February 8th, 1877.

C. W. MERRIFIELD, Esq., F.R.S., Vice-President, in the Chair.

Mr. G. W. Von Tunzelmann was admitted into the Society.

Mr. A. B. Kempe moved, and Mr. Roberts seconded, a vote of thanks to Lord Rayleigh and Mr. Spottiswoode for the presents of their portraits to the Society.

The following communications were made:—

"On the Area of the Quadrangle formed by the Four Points of Intersection of Two Conics:" Mr. Leudesdorf (read in part by Mr. Tucker).

"On a Certain Series:" Mr. J. W. L. Glaisher.

"On the Equation $\frac{dx}{\surd X}+\frac{dy}{\surd Y}=0$:" Prof. Cayley.

"Classification of Loci, and a Theorem in Residuation:" Prof. Clifford.

The following presents were made to the Society:—

"Reprint from Educational Times," Vol. xxvi.

"Educational Times," Feb.

"Nautical Almanac" for 1880, presented by Lords Commissioners of Admiralty.

"Haversines, Natural and Logarithmic, used in computing Lunar Distances for the Nautical Almanac," edited by Major-General Hannyngton, F.R.A.S. 1876.

"Proceedings of the Royal Society," Vol. xxv., No. 177.

"Vorlesungen über Geometrie," von Alfred Clebsch, bearbeitet und herausgegeben von Dr. F. Lindemann, ersten Bandes zweiter Theil; Leipzig, 1876: from the Editor.

"Weitere Untersuchungen über das Ikosaeder II.," von F. Klein, Erlangen (read Jan. 15, 1877).

On the General Differential Equation $\frac{dx}{\surd X}+\frac{dy}{\surd Y}=0$, *where* X, Y *are the same Quartic Functions of* x, y *respectively. By* Prof. CAYLEY, F.R.S.

[*Read February 8th,* 1877.]

Write $\Theta = a+b\theta+c\theta^2+d\theta^3+e\theta^4$, the general quartic function of θ; and let it be required to integrate by Abel's theorem the differential equation

$$\frac{dx}{\surd X}+\frac{dy}{\surd Y}=0.$$

We have
$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix} = 0,$$
a particular integral of
$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0;$$
and consequently the above equation, taking therein z, w as constants, is the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, viz., the two constants z, w must enter in such wise that the equation contains only a single constant; whence also, attributing to w any special value, we have the general integral with z as the arbitrary constant.

Take $w=\infty$; the equation becomes
$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ 1, & 0, & 0, & \sqrt{e} \end{vmatrix} = 0,$$
a relation between x, y, z which may be otherwise expressed by means of the identity
$$e(\theta^2+\beta\theta+\gamma)^2-(e\theta^4+d\theta^3+c\theta^2+b\theta+a)$$
$$=(2\beta e-d)(\theta-x)(\theta-y)(\theta-z),$$
or, what is the same thing,
$$\begin{aligned} e(2\gamma+\beta^2)-c &= -(2\beta e-d)(x+y+z), \\ e\,2\beta\gamma\quad -b &= (2\beta e-d)(yz+zx+xy), \\ e\,\gamma^2\quad -a &= -(2\beta e-d)\,xyz, \end{aligned}$$
where β, γ are indeterminate coefficients which are to be eliminated.

Write
$$x^2-\frac{\sqrt{X}}{\sqrt{e}}=P,\quad y^2-\frac{\sqrt{Y}}{\sqrt{e}}=Q;$$
then we have
$$\begin{aligned} &\beta x+\gamma+P=0, \quad \text{giving}\quad \beta:\gamma:1=Q-P:Py-Qx:x-y; \\ &\beta y+\gamma+Q=0; \end{aligned}$$
and substituting these values in the first of the preceding three equations, we have
$$e\,\frac{2(Py-Qx)(x-y)+(Q-P)^2}{(x-y)^2}-c=-\left(\frac{2(Q-P)e}{x-y}-d\right)(x+y+z),$$
that is,
$$e\left\{\frac{2(Qy-Px)}{x-y}+\frac{(Q-P)^2}{(x-y)^2}+\frac{2(Q-P)}{x-y}z\right\}=c+d(x+y+z);$$

