"The Mathematical Magazine, a Journal of Elementary Mathematics," edited and published by Artemas Martin, M.A., Vol. i., No. 2, April, 1882.

"The Pasturage Problem," by A. Martin, M.A. (Extract from above magazine.)

"Educational Times," June, 1882.

"Proceedings of Royal Society," Vol. xxiii., No. 219.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome v., Nov.–Dec., 1881.

"Bulletin de la Société Mathématique de France," Tome x., No. 3.

"Atti della R. Accademia dei Lincei," Vol. vi., Fasc. 12° (in duplicate).

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück v., No. 5, 1882.

"Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder i Aaret 1881," Kjobenhavn, No. 3, 1881: presented by Sir J. Lubbock, F.R.S.

"Washington Observations for 1877"—Appendix ii., "The Multiple Star Σ 748," by Ed. S. Holden; Washington, 1881.

"Annali di Matematica Pura ed Applicata," Serie iiª, Tomo x°, Fasc. 4° (Maggio, 1882); Milano.

"American Journal of Mathematics," Vol. iv., No. 3; Baltimore (Sept. 1881), 1882.

On the Extension of certain Theories relating to Plane Cubics to Curves of any Deficiency. By Arthur Buchheim, B.A.

[*Read June 8th*, 1882.]

The object of the following paper is the extension, by the use of Abelian functions, of certain theories which, in the case of plane cubics, are immediate consequences of the representation of the coordinates of a point of the curve as elliptic functions of a parameter. The theories considered are: (i.) The theory of Steiner's Polygons, and (ii.) Prof. Sylvester's theory of Derived Points. The curves considered are curves of order n and deficiency p in an $(n-p)$ flat.

I write k for $n-p$, and m, r respectively for the quotient and remainder of the division of k by p, so that $k = mp+r$ and $r<p$; we have also $n = (m+1)p+r$.

Every point on the curve has p parameters, viz., the values for that point of the p Abelian integrals of the first species appertaining to the curve; in the equations that follow, only one parameter is written down, so that every equation stands for p equations. We have to consider groups of p points, and sets of groups of $(m-1)$ groups each; each set therefore containing $(m-1)p$ points. A group of p points will generally be called simply a *group*, and a set of $(m-1)p$ points will generally be called simply a *set*.

*Steiner's Polygons.**

Take two sets, of $(m-1)p$ points each, on the curve: call the sum of their parameters a and b respectively; take a group of p points, and let the sum of the parameters be u; lastly, take a group of r points, and let the sum of their parameters be c.† Draw a $(k-1)$ flat through u, passing through c and a: this cuts the curve in a group v_1, containing p points: draw a $(k-1)$ flat through v_1, c and b cutting in a group v_2; draw a $(k-1)$ flat through v_2, c and a cutting in a group v_3, and so on. We thus get a configuration which we may call a polygon, each vertex of the "polygon" being a group of p points: the polygon will be closed if the last group coincides with the first. We have the congruences

$$u+a+c+v_1 \equiv 0,$$
$$v_1+b+c+v_2 \equiv 0,$$
$$v_2+a+c+v_3 \equiv 0,$$
$$\dots \quad \dots \quad \dots \quad \dots$$

It follows from these equations that for a "closed $2n$-gon" we must have

$$n(a-b) \equiv 0;$$

the theory of these polygons is thus seen to be entirely analogous to the theory of Steiner's polygons. I shall now show how Steiner's theorems‡ can be extended to the curves considered in this paper.

One of Steiner's theorems is the following:—"Let P, Q be two points, R their residual, S the point of contact of a tangent from R; then, if taking P, Q as base points, we get a polygon of $2n$ sides, we shall get a polygon of $4n$ sides if we take P and S as base points." The corresponding theorem is as follows:—"Let P and Q be two sets [of $(m-1)p$ points], T a fixed group of r points: divide the two sets into groups, P_i, Q_i, and establish a correspondence between the groups. Draw a $(k-1)$ flat having $(m-1)$-point contact in every point of P_i and passing through Q_i and T: let this cut in a group R_i: through R_i, T draw a $(k-1)$ flat, having m-point contact in a group S_i. We thus get $(m-1)$ groups S_i forming a set S; then, if P, Q give a $2n$-gon, P, S will give a $2mn$-gon. The proof is very simple; we have

$$(m-1)P_i+Q_i+T+R_i \equiv 0,$$
$$T+R_i+mS_i \equiv 0.$$

Therefore

$$(m-1)P_i+Q_i-mS_i \equiv 0,$$
$$Q_i \equiv mS_i-(m-1)P_i,$$

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* In his paper "Ueber die Anwendung der Abel'schen Functionen in der Geometrie," Clebsch gives a very general theorem, extending the theory of Steiner's polygons to curves in three-dimensional space which are the complete intersections of surfaces.

