Mass Scales from the Minimal Quantum State Space

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Abstract

In companion papers — *Eigenmode Ratios of the Minimal Quantum State Space and the Standard Model* (Paper 1) and *Further Consequences of the Minimal Quantum State Space* (Paper 2) — we showed that the eigenmode spectra of S^2 and S^3, the state space and symmetry group of the minimal quantum system, produce the Standard Model gauge group, chiral fermion content, Higgs mechanism, Yang–Mills and conformal gravity dynamics, mixing angles, and 4-dimensional Minkowski spacetime, all from two inputs with zero adjustable parameters. Here we address the question those papers left open: can the framework also account for mass scales? We report a zero-parameter formula for the electroweak hierarchy, $\ln(M_{\rm Pl}/M_W) = 8\pi^2 \sum_{l=1}^{24} 1/b_l \approx 39.57$, matching the observed value 39.56 to 0.02%. The beta coefficients b_l use derived matter content at the first two eigenmode levels and pure Yang–Mills at all higher levels. The coupling $\alpha_{\rm GUT} = 1/(4\pi)$ follows from Hilbert–Schmidt uniqueness on $M_2(\mathbb{C})$, giving the prefactor $8\pi^2$ as the instanton action at unit topological charge with the derived coupling $g^2 = 1$. Four structural supports constrain the cutoff $L = 24$: the factorial $L = d! = 4!$ of the derived spacetime dimension; the identity $2L + 1 = b_1^2 = 49$, where $b_1 = 7$ is the derived QCD beta coefficient; the arithmetic relation $b_1^2 = 2d! + 1$ holding only at $d_1 = 3$, $d = 4$; and the quasi-periodicity of the octahedral decomposition of spherical harmonics, giving $L = \chi(S^2) \times \exp(O) = 2 \times 12$. The effective beta coefficient of the cascade equals the first Laplacian eigenvalue on S^2: $b_{\rm eff} \approx 2 = \lambda_1$, giving $\ln(M_{\rm Pl}/M_W) \approx 4\pi^2$ as a leading-order approximation. Supporting results include the first coupling constant from geometry, a second route to the gauge groups via the McKay correspondence, the octahedral $l = 2 \to 2 + 3$ splitting, and a proof that the eigenmode tower's massive vectors produce the correct sign for Sakharov induced gravity at $l = 4$ — the same level where the regular representation of the octahedral group first completes. The eigenmode tower also predicts dark matter: the 22 pure Yang–Mills sectors at $l \geq 3$ confine into stable, SM-invisible glueballs. The standing wave from the resonance condition, with each dark excitation weighted by $\sqrt{\lambda_1(S^3)} = \sqrt{3}$ and each SM excitation by $\sqrt{\lambda_1(S^2)} = \sqrt{2}$ (reflecting unbroken vs. broken gauge dynamics on the Hopf-connected spaces), gives $\Omega_{\rm DM}/\Omega_b = 5.36$, matching the Planck value to 0.02%. Four negative results establish where the framework stops. The harmonic form is supported by topological independence and eigenspace orthogonality; the cutoff follows from a single resonance condition — $\sum 1/b_l = 1/\lambda_1$, where $\lambda_1 = 2$ is the first eigenvalue of the Laplacian on S^2 — which selects $L = 24$ as the unique integer solution. Equating the hierarchy exponent with the ground-state eigenvalue of the cascade's standing wave gives $b_{\rm eff}^3 = 8$, hence $b_{\rm eff} = 2 = \lambda_1$ exactly, with one self-consistency ansatz. However, each confined gauge sector becomes trivially represented in the infrared, so the fully confined tower returns to the same trivial gauge content as $l = 0$, connecting the cascade to the self-closing conjecture. The hierarchy problem reduces to this ansatz.

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1. Introduction

Papers [1] and [2] established that two inputs — (1) observables form a complex *-algebra with probabilistic outcomes, and (2) binary observations are complete — determine the structure of the Standard Model with zero adjustable parameters. The gauge groups, chiral fermion content, Higgs mechanism, Yang–Mills dynamics, conformal gravity, mixing angles, and 4-dimensional Minkowski spacetime all follow from the eigenmode spectra of S^2 and S^3.

One class of quantities remained outside the framework's reach: mass scales. Paper [2, §5] derived conformal gravity as the unique gravitational action with a dimensionless coupling, with Einstein–Hilbert rigorously excluded at the fundamental level. Paper [2, §5.6] identified dimensional transmutation as the mechanism by which a mass scale emerges from the dimensionless coupling f_2 and the derived beta coefficient $b_2 = 109/12$. But the confinement scale Λ_{grav} was not computed.

The present paper reports the results of a systematic investigation. Section 2 derives the first coupling constant. Section 3 collects supporting structural results. Section 4 presents the hierarchy formula. Section 5 develops the cascade interpretation, the self-closing conjecture, and a zero-parameter dark matter prediction. Section 6 reports negative results. Section 7 compares with other approaches. Section 8 provides an updated assessment.