or, reducing by

$$Qy-Px = y^3-x^3+\frac{x\sqrt{X}-y\sqrt{Y}}{\sqrt{e}},$$

$$Q-P = y^2-x^2+\frac{\sqrt{X}-\sqrt{Y}}{\sqrt{e}}, = y^2-x^2+(y-x)\frac{M}{\sqrt{e}}, \text{ if } M=\frac{\sqrt{X}-\sqrt{Y}}{x-y},$$

this is

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}+2xy+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}-2(x+y)z+2z\frac{M}{\sqrt{e}}\right\}$$
$$= c+d(x+y+z)+e(x+y)^2.$$

We have Euler's solution in the far more simple form

$$M^2 = C+d(x+y)+e(x+y)^2,$$

where C is the arbitrary constant. It is to be observed that, in the particular case where $e=0$, the first equation becomes

$$M^2 = c+d(x+y+z)+e(x+y)^2;$$

and the two results agree on putting $C = c+dz$.

But it is required to identify the two solutions in the general case where e is not $=0$. I remark that I have, in my Treatise on Elliptic Functions, further developed the theory of Euler's solution, and have shown that, regarding C as variable, and writing

$$\mathfrak{C} = ad^2+b^2e-2bcd+C[-4ae+bd+(C-c)^2],$$

then the given equation between the variables x, y, C corresponds to the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0,$$

a result which will be useful for effecting the identification. The Abelian solution may be written

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}-x^2-y^2+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}\right\}-c-d(x+y)$$
$$= z\{d+2e(x+y)-2M\sqrt{e}\};$$

and substituting for M its value, and multiplying by $(x-y)^2$, the equation becomes

$$2\sqrt{e}(x-y)(x\sqrt{X}-y\sqrt{Y})-e(x^2+y^2)(x-y)^2+(\sqrt{X}-\sqrt{Y})^2$$
$$-2(x^2-y^2)(\sqrt{X}-\sqrt{Y})\sqrt{e}-c(x-y)^2-d(x+y)(x-y)^2$$
$$= z(x-y)\{d(x+y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}.$$

On the left-hand side the rational part is

$$X+Y+c(-x^2+2xy-y^2)+d(-x^3+x^2y+xy^2-y^3)$$
$$+e(-x^4+2x^3y-2x^2y^2+2xy^3-y^4),$$

which, substituting therein for X, Y their values, becomes

$$= 2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2);$$

and the irrational part is at once found to be

$$=2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}.$$

The equation thus is

$$z=\frac{\begin{array}{c}2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\\ +2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\end{array}}{(x-y)\,\{d\,(x-y)+2e\,(x^2-y^2)-2\,(\sqrt{X}-\sqrt{Y})\,\sqrt{e}\}},$$

which equation is thus a form of the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, and also a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$.

Multiplying the numerator and denominator by

$$d\,(x-y)+2e\,(x^2-y^2)+2\,(\sqrt{X}-\sqrt{Y})\,\sqrt{e},$$

the denominator becomes

$$=(x-y)^2\left\{d+2e\,(x+y)^2-4e\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2\right\},$$

which, introducing herein the C of Euler's equation, is

$$=(x-y)^2\,(d^2-4eC).$$

We have therefore

$$\begin{aligned} z\,(x-y)^2\,(d^2-4eC) &= \{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\\ &\qquad +2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\\ &\quad\times\{d\,(x-y)+2e\,(x^2-y^2)+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\}.\end{aligned}$$

Using $\mathfrak{C}$ to denote the same value as before, the function on the right hand is in fact

$$=(x-y)^2\,\{2be-cd+dC+2\sqrt{e}\,\sqrt{\mathfrak{C}}\};$$

and, this being so, the required relation between z, C is

$$z\,(d^2-4eC)=\{2be-cd+dC+2\sqrt{e}\,\sqrt{\mathfrak{C}}\}.$$

To prove this, we have first, from the equation

$$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=C+d\,(x+y)+e\,(x+y)^2,$$

to express $\mathfrak{C}$ as a function of x, y. This equation, regarding therein C as a variable, gives

