same as that used in proving Theorem 3 may be used, only that, as M'_Q is not the highest but the lowest possible limit approached by $M_{n, Q}$, we cannot determine k_1 so that for *all* values of $n \geqslant k_1$, $$M_{n, Q} < M'_Q + e;$$

but we can insure that this is true for all values of n *belonging to a certain sequence* of constantly increasing integers

$$n_1, n_2, \dots .$$

The same is then true for the inequality

$$L_{n, Q} > L'_Q - e,$$

for all values of $n \geqslant k_2$, belonging to a certain sequence

$$n'_1, n'_2, \dots,$$

$L_{n, Q}$ denoting the lower limit of $f_n(x)$ in the interval PQ, and L'_Q the highest limit of $L_{n, Q}$.

If these two sequences are the same, or if they have a common sequence, the argument used in the proof of Theorem 3 applies; hence it follows that the condition for uniform convergence at P is satisfied, with the restriction of n to values belonging to a certain sequence. When this is the case the convergence at P is said to be *simply uniform.*

This mode of looking at simple uniform convergence shews that there is no essential difference between simple uniform convergence at one point and uniform convergence at that point. If the corresponding sequence is

$$n_1, n_2, \dots,$$

we only have to take, instead of the given series of functions

$$f_1, f_2, \dots,$$

the sub-series $$f_{n_1}, f_{n_2}, \dots$$

having the same limiting function f, and the convergence at the point under consideration will be uniform.

11. If the given series is simply uniformly convergent at every point of an interval or of a closed set, it may be, but this is not necessarily the case, that the same sequence will serve at every point of the interval, or closed set. By the extended Heine-Borel theorem, it then follows that a finite number of the intervals which are defined at each point by the simple uniform convergence will suffice to contain every point of the interval, or closed set. To these intervals correspond a finite number of integers k; thus, if k' denote the sum of these, it will be true that, for all points x of the interval, or closed set, and for all integers $n \geqslant k'$ belonging to the sequence in question,

$$|f(x)-f_n(x)| < A.$$

A being the quantity with which we started to determine the intervals and integers.

Conversely, if this is true, there is a sequence of integers which will serve at every point.

In this case it is customary to say that the given series is *simply uniformly convergent throughout the interval or closed set*, and it has been pointed out,* by the same argument as that used above, that by properly picking out the functions f_n, we can reduce this to uniform convergence at every point of the interval, or closed set.

It is clear, however, that this is only a particular case of simple uniform convergence at every point of the interval, or closed set, and it by no means follows that in the general case of simple uniform convergence at every point of an interval, or closed set, we can so pick out the functions as to make the convergence uniform at all the points in question simultaneously.

12.† We now proceed to prove for the π's and χ's the same theorems as those obtained in the paper already quoted for the ϕ's and ψ's.

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* Arzelà, *loc. cit.*

† In this article the assumption that f_n is continuous need not be made.

THEOREM 6.*—*Any limit approached by* $\pi(x)$, $\pi_L(x)$, *or* $\pi_R(x)$ *as* x *approaches a point* P *as limit on the right* $\leqslant \pi_L(P)$, *and, as* x *approaches* P *as limit on the left* $\leqslant \pi_R(P)$.

If $\pi_L(P) = +\infty$, this is certainly the case; if not, we can find a finite quantity $A > \pi_L(P)$.

Then, by the definition of $\pi_L(P)$, we can find an interval d with P as right-hand end-point, such that, Q being any point of this interval,

$$M_Q < A.$$

Therefore, by the definition of M_Q, we can find an integer k, such that, for all integers $n \geqslant k$, $$M_{n,Q} < A.$$

Therefore, $M_{n,Q}$ being the upper limit of $f_n(x)$ in the interval (P, Q),

$$f_n(x) < A,$$

for all integers $n \geqslant k$, and all points of (P, Q).

Now, let P_1 and Q_1 be any two points internal to PQ, and let us consider the quantities M_{Q_1} and M_{n,Q_1} referred, not to the interval PQ_1 but to P_1Q_1. The preceding inequality (3) gives us then

$$M_{n,Q_1} < A.$$

We then have $$M_{Q_1} \leqslant A.$$

Now, let Q_1 move up to P_1 as limit, we get

$$\pi_L(P_1) \leqslant A, \quad \text{or} \quad \pi_R(P_1) \leqslant A,$$

according as Q_1 lay on the left or the right of P_1.