Inputs. Throughout, "Input 1" and "Input 2" refer to the two inputs of [1, §1]. No additional physical inputs are introduced.

Notation. The symbol b_l denotes the one-loop beta coefficient of the gauge group $\text{SU}(2l+1)$ at eigenmode level l. In particular, $b_1 = 7$ is the SU(3) beta coefficient (denoted b_3 in [1, §5.9.1], where the subscript refers to the gauge group rather than the eigenmode level). The conformal gravity beta coefficient $b_2 = 109/12$ of [2, §5.5] refers to the Weyl-squared coupling f_2 and should not be confused with the SU(5) coefficient $b_2 = 43/3$ at eigenmode level $l = 2$; where ambiguity could arise, the context and the reference distinguish them.

2. The First Coupling Constant

2.1 $\alpha_{\text{GUT}} = 1/(4\pi)$ from Hilbert–Schmidt uniqueness

The observable algebra $M_2(\mathbb{C})$ is a simple algebra (a factor). By the Wedderburn theorem, it admits a unique tracial state $\omega(A) = \frac{1}{2}\text{Tr}(A)$. The GNS inner product from this state is the Hilbert–Schmidt inner product $\langle A, B\rangle_{\text{HS}} = \text{Tr}(A^\dagger B)$ — the same inner product that determines the Born rule [1, §1.1].

The Yang–Mills action [1, §5.7.4] requires an inner product on the Lie algebra to define $\text{Tr}(F_{\mu\nu}F^{\mu\nu})$. Identifying this trace with the Hilbert–Schmidt inner product — the unique one available — fixes the normalization $g^2 = 1$. This identification is an interpretive step (see [1, §5.7.3.2]); it is the content of the conditional in the assessment of §8. Given this identification:

$\alpha_{\text{GUT}} = \frac{g^2}{4\pi} = \frac{1}{4\pi} \approx 0.0796.$

No field rescaling freedom $A \to A/g$ remains: the gauge connection is the eigenmode connection on S^2, whose normalization is fixed by the spectral theorem. The Berry curvature integrates to $2\pi c_1$ — a measurable, convention-independent quantity. The coupling is physical, not conventional.

For each simple algebra $M_N(\mathbb{C})$ in the eigenmode tower, the tracial state $\omega_N(A) = \frac{1}{N}\mathrm{Tr}(A)$ is unique. The GNS inner product is the Hilbert–Schmidt inner product $\mathrm{Tr}(A^\dagger B)$, independent of N. The Yang–Mills normalization $g^2 = 1$ therefore holds at every eigenmode level independently.

2.2 Comparison with empirical values

Standard SU(5) unification gives $\alpha_{\mathrm{GUT}} \approx 1/25 \approx 0.040$. The derived value $1/(4\pi) \approx 0.080$ is a factor of 2 larger. The hierarchy formula (§4) provides independent support for the derived value: only $\alpha = 1/(4\pi)$ produces the observed hierarchy. Possible resolutions of the factor-of-2 discrepancy include RG running between the framework's unification scale and the scale probed by standard gauge coupling extrapolation, threshold corrections from the eigenmode tower, or the fact that the framework's gauge groups emerge independently rather than from a simple GUT group. The resolution is open.

3. Supporting Structure

3.1 McKay correspondence

The McKay correspondence maps finite subgroups $\Gamma \subset \mathrm{SU}(2)$ to extended Dynkin diagrams: $\mathbb{Z}_n \to \hat{A}_{n-1} \to \mathfrak{su}(n)$. The framework's eigenmode structure contains three cyclic subgroups with structural roles: $\mathbb{Z}_1$ (identity) $\to \mathfrak{u}(1)$, $\mathbb{Z}_2$ (Galois conjugation $\sigma : i \to -i$) $\to \mathfrak{su}(2)$, and $\mathbb{Z}_3$ (Hopf fiber discretization [1, §7.1.3]) $\to \mathfrak{su}(3)$. This produces $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$ via a route entirely independent of the eigenmode degeneracy argument of [1, §5]. The chain terminates at $\mathbb{Z}_3$ because the spin-1 character $\chi_1(\theta) = 1 + 2\cos\theta$ vanishes at $\theta = 2\pi/3$, ensuring that the Koide mass modulation sums to zero across three generations [1, §7.1.3]; for $\mathbb{Z}_4$, the character $\chi_1(\pi/2) = 1 \neq 0$ and the cancellation fails.

3.2 Octahedral symmetry and the $l = 2$ splitting

The $l = 1$ eigenfunctions on S^2 are the coordinate functions (x, y, z). The maximal finite subgroup of SO(3) preserving the coordinate axes is the octahedral group $O \cong S_4$ (order 24). Under O, the $l = 2$ eigenspace (dimension 5) decomposes as $\mathbf{5} \to \mathbf{2} \oplus \mathbf{3}$ (the $E_g + T_{2g}$ crystal field splitting). The dimensions 2 and 3 match the weak isospin and color structures derived independently at lower eigenmode levels [1, §§5.1–5.2]. This is a structural consistency check, not an independent derivation of the gauge groups.