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0;$$

and we have therefore

$$-\sqrt{\mathfrak{C}}=\sqrt{X}\,\frac{dC}{dx}=\sqrt{Y}\,\frac{dC}{dy},$$

viz., $\sqrt{X}\dfrac{dC}{dx}$ will be a symmetrical function of x, y. Putting, as before, $M=\dfrac{\sqrt{X}-\sqrt{Y}}{x-y}$, we have

$$C = M^2 - d(x+y) - e(x+y)^2,$$

and thence
$$\frac{dC}{dx} = 2M\frac{dM}{dx} - d - 2e(x+y).$$

We have
$$\frac{dM}{dx} = \frac{1}{x-y}\,\frac{X'}{2\sqrt{X}} - \frac{\sqrt{X}-\sqrt{Y}}{(x-y)^2},$$

and hence

$$\begin{aligned}\sqrt{\mathfrak{C}}\,(x-y)^3 &= -\sqrt{X}\,(x-y)^3\left\{2M\frac{dM}{dx} - d - 2e(x+y)\right\}\\ &= -(x-y)X'(\sqrt{X}-\sqrt{Y}) + 2(X+Y-2\sqrt{XY})\sqrt{X}\\ &\qquad + (d+2e\overline{x+y})(x-y)^3\sqrt{X}\\ &= [(x-y)X'+2X+2Y+(d+2e\overline{x+y})(x-y)^3]\sqrt{X}\\ &\quad + [(x-y)X'-4X]\sqrt{Y}.\end{aligned}$$

We obtain at once the coefficient of $\sqrt{Y}$, and with little more difficulty that of $\sqrt{X}$; and the result is

$$\begin{aligned}\sqrt{\mathfrak{C}}\,(x-y)^3 &= -[4a+3bx+2cx^2+dx^3+y(b+2cx+3dx^2+4ex^3)]\sqrt{Y}\\ &\quad + [4a+3by+2cy^2+dy^3+x(b+2cy+3dy^2+4ey^3)]\sqrt{X}.\end{aligned}$$

We have also

$$\begin{aligned}C(x-y)^2 &= (\sqrt{X}-\sqrt{Y})^2 - d(x+y)(x-y)^2 - e(x+y)^2(x-y)^2\\ &= X+Y-d(x^3-x^2y-xy^2+y^3) - e(x^4-2x^2y^2+y^4) - 2\sqrt{XY}\\ &= 2a+b(x+y)+c(x^2+y^2)+d\,xy(x+y)+2e\,x^2y^2-2\sqrt{XY},\end{aligned}$$

or say

$$\begin{aligned}C(x-y)^3 &= 2a(x-y)+b(x^2-y^2)+c(x^3-x^2y+xy^2-y^3)+d\,xy(x^2-y^2)\\ &\qquad + 2e\,x^2y^2(x-y) - 2(x-y)\sqrt{XY},\end{aligned}$$

and we can hence form the expression of

$$(x-y)^3\,\{2be-cd+dC+2\sqrt{e}\sqrt{\mathfrak{C}}\},$$

viz., this is

$$\begin{aligned}&= (2be-cd)(x-y)^3\\ &\quad + 2ad(x-y)+bd(x^2-y^2)+cd(x^3-x^2y+xy^2-y^3)+d^2xy(x^2-y^2)\\ &\qquad\qquad + 2de\,x^2y^2(x-y) - 2d(x-y)\sqrt{XY}\\ &\quad + 2\sqrt{e}\,\{[-(4a+3bx+2cx^2+dx^3)-y(b+2cx+3dx^2+4ex^3)]\sqrt{Y}\\ &\qquad + [(4a+3by+2cy^2+dy^3)+x(b+2cy+3dy^2+4ey^3)]\sqrt{X}\},\end{aligned}$$

and this should be

$$\begin{aligned}&= \{2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2)\\ &\qquad + 2\sqrt{e}\,(x-y)(x\sqrt{Y}-y\sqrt{X}) - 2\sqrt{XY}\}\\ &\quad\times\{d(x-y)+2e(x^2-y^2)+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\}.\end{aligned}$$