† A group or set for which the sum of the parameters is v, will be referred to as the group or set v.

‡ Steiner, "Geometrische Lehrsätze," *Crelle*, t. xxxii., p. 182; Ges. Werke, t. ii., p. 371.

$$Q \equiv mS-(m-1)P,$$

$$P-Q \equiv m(P-S).$$

But $$n(P-Q) \equiv 0.$$

Therefore $$nm(P-S) \equiv 0,$$

and hence the theorem follows.

It seems hardly worth while to give further examples; it is, however, worth while to remark that, if $m=2$, $r=0$, Steiner's theorems can be extended by replacing "point" by "group of p points," and straight line by $(k-1)$ flat: these are the only alterations required.*

Derived Points.

In his paper "On the Classification of Loci,"† Clifford gives an account of Prof. Sylvester's theory of derived points on a plane cubic, and remarks that the theory is really a geometric representation of the multiplication of elliptic functions. He says:—"I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* .... of deficiency greater than 1. .... It may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1. ... Thus the impossibility of extending the theory of derivation to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k-1$ dimensions is at most of deficiency 1." It is, however, possible to extend the theory to curves of any deficiency, if, instead of seeking for derived *points*, we seek for derived *groups* (of p points); and, indeed, this seems the most natural extension of the theory, for the form in which the problem of the multiplication of Abelian functions is ordinarily presented‡ requires us to find a group, such that the sum of the parameters is a multiple of the sum of the parameters of a given group. If we seek for such an extension of the theory, we are at once led to the curves considered in this paper; for, suppose we have a curve (without singular points) in a k flat, such that k points being given a single group of p points is determined by drawing a $(k-1)$ flat through the k points; then, since a $(k-1)$ flat cuts a curve of order n in n points, the order of the curve must be $k+p$.

The extension of Prof. Sylvester's theory is now quite obvious. We

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* The following theorem (enunciated for simplicity for the case of $m=2$, $r=0$, but true *mutatis mutandis* for all values) includes the rest of Steiner's theorems: "Take two groups P, Q, draw $(k-1)$ flats touching the curve in every point of P, Q, and cutting in P_1Q_1; draw $(k-1)$ flats through PQ_1, P_1Q, cutting in P_2Q_2; derive P_3Q_3 from these as they were got from P_1Q_1, and so on. Then, for a $2(n+1)$-gon, P_nQ_n must coincide, n being any integer."

† *Phil. Trans.*, 1878, p. 663; Mathematical Papers, p. 304.

‡ Clebsch and Gordan, *Abel'sche Functionen*, p. 230.

take a group U and r fixed points, R on the curve; we draw a $(k-1)$ flat, having m-point contact in every point of U, passing through R and cutting in a new group V_1; from V_2 we derive a new group in the same way, and so on. We thus get a series of groups, and by drawing a $(k-1)$ flat through any m groups, no two of which are consecutive, and r points, we get a new group.

We have

$$V_1 \equiv -mU-R_1,$$
$$V_2 \equiv -mV_1-R_2 \equiv m^2U+mR_1-R_2,$$
$$V_3 \equiv -m^3U-m^2R_1+mR_2-R_3,$$
$$V_n \equiv (-)^n\{m^nU+m^{n-1}R_1-m^{n-2}R_2+m^{n-3}R_3\dots\}-R_n,$$

and it is easy to see that no coefficient in any derived group is divisible by $(m+1)$.

In the case of the cubic we have the theorem—The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, and the order of the functions to which they are proportional is always a square number. The corresponding theorem is—Given a derived group, the determination of the original group depends on an equation of order p, whose coefficients depend on an equation of which the order is always of the form n^{2p}, and the coefficients of this last equation are rational functions of the coordinates of the given group.

Application to Plane Curves.

I show, in this section, how the curves here considered may be derived from plane curves of deficiency p. Suppose we have a ∞^k-series of curves cutting a given curve C_n in m variable points; let $\psi_1 \dots \psi_{k+1}$ be $(\kappa+1)$ asyzygetic curves of the series; then, if we take

$$x_1 : x_2 : x_3 : \&c. = \psi_1 : \psi_2 : \psi_3 : \&c.,$$

$(x_1 : x_2 : x_3 \dots : x_\kappa)$ will determine a point on a curve in a space of k dimensions, and the order of the curve will be the number of points in which the C_n is cut by a curve $\Sigma a\psi = 0$; viz., the new curve will be of order m.