Since A is at our disposal, provided only it is $> \pi_L(P)$, this proves the first part of the theorem; similarly, working on the right, the second part follows.

COR.—π_L *is upper semi-continuous on the left and* π_R *on the right, while* π *is an upper semi-continuous function, and, as such, at most pointwise discontinuous.*

The proofs of the succeeding theorems in my paper quoted, depending, as they do, solely on the theorem correlative to the above theorem, can now be transferred verbatim, changing only the symbol ϕ into π. It is therefore only necessary to give the enunciations here.

THEOREM 7.—*At every point of continuity of* π,

$$\pi_L = \pi_R = \pi,$$

and both π_L *and* π_R *are continuous.*

COR.—π_L *and* π_R, *as well as* π, *are at most pointwise discontinuous.*

THEOREM 8.—*The only points at which both* π_L *and* π_R *are continuous are the points of continuity of* π.

* The correlative of Theorem 1, *loc. cit.*

THEOREM 9.*—*The points, if any, at which π_R differs from π_L, are countable.*

Similar results, interchanging the signs $>$ and $<$, hold, of course, for the χ's.

13. Thus we see that, except possibly at a countable set of points,

$$\pi_L = \pi_R = \pi \quad \text{and} \quad \chi_L = \chi_R = \chi,$$

while

$$\chi \leqslant \psi \leqslant f \leqslant \phi \leqslant \pi.$$

Thus *the distinction of right and left with respect to peaks and chasms exists at most at a countable set of points.*

When the functions f_n are continuous, it is, as we saw, the same to say that there is no distinction of right- and left-handed non-uniform convergence, except at most at a countable set of points.

14. THEOREM 10.—*At any point P where the peak and chasm functions are equal, both these functions are continuous; conversely, at any point where they are both continuous, provided the f_n's are continuous, the peak and chasm functions are equal.*

To prove the first part of the theorem, we only need Theorem 6. For, let P be a point at which

$$\chi(P) = \pi(P).$$

Since, as x approaches P as limit,

$$\chi(P) \leqslant \text{Lt}\, \chi(x) \leqslant \text{Lt}\, \pi(x) \leqslant \pi(P),$$

we must have the sign of equality throughout, which proves that both χ and π are continuous at P.

To prove the second part of the theorem, we proceed as follows.

Suppose, if possible, that there were a common point of continuity of the peak and chasm functions at which the peak and chasm functions were not equal, let this point be P, then

$$\pi(P) > \chi(P), \tag{1}$$

by Theorem 1.

By the sense of this equation $\pi(P)$ cannot be $-\infty$, nor $\chi(P) + \infty$; therefore we can find two numbers a and β, such that

$$\chi(P) < \beta, \quad a < \pi(P); \tag{2}$$

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* The more specialised results denoted by Theorem 5*a* and 5*b* *loc. cit.*, are not given here; they hold, of course, and the proofs only require the change of ϕ into π.

The correlative of Theorem 6, viz., *the points, if any, at which $f > \pi$ are countable*, follows, of course, at once from our Theorem 1, using the results of the former paper. This was already referred to in the footnote on p. 34.

further we can so choose these numbers that

$$\beta < \alpha. \tag{3}$$

Since P is a common point of continuity of the peak and chasm functions, we can find a whole interval (A, B), at every internal point x of which

$$\chi(x) < \beta, \qquad \alpha < \pi(x), \tag{4}$$

the point P being internal to this interval.

From the definition of the peak function, it now follows from (2) that we can find a point Q in (A, B), such that

$$\alpha < M_Q;$$

and therefore we can find a value n_1 of n, greater than any assigned integer, such that

$$\alpha < M_{n_1, Q}.$$

Since $M_{n_1, Q}$ is the upper limit of the values of $f_{n_1}(x)$ in the interval (P, Q), there is certainly a point of this interval where $f_{n_1}(x) > \alpha$; hence, since f_{n_1} is continuous, there is a whole interval (A_1, B_1) internal to (A, B), at every point x of which

$$\alpha < f_{n_1}(x), \tag{5}$$

while, of course, the relations (4) hold throughout the interval.