The function on the right hand is in fact

$$= \{2a+b(x+y)+c.2xy+d\,xy(x+y)+e.2xy(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$\times\{d(x-y)+2e(x^2-y^2)\}$$
$$+4e(x-y)(\sqrt{X}-\sqrt{Y})(x\sqrt{Y}-y\sqrt{X})$$
$$+2\sqrt{e}(\sqrt{X}-\sqrt{Y})\{2a+b(x+y)+c.2xy+d\,xy(x+y)$$
$$+e.2xy(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$+2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})\{d(x-y)+2e(x^2-y^2)\},$$

viz., this is

$$= \{2a+b(x+y)+c.2xy+d\,xy(x+y)+e.2xy(x^2-xy+y^2)\}$$
$$\times[d(x-y)+2e(x^2-y^2)]+4e(x-y)(-xY-yX)$$
$$-2\sqrt{XY}[d(x-y)+2e(x^2-y^2)]+4e(x-y)(x+y)\sqrt{XY}$$
$$+2\sqrt{e}\left\{\begin{array}{l}\sqrt{X}\{2a+b(x+y)+c.2xy+d\,xy(x+y)+e.2xy(x^2-xy+y^2)\\ \qquad +2Y-(x-y)y[d(x-y)+2e(x^2-y^2)]\}\\ -\sqrt{Y}\{2a+b(x+y)+c.2xy+d\,xy(x+y)+e.2xy(x^2-xy+y^2)\\ \qquad +2X-(x-y)x[d(x-y)+2e(x^2-y^2)]\}\end{array}\right\}$$

which is, in fact, equal to the expression on the left-hand side.

To complete the theory, we require to express $\sqrt{Z}$ as a function of x, y. It would be impracticable to effect this by direct substitution of the foregoing value of z; but, observing that the value in question is a solution of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$, or, what is the same thing, that $\frac{1}{\sqrt{X}}+\frac{1}{\sqrt{Z}}\frac{dz}{dx}=0$, $\frac{1}{\sqrt{Y}}+\frac{1}{\sqrt{Z}}\frac{dz}{dy}=0$, we can from either of these equations, considering therein z as a given function of x, y, calculate $\sqrt{Z}$.

Writing for shortness

$$z=\frac{J-2\sqrt{e}\,y(x-y)\sqrt{X}+2\sqrt{e}\,x(x-y)\sqrt{Y}-2\sqrt{XY}}{R-2\sqrt{e}(x-y)\sqrt{X}+2\sqrt{e}(x-y)\sqrt{Y}},$$

where $\quad R=(x-y)^2\{d+2e(x+y)\},$

$$J=2a+b(x+y)+2c\,xy+d\,xy(x+y)+2e\,xy(x^2-xy+y^2);$$

or, if for a moment $z=\frac{N}{D}$, then

$$\frac{dz}{dx}=\frac{1}{D^2}\left(D\frac{dN}{dx}-N\frac{dD}{dx}\right)=-\frac{\sqrt{Z}}{\sqrt{X}},$$

that is, $\quad \sqrt{Z}=\frac{\sqrt{X}}{D^2}\left(N\frac{dD}{dx}-D\frac{dN}{dx}\right), =\frac{\Omega}{D^2}$ suppose;

or writing for shortness X′, R′, J to denote the derived functions

$\frac{dX}{dx}, \frac{dR}{dx}, \frac{dJ}{dx}$ (Y′ is afterwards written to denote $\frac{dY}{dy}$, but as the final formulæ contain only X′, $= \frac{X}{dx}$, and Y′, $= \frac{dY}{dy}$, this does not occasion any defect of symmetry), we find

$$\begin{aligned}\Omega = \; & N\{R'\sqrt{X}-2\sqrt{e}\,X-\sqrt{e}\,(x-y)\,X'+2\sqrt{e}\,\sqrt{XY}\} \\ & -D\{J'\sqrt{X}-2\sqrt{e}\,yX-\sqrt{e}\,(x-y)\,yX' \\ & \qquad +2\sqrt{e}\,(2x-y)\sqrt{XY}-X'\sqrt{Y}\};\end{aligned}$$