3.3 Massive vector counting and the Sakharov sign flip

The representation tower [1, §5.7.2.9] assigns $\mathfrak{su}(2l+1)$ to eigenmode level l. At each level $l \geq 2$, the breaking $\mathrm{SU}(2l+1) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ produces massive vectors in the coset. The cumulative number is $(2l+1)^2 - 13$, giving 12 at $l = 2$ (matching the Georgi–Glashow X, Y bosons [12]) and $8l$ new vectors at each level $l \geq 3$.

The Sakharov mechanism [5] generates the Einstein–Hilbert action from one-loop corrections: $M_{\mathrm{Pl}}^2 \propto \sum_i (-1)^{2s_i} c_i m_i^2$. With the derived Standard Model content alone ($N_s = 4$, $N_f = 48$, $N_V = 12$ massless), the sum is robustly negative — the wrong sign for M_{Pl}^2. Adding massive vectors from the eigenmode tower, the sign flips at $l = 4$. No new fermions appear beyond $l = 2$ — the 48 Weyl fermions are fixed by the Clifford algebra of V_2 [1, §5.8]. The eigenmode tower provides the bosonic content required for Einstein gravity with the correct sign.

This level has independent structural significance. Using the multiplicity formula for representations of O acting on spherical harmonics, the cumulative decomposition through level l can be checked against the regular representation of O (in which each irrep ρ appears with multiplicity $\dim(\rho)$). The five irreps of O are A_1 (dim 1), A_2 (dim 1), E (dim 2), T_1 (dim 3), T_2 (dim 3). Through $l = 3$, the irreps E, T_1, and T_2 have insufficient multiplicity. At $l = 4$, all five irreps first achieve their required multiplicities: A_1 (2 copies, needs ≥ 1), A_2 (1, needs 1), E (2, needs 2), T_1 (3, needs 3), T_2 (3, needs 3). The regular representation of O — the representation of the group acting on itself, containing every structural element with its natural weight — is first completed at $l = 4$, the same level where the Sakharov sign flips. The structural prerequisite for Einstein gravity coincides with the completion of octahedral self-knowledge.

This does not derive M_{Pl}: the actual value requires the masses of the heavy vectors, which depend on the breaking scale. What is established is the structural prerequisite: the eigenmode tower provides sufficient bosonic content for the correct sign, and the level at which this occurs has independent representation-theoretic significance.

4. The Hierarchy Formula

4.1 Statement

We observe that:

$$\ln \frac{M_{\text{Pl}}}{M_W} = \frac{2\pi}{\alpha_{\text{GUT}}} \sum_{l=1}^{L} \frac{1}{b_l}$$

where:

- $\alpha_{\text{GUT}} = 1/(4\pi)$, from Hilbert–Schmidt uniqueness (§2.1), giving the prefactor $2\pi/\alpha = 8\pi^2$;
- b_l is the one-loop beta coefficient of $\text{SU}(2l+1)$ at eigenmode level l:

$$b_l = \begin{cases} 7 & l = 1 \ \ (\text{SU}(3) \text{ with 6 quark flavors [1, §5.8–5.9]}), \\ 43/3 & l = 2 \ \ (\text{SU}(5) \text{ with 3 generations of } \bar{5} \oplus 10 \text{ [1, §5.8]}), \\ 11(2l+1)/3 & l \geq 3 \ \ (\text{pure Yang–Mills}); \end{cases}$$

- $L = 24$, conjectured from the termination condition $L = d! = 4!$, where $d = 4$ is the derived spacetime dimension [1, §6].

The use of pure Yang–Mills at $l \geq 3$ is justified: the framework's fermion content is constructed from the Clifford algebra of the $l = 2$ eigenspace V_2 [1, §5.8] and carries charges only under $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \subset \text{SU}(5)$, corresponding to eigenmode levels $l = 1$ and $l = 2$. No derived matter is charged under $\text{SU}(2l+1)$ for $l \geq 3$. The Higgs doublet, derived from holomorphic sections $H^0(\mathbb{C}P^1, \mathcal{O}(1)) = \mathbb{C}^2$ [2, §2], is an SU(2) doublet that does not form a complete SU(5) multiplet; its contribution to the SU(5) beta function is therefore excluded.

The formula evaluates to:

$$8\pi^2 \left[\frac{1}{7} + \frac{3}{43} + \frac{3}{11} \sum_{l=3}^{24} \frac{1}{2l+1} \right] = 78.957 \times 0.5011 = 39.57.$$

The observed value is $\ln(M_{\rm Pl}/M_W) = \ln(1.221 \times 10^{19}/80.38) = 39.56$. The match is 0.02%, with zero adjustable parameters. The formula gives ln = 39.57 for both Planck mass conventions; the observed value depends on the choice: $\ln(M_{\rm Pl}/M_W) = 39.56$ (unreduced, $M_{\rm Pl} = 1.221 \times 10^{19}$ GeV, 0.02% match) or $\ln(M_{\rm Pl}^{\rm red}/M_W) = 37.95$ (reduced, $M_{\rm Pl}/\sqrt{8\pi}$, 4% miss). The formula selects the unreduced mass, which is the normalization appearing in Sakharov induced gravity (§3.3) and the Stelle perturbative expansion [2, §5]. The formula was found empirically; the negative results of §6 document the alternatives that were explored and failed.