By precisely the same argument, using χ instead of π, and interchanging the signs $>$ and $<$, we shew that there is an interval (A_1', B_1') inside (A, B), such that at every point x of it

$$f_{n'_1}(x) < \beta, \tag{6}$$

n_1' being also an integer greater than the assigned integer.

We now take the interval (A_1, B_1) and shew, by the same argument, that there is inside it an interval (A_2', B_2') at every point of which

$$f_{n'_2}(x) < \beta,$$

n_2' being a certain integer greater than n_1.

Similarly, inside this we get an interval (A_3, B_3), at every point of which

$$\alpha < f_{n_3}(x).$$

Proceeding thus we get a countably infinite set of intervals, each lying inside the preceding,

$$(A_1, B_1),\ (A_2', B_2'),\ (A_3, B_3),\ (A_4', B_4'),\ \ldots,$$

and a corresponding series of constantly increasing integers

$$n_1,\ n_2',\ n_3,\ n_4',\ \ldots,$$

such that for the intervals and integers denoted by dashed letters we have relations of the form (6), and for the others of the form (5).

Now, there is at least one point internal to all these intervals and at this point we shall have both

$$f(x) = \underset{i=\infty}{\mathrm{Lt}}\, f_{n_i}(x) \geqslant a,$$

and, using (3),

$$f(x) = \underset{i=\infty}{\mathrm{Lt}}\, f_{n'_i}(x) \leqslant \beta < a,$$

which is impossible, $f(x)$ being, by hypothesis, determinate. Thus the supposition made at the beginning is untenable, which, by a *reductio ad absurdum*, proves the theorem.

COR. 1.—*The f_n's being continuous, the points at which*

$$\pi > \chi$$

*form a set of the first category; in other words, the points of non-uniform convergence and divergence form a set of the first category; this set is none other than the set of points at which one at least of the functions χ and π is discontinuous, and is therefore an ordinary outer limiting set.**

It should be noted that in an interval in which there are no points of uniform divergence, $\pi = \chi$ is a null function whose zeros are its points of continuity and are the points of uniform convergence of the series in that interval.

COR. 2.—*In an interval throughout which the series converges, the points at which*

$$\pi = +\infty$$

form a closed set nowhere dense; the same is true of the points at which

$$\chi = -\infty.$$

First, either of these sets must be closed because the peak function is upper semi-continuous and the chasm function is lower semi-continuous.† Hence, unless nowhere dense, either of these sets would fill up an interval, and therefore throughout that interval π would be greater than f or f greater than χ, that is, in either contingency π would be greater than χ throughout that interval, contrary to Cor. 1.

* That is, the outer limiting set of a series of closed sets.

† See my paper in the *Quarterly Journal* already cited.

15. THEOREM 11.—*The points of uniform divergence of a series of continuous functions form an inner limiting set.*

At a point of uniform divergence either

$$\chi = \pi = +\infty \quad \text{or} \quad \chi = \pi = -\infty.$$

Consider the first of these sets; since χ is a lower semi-continuous function, the points $\chi = +\infty$ form an inner limiting set.* But, if

$$\chi = +\infty, \qquad \pi = +\infty;$$

since

$$\pi \geqslant \chi.$$

Thus the first set is an inner limiting set; similarly, the second set is an inner limiting set, and therefore the sum of the two sets is an inner limiting set.

COR.—*If a series of continuous functions diverges at a set of points dense everywhere in an interval, it diverges uniformly at points which form a set of the second category.*

16. In § 13 it was proved that the points at which there can be a distinction of right and left with respect to non-uniform convergence are at most countable. It remains to shew, by an example, that this is the utmost that can be said. In the one we proceed to give, it will be noticed that all four peak and chasm functions are different from one another and from f at every point of a countable set which is everywhere dense in the segment (0, 1).

Ex.—Let us construct the continuous function f_n as follows:—

Divide the segment (0, 1) of the x-axis into ten parts. At the first point of division, let the value assigned to f_1 be any chosen quantity A_0; at the second point of division ·2, let the value be any chosen quantity B_0; at the last point of division ·9, let the value be any chosen quantity A_1; and at ·8, any chosen quantity B_1. At the remaining points of division the values assigned are as follows:—At ·5 the value of f_1 is ·5, and this value will remain fixed for every f_n; at ·3 it is ·8; at ·4 it is ·2; at ·6 it is ·4; at ·7 it is ·6.