and substituting herein for N, D their values, and arranging the terms, we find

$$\Omega = \sqrt{e}\,\mathfrak{A}+\mathfrak{B}\sqrt{X}+\mathfrak{C}\sqrt{Y}+\sqrt{e}\,\mathfrak{D}\sqrt{XY},$$

where

$$\begin{aligned}\mathfrak{A} = & -J\{2X+(x-y)X'\} \\ & -2(x-y)\,yR'X \\ & -4XY \\ & +Ry\{2X+(x-y)X'\} \\ & +2(x-y)XJ' \\ & +2(x-y)X'Y,\end{aligned} \qquad \begin{aligned}\mathfrak{B} = \; & JR' \\ & +2e(x-y)y\{2X+(x-y)X'\} \\ & +4ex(x-y)Y \\ & -RJ' \\ & -2e(x-y)y\{2X+(x-y)X'\} \\ & -4e(x-y)(2x-y)Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{C} = & -4ey(x-y)X \\ & -2e(x-y)x\{2X+(x-y)X'\} \\ & -2R'X \\ & +RX' \\ & +2e(x-y)y\{2X+(x-y)X'\} \\ & +4e(x-y)(2x-y)X,\end{aligned} \qquad \begin{aligned}\mathfrak{D} = \; & 2J \\ & +2(x-y)xR' \\ & +2\{2X+(x-y)X'\} \\ & -2(2x-y)R \\ & -2(x-y)X' \\ & -2(x-y)J',\end{aligned}$$

where the terms have been written down as they immediately present themselves, but collecting and arranging, we have

$$\begin{aligned}\mathfrak{A} &= 2X(-J+Ry-2Y)+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\} \\ \mathfrak{B} &= JR'-J'R-4e(x-y)^2Y, \\ \mathfrak{C} &= -2XR'+X'R+4e(x-y)^2X-2e(x-y)^3X', \\ \mathfrak{D} &= 2J+4X-2Rx+2(x-y)(xR'-R-J').\end{aligned}$$

For reducing these expressions, writing

$$\begin{aligned}M &= d+2e(x+y), \\ \Lambda &= c+d(x+y)+e(x^2+y^2),\end{aligned}$$

we have $R=(x-y)^2M$, and therefore $R'=2(x-y)M+2e(x-y)^2$; also

$$J = X+Y-(x-y)^2\Lambda;$$

also, from the original form,

$$J' = b+2cy+d(2xy+y^2)+e(6x^2y-4xy^3+2y^3).$$

The final values are

$$\begin{aligned}
\mathfrak{A} &= \quad -X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\},\\
\mathfrak{B} &= \quad (x-y)M\{4Y+(x-y)Y'\}+2e(x-y)^3Y',\\
\mathfrak{C} &= -(x-y)M\{4X-(x-y)X'\}-2e(x-y)^3X',\\
\mathfrak{D} &= \quad 4(X+Y)+4e(x-y)^4,
\end{aligned}$$

which, once obtained, may be verified without difficulty.

Verification of $\mathfrak{A}$.—The equation is

$$\begin{aligned}
-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\}\\
= 2X(-J+Ry-2Y)\\
+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};
\end{aligned}$$

or, putting for shortness $\Lambda^2+(-b+dxy)M+xyM^2=\nabla$, this is

$$\begin{aligned}
(x-y)^4\nabla &= X^2+6XY+Y^2\\
&\quad +2X\{-X-3Y+(x-y)^2\Lambda+(x-y)^2yM\}\\
&\quad +(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\},\\
&= -X^2+Y^2+2(x-y)^2X\Lambda+2(x-y)^2yXM\\
&\quad +(x-y)\{2XJ+2X'Y-X'J-2yR'X+yRX'\};
\end{aligned}$$