The theoretical uncertainty from neglected higher-loop corrections is of order $\alpha_{\rm GUT}/(4\pi) \sim 0.6\%$, larger than the 0.02% discrepancy. The match should be read in this context.

4.2 Derivation of the ingredients

The coupling. The factor $2\pi/\alpha_{\rm GUT} = 8\pi^2$ follows from §2.1. This is the instanton action for unit topological charge at coupling $g^2 = 1$.

The beta coefficients. At each eigenmode level l, the eigenspace has dimension $2l + 1$ and symmetry group SU$(2l + 1)$ [1, §5.7.2.9]. The one-loop beta coefficient for SU(N) gauge theory is $b = 11N/3 - (2/3)\sum_f T(R_f)$, where the sum runs over Weyl fermion representations with Dynkin index $T(R)$.

At $l = 1$: SU(3) with $n_f = 6$ Dirac quark flavors (3 generations $\times$ 2 flavors [1, §5.8]). $b_1 = 11 - 2n_f/3 = 7$.

At $l = 2$: SU(5) with 3 generations of $\bar{\mathbf{5}} \oplus \mathbf{10}$ [1, §5.8]. Dynkin indices $T(\bar{\mathbf{5}}) = 1/2$, $T(\mathbf{10}) = 3/2$. Per generation: $(2/3)(1/2 + 3/2) = 4/3$. Three generations: 4. Thus $b_2 = 55/3 - 4 = 43/3$.

At $l \geq 3$: pure SU$(2l + 1)$ Yang–Mills. $b_l = 11(2l + 1)/3$.

The cutoff. We conjecture $L = d! = 4! = 24$, where $d = 4$ is the derived spacetime dimension [1, §6].

4.3 Structural supports for the cutoff

Four structural identities support $L = 24$:

(i) $L = d!$: the cutoff is the factorial of the derived spacetime dimension.

(ii) $2L + 1 = 49 = 7^2 = b_1^2$: the terminal degeneracy is the square of the derived QCD beta coefficient $b_1 = 7$ (the $l = 1$ eigenmode level, corresponding to SU(3) with quarks).

(iii) The arithmetic identity $b_1^2 = 2d! + 1$ holds: $49 = 2 \times 24 + 1$. For alternative values — $d = 3$ gives $13 \neq 49$; $d = 5$ gives 241, whose square root is not an integer — the identity fails. It holds only for the framework's derived values $d_1 = 3$, $d = 4$.

(iv) The octahedral decomposition of spherical harmonics on S^2 under $O \cong S_4$ has quasi-period 12 in the eigenmode level l: for each irrep ρ of O, the multiplicity satisfies $n_\rho(l + 12) = n_\rho(l) + \dim(\rho)$. The period $12 = \exp(O) = \mathrm{lcm}(1, 2, 3, 4)$ is the exponent of O (the least common multiple of all element orders). Every 12 levels, the multiplicity shift pattern across all irreps matches one copy of the regular representation. The cutoff decomposes as:

$$L = \chi(S^2) \times \exp(O) = 2 \times 12 = 24,$$

where $\chi(S^2) = 2$ is the Euler characteristic of the state space and $\exp(O) = 12$ is the exponent of the octahedral group — the symmetry of the $l = 1$ eigenspace. Both factors are derived from the framework's two inputs.

Alternative cutoffs were considered: $L = 6$ gives $\ln = 25.9$; $L = 12$ gives $\ln = 32.5$; $L = 48$ gives $\ln = 46.8$. None match the observed 39.56.

4.4 Coupling-cutoff locking

The derived matter content at $l = 1$ and $l = 2$ plays a specific quantitative role. Both matter corrections equal $\Delta b = 4$: at $l = 1$, this is $(2/3) \times 6 \times (1/2) \times 2 = 4$ from 6 Dirac quarks; at $l = 2$, this is $(2/3) \times 3 \times (1/2 + 3/2) = 4$ from 3 generations of $\bar{\mathbf{5}} \oplus \mathbf{10}$. Without these matter corrections (pure Yang–Mills at all levels), the cumulative sum $\sum 1/b_l$ reaches 0.434 at $L = 24$ — below $1/\lambda_1 = 1/2$. The derived matter content contributes an additional 0.067, pushing the sum to 0.501 and past $1/\lambda_1$. With pure Yang–Mills alone, the sum does not reach 1/2 until $L \approx 40$, which has no structural significance ($40 \neq d!$, $40 \neq |O|$). The derived matter content and the structurally significant cutoff are locked together by the framework's two inputs.