We then draw a polygonal line starting from the point (0, 0) and passing in order from left to right through the points whose ordinates are the values of f_1 at the points of division, and ending at the point

* *Loc. cit.* in the preceding note.

(1, 1); in the figure, the first two and the last two straight lines are not drawn; the values of the A's and B's may be such that they do not lie entirely within the square. The equation to this polygonal line is to be

$$y = f_1(x).$$

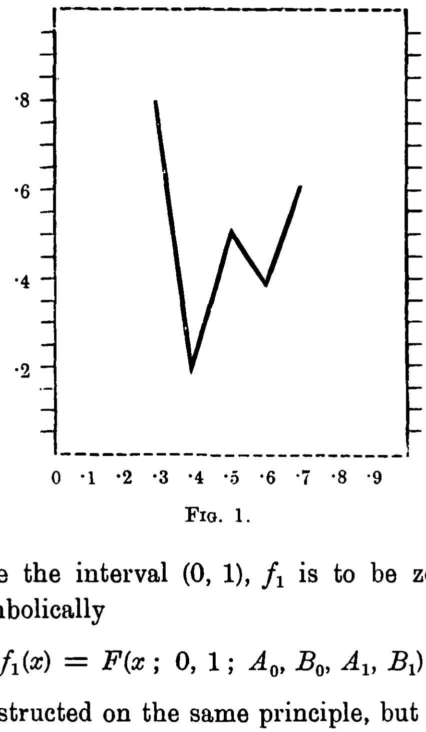

FIG. 1.

Everywhere outside the interval (0, 1), f_1 is to be zero. We may conveniently write symbolically

$$f_1(x) = F(x\,;\ 0,\,1\,;\ A_0,\,B_0,\,A_1,\,B_1),$$

and a function constructed on the same principle, but to a different scale, the interval being (a, b) instead of (0, 1), and the left-hand bottom corner of the square the point (a, a) instead of (0, 0), will be denoted by

$$F(x\,;\ a,\,b\,;\ A_a,\,B_a,\,A_b,\,B_b).$$

The function f_2 is now the sum of two such functions, corresponding to the two halves of the interval (0, 1), which determined f_1. The quantities A_0, B_0, A_1, B_1 are those used in constructing f_1; the quantities $A_{\cdot 5}$, $B_{\cdot 5}$ corresponding to each interval are determined by the values assigned at the four points of division nearest to ·5 in constructing f_1; we have, in fact,

$$f_2 = F(0,\,\cdot 5\,;\ A_0,\,B_0,\,\cdot 2,\,\cdot 8) + F(\cdot 5,\,1\,;\ \cdot 4,\,\cdot 6,\,A_1,\,B_1).$$

In each half interval we now repeat the construction, and so get f_3 defined, and then f_4, and so all the f_n's in succession. The value of f_n at the point $\frac{1}{2}$ will then always be $\frac{1}{2}$, and the corresponding peaks and chasms will always be ·8 and ·2 on the left and ·6 and ·4 on the right. Thus

f will be $\frac{1}{2}$ at the point $\frac{1}{2}$, π_L will be ·8, χ_L will be ·2, π_R will be ·6, and χ_R will be ·4.

The mode of construction shews that the same will be true with different numbers at all the rational points whose denominators are powers of 2. The difference between any two of the five functions at any point is the same as the same difference at any other point with the same power of 2 as denominator, while this difference is halved when the power of 2 is increased by 1; the value of the limiting function f at any such point x will then itself be x.

If x be any number other than one of the fractions whose denominators are powers of 10, it follows therefore that the values of the peak and chasm functions corresponding to the end-points of the interval (a, b), to which x is internal at each successive stage, will differ by less and less and have the same limiting values. In other words, the segments of the broken line f_n corresponding to this interval will become shorter and shorter without limit. Taking, therefore, any little interval with our point x as end-point, the upper and lower limits of f_n will more and more nearly coincide the greater n is, and this will be still more so the smaller the little interval; there is no positive lower limit to this limit, hence the peak and chasm functions at x coincide at x. Thus x is a point of uniform convergence of the series, and therefore a point of continuity of the limiting function f; it follows that throughout the interval (0, 1) considered

$$f(x) = x.$$

17. We now proceed to give examples showing that the results obtained in Cor. 2 of § 14 are the utmost that can be stated with regard to the points in question. We content ourselves with giving a series for which π_L and π_R are both $+\infty$ and χ_L and χ_R both $-\infty$ at any arbitrarily selected closed set of points nowhere dense. It is at once obvious how the construction may be modified so as to give a series for which χ, for example, is everywhere finite and π everywhere infinite at the points of such a set.