We have now to find curves ψ giving $m = k+p$. As we wish to apply Jacobi's problem of inversion to the intersections with a curve ψ, it must be a curve passing $i-1$ times through each i-tuple point of the C_n;* such curves may be called *associate*† curves. Suppose we have a series of associate curves of order λ passing through a fixed points on the curve, and transforming the curve into a C_{k+p} in a k-flat.

* Brill and Nöther, *Math. Ann.*, t. vii., or Clebsch's *Lectures*, pp. 430, &c.
† Brill and Nöther's "Adjungirte Curven."

We must have (if the multiple points are equivalent to δ double points),

$$2\delta+p+k+a = n\lambda,$$

$$\delta+k+a = \frac{\lambda\,.\,\lambda+3}{2},$$

therefore

$$+p = n\lambda - \frac{\lambda\,.\,\lambda+3}{2},$$

or

$$n-1\,.\,n-2 = 2n\lambda-\lambda\,.\,\lambda+3\,;$$

this gives

$$\lambda^2-(2n-3)\,\lambda+n-1\,.\,n-2 = 0,$$

$$\lambda = \frac{2n-3\pm 1}{2}$$

$$= n-1 \text{ or } n-2.$$

The first value of λ gives

$$k+a = \frac{n-1\,.\,n+2}{2}-\delta$$

$$= \frac{n-1\,.\,n-2}{2}-\delta+2\,(n-1)$$

$$= p+2\,(n-1),$$

and the second value of λ gives

$$k+a = p+n-2.$$

Since a must be positive or zero, we must have, for $\lambda = n-1$,

$$k \leqq p+2\,(n-1),$$

and for $\lambda = n-2$,

$$k \leqq p+n-2,$$

we can always transform the curve into a C_{2p} in a $2p$-flat; in certain cases we can do this in one step, but it can always be done in two steps. If the transformation is to be effected in one step, we must have

$$2p \leqq p+2\,(n-1), \text{ or } p+n-2,$$

$$p \leqq 2\,(n-1), \text{ or } n-2,$$

This is always satisfied if n is less than 6.

To effect the transformation in two steps, we first transform the curve into a plane curve of order $p+2$. This is always possible: we have to take

$$a = p+2n-4 \text{ for } \lambda = n-1,$$

and

$$a = p+n-4 \text{ for } \lambda = n-2.$$

The plane curve of order $p+2$ can be transformed into the curve of order $3p$ in a $2p$-flat, by taking

$$a = p+2,$$

and transforming by curves of order $p+1$. The transformation by curves of order p is inapplicable, as it gives a negative value for a.

Two Notes:—(1) *A Definite Integral;* (2) *Equation of the Director-Circle of a Conic. By* Prof. WOLSTENHOLME.

[*Read June* 8*th*, 1882.]

I. If $$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^n}{\lfloor n} X,$$

and r be any positive quantity,

$$\int_0^\infty \frac{X}{x^r}\,dx = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(x)}{x^r}\,dx.$$

In general,

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{1}{\lfloor n-1} \int_0^x z^{n-1} f^n(x-z)\,dz;$$

hence $$X = \frac{n}{x^n} \int_0^x z^{n-1} f^n(x-z)\,dz = \frac{n}{x^n} \int_0^x (x-z)^{n-1} f^n(z)\,dz,$$

and $$\int_0^\infty \frac{X}{x^r}\,dx = n \int_0^\infty \frac{dx}{x^{n+r}} \int_0^x (x-z)^{n-1} f^n(z)\,dz,$$

which, changing the order of integration, is

$$n \int_0^\infty f^n(z)\,dz \int_z^\infty \frac{(x-z)^{n-1}}{x^{n+r}}\,dx,$$

and $$\int_z^\infty \frac{(x-z)^{n-1}}{x^{n+r}}\,dx = \frac{1}{z^r} \int_1^\infty \frac{(x-1)^{n-1}}{x^{n+r}}\,dx = \frac{1}{z^r} \int_0^1 \left(\frac{1}{x}-1\right)^{n-1} . x^{n+r} \frac{dx}{x^2}$$

$$= \frac{1}{z^r} \int_0^1 x^{r-1} (1-x)^{n-1}\,dx = \frac{1}{z^r} \frac{\Gamma(r)\,\Gamma(n)}{\Gamma(n+r)},$$

if r be any positive quantity, but is ∞ if r be zero or negative.

We have then

$$\int_0^\infty \frac{X}{x^r}\,dx = n \frac{\Gamma(r)\,\Gamma(n)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(z)}{z^r}\,dz = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(x)}{x^r}\,dx.$$

The integrals will generally be finite, if r be a proper fraction.