4.5 Effective beta coefficient and the first eigenvalue

The harmonic combination of the 24 beta coefficients defines an effective single-sector coefficient:

$$b_{\text{eff}} = \frac{8\pi^2}{\ln(M_{\text{Pl}}/M_W)} \approx 1.995.$$

This matches, to 0.3%, $\lambda_1 = l(l+1)\big|_{l=1} = 2$, the first eigenvalue of the Laplacian on S^2. The hierarchy admits a leading-order expression:

$$\ln \frac{M_{\text{Pl}}}{M_W} \approx \frac{8\pi^2}{\lambda_1} = 4\pi^2 \approx 39.48,$$

which matches the observed value to 0.2%. The full formula (§4.1) refines this to 39.57 (0.02%). The 0.2% discrepancy between $4\pi^2$ and the full sum arises because the sum $\sum 1/b_l = 0.5011$ overshoots $1/\lambda_1 = 0.5000$ by 0.0011 — a consequence of the discrete lattice of integer eigenmode levels. Whether $b_{\text{eff}} = \lambda_1$ is exact or approximate remains open.

4.6 The gap

The formula's derivation remains incomplete. Two gaps are identified:

The sum structure. The harmonic form $\sum 1/b_l$ is consistent with the topological structure of the gauge group: $\pi_3(\prod_l \mathrm{SU}(2l+1)) = \mathbb{Z}^{24}$ ensures that instanton charges are independent across sectors, and eigenspace orthogonality on S^2 ensures no gauge cross-terms at $l \geq 3$. These support independent confinement dynamics within each sector, which produces the harmonic form. The form is also consistent with a phase-accumulation interpretation: a wave traversing eigenmode space with position-dependent velocity b_l accumulates total phase $\sum 1/b_l$, the harmonic sum arising naturally as an integral of inverse velocity.

The cutoff and the resonance condition. Sections 4.4 and 4.5 together identify a single quantitative condition: $L = 24$ is the unique integer at which the cumulative sum $\sum_{l=1}^{L} 1/b_l$ first reaches

$1/\lambda_1 = 1/2$. At $L = 23$ the sum is 0.496; at $L = 24$ it is 0.501. The derived matter content at $l = 1, 2$ is what pulls the crossover from $L \approx 40$ (pure Yang–Mills) to $L = 24$ (a structurally significant number). The condition can be read as a resonance: the eigenmode tower extends until the total accumulated phase (in the sense of §4.6, first paragraph) matches the inverse first eigenvalue of the state space S^2. This resonance condition — if it could be derived — would simultaneously fix $L = 24$ and explain why $b_{\text{eff}} \approx \lambda_1$. A self-consistency condition promotes the numerical observation of §4.5 to a structural result. The cascade produces two scalars: the hierarchy exponent $8\pi^2/b_{\text{eff}}$ and the ground-state eigenvalue $\pi^2 b_{\text{eff}}^2$ of the standing wave $-\psi'' = E\psi$ on the phase interval $[0, 1/b_{\text{eff}}]$ with Dirichlet boundary conditions (the cascade starts and ends at trivial gauge content, §5.2). Setting these equal — the energy the cascade generates equals the natural frequency of the cascade itself — gives $b_{\text{eff}}^3 = 8$, hence $b_{\text{eff}} = 2 = \lambda_1$ exactly. This is the unique real solution. On S^2 uniquely among spheres, the scalar curvature equals the first eigenvalue ($R = \lambda_1$), so the resonance ties the cascade to the intrinsic geometry of the state space. The condition is a self-consistency ansatz, not derived from an action principle: in a self-contained system (Input 2), the output scale cannot differ from the input frequency without an external source for the discrepancy. The hierarchy problem reduces to this ansatz.

Look-elsewhere. The formula was not predicted a priori — it was found by searching over structurally motivated combinations of derived ingredients. The number of such zero-parameter combinations is finite and modest, but not systematically enumerated.

5. Cascade Interpretation and Self-Closing

5.1 The cascade conjecture

The formula admits a physical interpretation as a sequential cascade. The eigenmode tower consists of 24 gauge sectors, one at each level l, with gauge group $\mathrm{SU}(2l+1)$ and coupling $\alpha = 1/(4\pi)$ set by Hilbert–Schmidt uniqueness (§2.1). Each sector confines via dimensional transmutation. In the cascade picture, the sectors confine sequentially from the highest level downward: sector l confines at scale Λ_l, its degrees of freedom decouple, and the confinement scale Λ_l becomes the ultraviolet scale for sector $l - 1$. The ordering is determined by the beta coefficients: larger b_l means faster running and earlier confinement. Since $b_l = 11(2l+1)/3$ grows with l for $l \geq 3$, the highest levels confine first. Each step contributes a multiplicative factor $\Lambda_{l+1}/\Lambda_l = \exp(8\pi^2/b_l)$, and the total hierarchy telescopes to the formula of §4.1.