Ex. 2.—We only need to shew how to construct a function uniformly convergent throughout an open interval and having at both end-points

$$\chi = -\infty, \qquad \pi = +\infty,$$

f_n having at the end-points any the same assigned value which has a definite limit as n increases indefinitely.

If we can do this, we can do it for every black interval of any given closed set; we can then ascribe to f_n the above selected value at every remaining point of the closed set. We thus have a series of continuous functions defined for the whole segment whose peak and chasm functions are respectively $+\infty$ and $-\infty$ at every end-point of a black interval, and therefore at all the limiting points of the set of these end-points, that is, at all the points of the given closed set. In this example, f will be usually discontinuous, but it is easy to arrange it otherwise if we please.

Take the interval (0, 1) and let

$$f_n = \frac{n^2x+n}{1+n(nx+1)^2}$$

from 0 to $\frac{1}{2}$ both inclusive, while from $\frac{1}{2}$ to 1 we use the same expression, changing x into $(1-x)$.

Here f is zero except at the extremities of the interval where it is unity.

Or, again, put

$$f_n = \frac{n^{\frac{3}{2}}x+n^{\frac{1}{4}}}{1+(n^{\frac{3}{2}}x+n^{\frac{1}{4}})^2}$$

from 0 to $\frac{1}{2}$ both inclusive, while from $\frac{1}{2}$ to 1 we use again the same expression, changing x to $(1-x)$.

Here f is always zero, and therefore continuous. At the ends of the interval the peak and chasm functions are in both cases $+\infty$ and $-\infty$ respectively.

18. Can $\pi = +\infty$ throughout an interval?

If so, the series must, as we have seen, be divergent at a set of points dense everywhere in that interval. With this *proviso* it is easy to construct such a series. Such a series, for example, is that given by Borel,*

$$\frac{A_1}{r_1}+\frac{A_2}{r_2}+\ldots,$$

where $r_1, r_2, \ldots$ are the distances, taken positively, of the point x from the rational points of the segment (0, 1) arranged in countable order, and the A's are constants suitably chosen, viz., in such a way that the series whose general term is A_n converges.

Notice further that it follows from Theorem 11 that this series diverges uniformly at a set of the second category.

19. For the sake of completeness we shew how to form a series in which the peak and chasm functions as well as the limiting functions are

* Borel, *Comptes Rendus*, Vol. CXVIII., p. 540; Schoenflies's *Bericht*, p. 243. See also my paper in the *Messenger of Mathematics*, Vol. XXXVII., "On a New Proof of a Theorem of Baire's."

all infinite, say $= +\infty$, throughout an interval, so that the series diverges uniformly at every point of the interval. We have merely to take any series which converges uniformly to zero positively, say

$$u_0+u_1+\ldots,$$

the required series is $$v_0+v_1+\ldots,$$

where $$v_n = \frac{1}{u_0+u_1+\ldots+u_{n+1}} - \frac{1}{u_0+u_1+\ldots+u_n}.$$

We content ourselves with the examples already given; it is not difficult to construct others illustrating more fully the various theorems above given; for instance, the fundamental one with respect to the distinction of right and left. Thus we could arrange that the series diverges uniformly on the right and non-uniformly on the left at every point of a countable set nowhere dense.

20. All these theorems still hold *mutatis mutandis* if we consider the behaviour of the series at points of a perfect set of points contained in the segment in which the functions are defined, instead of at all the points of that segment itself. We shall have, of course, to replace "uniform convergence at a point" by "uniform convergence at a point with respect to the perfect set." The new peak and chasm functions relative to the perfect set will be obtained by letting the variable point Q describe not the continuum, but the perfect set in the neighbourhood of a point P of the set.

Thus, for instance, the new and extended form of Theorem 10 gives us the following statement:—

The f_n's being continuous, the series is only pointwise non-uniformly convergent (including divergent) with respect to every perfect set.

Using the result of § 5 relative to a perfect set, this includes Baire's theorem that *the limiting function f of a series of continuous functions f_n is only pointwise discontinuous with respect to every perfect set.*