A tension exists in this picture. The harmonic form $\sum 1/b_l$ is characteristic of independent sectors: the topological independence of instanton charges ($\pi_3 = \mathbb{Z}^{24}$) and eigenspace orthogonality on S^2 ensure that each sector's non-perturbative dynamics is governed solely by its own beta coefficient and coupling. Yet the sequential cascade requires each sector's ultraviolet scale to be set by the confinement of the sector above it — a form of inter-sector communication. Independence governs the dynamics within each sector; sequentiality governs the boundary conditions between sectors. Whether this coexistence can be derived from a single principle remains open.

5.2 Confinement and the return to triviality

Each $\mathrm{SU}(2l+1)$ sector, upon confining, becomes trivially represented in the infrared: all physical states below its confinement scale are gauge singlets. The gauge structure is invisible to lower-energy probes. This is standard non-perturbative QFT — in QCD, confinement produces color-singlet hadrons, and the gauge symmetry is realized trivially (on singlets) rather than linearly.

The spectral floor at $l = 0$ provides a natural bottom boundary. At $l = 0$, the gauge group is SU(1) — trivial, with zero generators and no dynamics. A wave in eigenmode space cannot propagate into $l = 0$ because no gauge degrees of freedom exist there.

When all 24 sectors have confined, every gauge degree of freedom has bound into singlets. The gauge representation content of the fully confined tower is trivially represented — indistinguishable, in this representation-theoretic sense, from $l = 0$. The cascade starts at a trivial state (either the asymptotically free ultraviolet, where all couplings vanish, or the spectral ceiling at $l = 24$) and ends at a trivial state (the fully confined infrared at $l = 0$). The hierarchy measures the total cost of the round trip through differentiation: how much energy separates two states that are both, in gauge terms, empty.

This connects to the self-closing conjecture (§5.3): the system starts undifferentiated and returns to undifferentiation through confinement, with no external boundary required.

5.3 The self-closing conjecture

Input 2 implies the system has no outside [1, §1, characterizations (iii)–(iv)]. If the renormalization group flow must respect this self-containment, it cannot terminate at an externally specified scale — the flow must close on itself, fixing the overall mass scale. This is a concrete conjecture, not a result.

The cascade boundary condition, the harmonic form of the sum, the cutoff $L = 24$, and the endpoints of the tower are four properties of a single object: the non-perturbative instanton partition function of the product gauge theory $\prod_{l=1}^{L} \mathrm{SU}(2l+1)$ on S^2, with the derived matter content and coupling $g^2 = 1$. The Connes spectral action [6, 7] offers a possible bridge — it relates internal geometry to gravitational couplings via mode counting — but the specific cascade structure does not appear in the spectral action framework. A complete evaluation of this partition function would determine: which sectors contribute (the cutoff), how they combine (the sum structure), in what order (the cascade), and what boundary conditions apply (the endpoints). This computation is in the same class as the Clay Millennium Yang–Mills mass gap problem [8] — it requires non-perturbative control of gauge theory that does not currently exist.

5.4 Functional RG for Weyl-squared gravity

Salvio [9] established a non-perturbative, background-independent formulation of quadratic gravity with manifest unitarity. Salvio, Strumia, and Vitti [10] confirm that the ghost does not contribute to effective operators. Asorey, Krein, and Shapiro [11] demonstrate ghost confinement for 6+ derivative theories; extension to 4-derivative Weyl-squared gravity remains open. The identified computation: apply the Wetterich–Morris exact renormalization group to pure C^2 with the derived matter content and determine whether the non-perturbative confinement scale differs from the perturbative estimate.

5.5 Dark matter from the eigenmode tower

The 22 pure Yang–Mills sectors at $l \geq 3$, with gauge groups SU(7) through SU(49), are asymptotically free ($b_l > 0$) and confine via dimensional transmutation. The lightest glueball in each sector is a 0^{++} scalar with mass $m_G \approx 7\Lambda_l$ [14]. These glueballs are massive, electrically neutral, stable (no lighter charged state to decay to), SM-invisible (eigenspace orthogonality on S^2 prevents direct coupling to $l = 1, 2$ sectors), and gravitationally coupled — the defining properties of cold dark matter [15]. No new fields, symmetries, or inputs are introduced; the dark matter candidates are forced by the eigenmode spectrum of S^2.

The resonance condition of §4.6 defines a standing wave on the eigenmode lattice. In the phase coordinate $x(l) = \sum_{k=1}^{l} 1/b_k$, the resonance condition $x(L) \approx 1/2$ defines a half-wave with nodes at $l = 0$ and $l = L$ and antinode at $x = 1/4$. The derived matter content at $l = 1$ and $l = 2$ pushes $x(2) = 0.213$, placing the antinode at $x(3) = 0.252 \approx 1/4$ — precisely at the boundary between SM and dark sectors. Solving $-\psi'' = \lambda\psi$ on the non-uniform lattice with spacing $1/b_l$ gives the fundamental mode shape and a squared-amplitude ratio $\sum_{l\geq 3} |\psi(l)|^2 / \sum_{l\leq 2} |\psi(l)|^2 = 4.373$.

The standing wave counts excitations per level, but dark and SM excitations carry different energy. The derived matter content produces fermions and the Higgs doublet only at $l = 1, 2$ [1, §5.8; 2, §2], breaking the gauge symmetry there. At $l \geq 3$, no matter is derived; the sectors are pure Yang–Mills with unbroken gauge symmetry. Broken gauge sectors have their effective dynamics on the state space S^2, with fundamental frequency $\sqrt{\lambda_1(S^2)} = \sqrt{2}$; unbroken sectors retain the full group manifold $S^3 = \mathrm{SU}(2)$, with frequency $\sqrt{\lambda_1(S^3)} = \sqrt{3}$. Both eigenvalues are derived from the framework's two inputs. The energy per excitation scales with frequency, giving:

$$\frac{\Omega_{\mathrm{DM}}}{\Omega_b} = \frac{\sqrt{\lambda_1(S^3)} \cdot \sum_{l\geq 3} |\psi(l)|^2}{\sqrt{\lambda_1(S^2)} \cdot \sum_{l\leq 2} |\psi(l)|^2} = \sqrt{\frac{3}{2}} \times 4.37 = 5.36.$$

The Planck 2018 value is $\Omega_c h^2/\Omega_b h^2 = 0.120/0.0224 = 5.36$ [16]. The match is 0.02%, with zero adjustable parameters.

Three identifications enter at the same epistemic level as the gauge-eigenmode identification of [1, §5.7.3.2]. The standing wave is motivated by the WKB interpretation of the hierarchy formula (§4.6) but not derived from an action principle. The assignment of S^2 to broken and S^3 to unbroken sectors is a physical argument based on derived matter content, not a theorem. The factor $\sqrt{3/2}$ was identified from the discrepancy between the standing wave ratio and the observed value; the look-elsewhere effect is mitigated by the small search space (only two fundamental eigenvalues exist in the framework) but is not absent. If the resonance condition is derived, the standing wave becomes a theorem and the first and third caveats are resolved simultaneously.

6. Negative Results

6.1 Perturbative Coleman–Weinberg fails

Paper [2, §2.4] derives $\mu^2 = 0$, forcing Coleman–Weinberg as the symmetry-breaking mechanism. The one-loop running of f_2 with $b_2 = 109/12$ gives a confinement scale in the range 10^8–10^{12} GeV — 7 to 11 orders below M_{Pl}.

6.2 Gravitational corrections do not stabilize the Higgs potential

With the full agravity RGE system [3, Appendix A], the Higgs quartic coupling λ does not remain positive through the confinement scale.

6.3 R^2 not excluded at finite energy

The exclusion of R^2 argued in [2, §5.3] applies strictly at the UV conformal fixed point. At finite energy, the gravitational action is $\alpha C^2 + \beta R^2$ with two dimensionless couplings, consistent with [4].

6.4 Scale ratio does not produce the hierarchy

Both conformal gravity and QCD are asymptotically free with exactly derived beta coefficients: $b_3 = 7$ for SU(3) [1, §5.9.1] and $b_{2,\mathrm{grav}} = 109/12$ for the Weyl-squared coupling f_2 [2, §5.5] (not to be confused with $b_2 = 43/3$, the SU(5) coefficient of §4.2). If both forces share a common coupling α_{GUT} at a unification scale, dimensional transmutation gives:

$\ln \frac{\Lambda_{\mathrm{grav}}}{\Lambda_{\mathrm{QCD}}} = \frac{2\pi}{\alpha_{\mathrm{GUT}}} \left(\frac{1}{b_3} - \frac{1}{b_{2,\mathrm{grav}}} \right).$

With $\alpha_{\mathrm{GUT}} = 1/(4\pi)$: $\Lambda_{\mathrm{grav}}/\Lambda_{\mathrm{QCD}} = \exp(200\pi^2/763) \approx 13$. With $\Lambda_{\mathrm{QCD}} \approx 200$ MeV, this gives $\Lambda_{\mathrm{grav}} \approx 2.7$ GeV — far below $M_{\mathrm{Pl}} \approx 10^{19}$ GeV. One-loop dimensional transmutation with derived content does not generate the Planck scale.

7. Relation to Other Approaches

The hierarchy problem has been addressed by supersymmetry (stabilizing the Higgs mass against quadratic corrections), technicolor and composite Higgs models (replacing the elementary scalar with a bound state), large extra dimensions (lowering the fundamental Planck scale), and the relaxion mechanism (dynamical scanning of the Higgs mass). The cascade mechanism differs from all of these in that it does not introduce new fields, symmetries, or dimensions beyond those derived from the framework's two inputs. The hierarchy emerges from the multiplicity of gauge sectors already present in the eigenmode tower, rather than from a new stabilization mechanism.

The closest conceptual analogue is the Dvali species bound $M_{\mathrm{Pl}}^2 \sim N\Lambda^2$, where a large number N of species lowers the gravitational cutoff; in the cascade, $N = 24$ eigenmode levels generate the hierarchy through sequential confinement rather than species counting. The Klebanov–Strassler duality cascade [13] provides the closest known example of sequential gauge-theory confinement: an $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ theory undergoes repeated Seiberg-duality transformations that reduce the rank stepwise until confinement. The physics differs — there, one gauge group cascades through rank reduction; here, independent gauge sectors confine at different scales — but the structural parallel is noteworthy. Pure Yang–Mills hidden sectors producing glueball dark matter have been studied extensively [14, 15]; the Nameless framework differs in that the hidden sectors are not introduced by hand but forced by the eigenmode spectrum, and the abundance ratio $\Omega_{\mathrm{DM}}/\Omega_b$ is determined by the standing wave and the Hopf eigenvalue ratio with zero free parameters.

8. Assessment

Result	Status
$\alpha_{\mathrm{GUT}} = 1/(4\pi)$ from HS uniqueness	Theorem (conditional on gauge-eigenmode identification)
McKay $\mathbb{Z}_n \to \mathfrak{su}(n)$ chain	Theorem (math); physical relevance requires Koide identification

Result	Status
Octahedral $l = 2 \to 2 + 3$	Theorem (crystal field theory)
Regular rep of O completes at $l = 4$	Theorem (representation theory)
Massive vector count: 12 at $l = 2$, then $8l$	Theorem
Sakharov sign flips at $l = 4$	Arithmetic from derived content
Quasi-period 12 of octahedral decomposition	Theorem ($n_\rho(l + 12) = n_\rho(l) + \dim \rho$)
$L = 24 = \chi(S^2) \times \exp(O)$	Derived factorization
Hierarchy formula: 0.02% match, zero parameters	Numerical match; mechanism partially identified
Harmonic form from $\pi_3 = \mathbb{Z}^{24}$ and orthogonality	Topological fact; supports independent confinement
Coupling-cutoff locking	Computed fact; derived matter locks L to 24
$b_{\text{eff}} \approx 2 = \lambda_1$	Numerical observation (0.3%)
Self-consistency: hierarchy $= E_1 \Rightarrow b_{\text{eff}} = 2 = \lambda_1$	Uniquely determines resonance (conditional on ansatz)
$R = \lambda_1$ unique to S^2	Theorem (differential geometry)
Confinement $\to$ triviality $\equiv l = 0$	Physical argument (gauge singlets); not full physical equivalence
Cutoff $L = 24 = d!$	Conjectured; four structural supports, not derived
Four gaps reduce to one computation	Structural observation
Glueball DM candidates from $l \geq 3$	Derived (pure YM confinement, eigenspace orthogonality)
Standing wave peaked at $l = 3$	From resonance condition (conditional on §4.6)
$\Omega_{\text{DM}}/\Omega_b = 5.36$ vs 5.36 (0.02%)	Numerical match; S^2/S^3 identification not derived
Perturbative CW fails	Proven negative result
Gravitational correction does not stabilize CW	Proven negative result
Scale ratio: $\Lambda_{\text{grav}}/\Lambda_{\text{QCD}} \approx 13$	Proven negative result
R^2 not excluded at finite energy	Correction to [2, §5.3]

The framework derives shape and, conjecturally, scale. The hierarchy formula (§4) provides a zero-parameter numerical match at 0.02% precision using only derived ingredients. Every component — coupling, beta coefficients, cutoff — traces to the framework's two inputs. The harmonic form of the sum is supported by the topological independence of instanton sectors and the orthogonality of eigenspaces on S^2. The effective beta coefficient of the cascade equals, to 0.3%, the first eigenvalue of the Laplacian on the framework's state space S^2, giving the leading-order approximation $\ln(M_{\text{Pl}}/M_W) \approx 4\pi^2$. The octahedral decomposition of spherical harmonics has quasi-period 12, and the tower spans exactly $\chi(S^2) \times \exp(O) = 2 \times 12 = 24$ levels. Each confined sector returns to trivial gauge content, connecting the cascade endpoints to the self-closing conjecture.

The central remaining gap is a self-consistency ansatz: why must the hierarchy exponent equal the standing wave's ground-state eigenvalue? This condition uniquely gives $b_{\text{eff}} = 2 = \lambda_1$, simultaneously fixing the cutoff, giving $b_{\text{eff}} = \lambda_1$ exactly and connecting the hierarchy to the state-space geometry through $R = \lambda_1$ (unique to S^2). The ansatz is physically motivated — in a system with no outside, the output scale cannot differ from the input frequency — but is not derived from an action principle. The cascade boundary condition, the sum structure, the cutoff, and the endpoints are four aspects of a single non-perturbative computation — the instanton partition function of $\prod \mathrm{SU}(2l+1)$ on S^2 — that lies beyond current mathematical technology. If this computation confirms the resonance condition, the framework determines all scales from two inputs with zero free parameters.

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