we have $-X^2+Y^2=-(X-Y)(X+Y)$, where $X-Y$ divides by $x-y$, $=(x-y)\Omega$ suppose; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}
(x-y)^3\nabla &= -\Omega(X+Y)+2(x-y)X\Lambda+2(x-y)yXM\\
&\quad +2XJ'+2X'Y-X'\{X+Y-(x-y)^2\Lambda\}\\
&\quad -2yX\{2(x-y)M+2(x-y)^2e\}+(x-y)^2yMX',\\
&= -\Omega(X+Y)+2XJ'-X'(X-Y)\\
&\quad +2(x-y)X\Lambda-2(x-y)yXM\\
&\quad +(x-y)^2X'\Lambda-4(x-y^2)eyX+(x-y)^2yMX'.
\end{aligned}$$

We have $2XJ'=J'(X+Y)+J'(X-Y)$, and hence the first line is $=(-\Omega+J')(X+Y)+J'(X-Y)$; $-\Omega+J'$, as will be shown, divides by $x-y$, or say it is $=(x-y)\Phi$, and, as before, $X-Y$ is $=(x-y)\Omega$; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}
(x-y)^2\nabla &= \Phi(X+Y)+\Omega(J'-X')+2X\Lambda-2yXM\\
&\quad +(x-y)\{X'\Lambda-4eyX+yMX'\}.
\end{aligned}$$

We have

$$\Omega=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3),$$

and thence

$$-\Omega+J'=c(-x+y)+d(-x^2+xy)+e(-x^3+5x^2y-5xy^2+y^3);$$

or, dividing this by $(x-y)$, we find

$$\Phi = -c-dx-e\,(x^2-4xy+y^2),$$

or, as this may be written, $\Phi = -\Lambda+dy+4exy$. We find, moreover,

$$J'-X' = 2c(-x+y)+d(-3x^2+2xy+y^2)+e(-4x^3+6x^2y-4xy^2+2y^3),$$

which divides by $(x-y)$, the quotient being

$$-2c-d\,(3x+y)-e\,(4x^2-2xy+2y^2),$$

viz., this is $\qquad = -2\Lambda-(x-y)\,(d+2ex).$

Hence the equation is

$$\begin{aligned}(x-y)^2\,\nabla = (X+Y)\,\{-\Lambda+dy+4exy\}+2X\Lambda-2yXM \\ +(x-y)\,\Omega\,\{-2\Lambda-(x-y)\,(d+2ex)\} \\ +(x-y)\quad\{\ X'\Lambda-4eyX+yMX'\ \}.\end{aligned}$$

The first line is $(X+Y)\,\{-\Lambda+yM+2\,(x-y)\,ye\}+2X\Lambda-2yXM$, which is $=(\Lambda-yM)\,(X-Y)+2\,(x-y)\,ey\,(X+Y)$; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}(x-y)\,\nabla &= (\Lambda-yM)\,\Omega+2ey\,(X+Y)-2\Lambda\Omega+X'\Lambda-4eyX+yMX' \\ &\qquad -(x-y)\,\Omega\,(d+2ex) \\ &= (\Lambda+yM)\,(-\Omega+X')-2ey\,(X-Y)-(x-y)\,\Omega\,(d+2ex).\end{aligned}$$

We have $-\Omega+X' = c(x-y)+d(2x^2-xy-y^2)+e(3x^3-x^2y-xy^2-y^3)$, which is $=(x-y)\,(\Lambda+xM)$; also $(X-Y)=(x-y)\,\Omega$, as before, whence, throwing out $x-y$, the equation is

$$\nabla = (\Lambda+xM)\,(\Lambda+yM)-2ey\Omega-(d+2ex)\,\Omega,$$

that is, $\qquad \nabla = (\Lambda+xM)\,(\Lambda+yM)-M\Omega\,;$

viz., substituting for ∇ its value, reducing, and throwing out the factor M, the equation becomes

$$-b+dxy = (x+y)\,\Lambda-\Omega,$$

which is right.

Verification of $\mathfrak{B}$.—The equation is

$$\begin{aligned}J\,\{2\,(x-y)\,M+2e\,(x-y)^2\}-J'\,(x-y)^2\,M-4e\,(x-y)^2\,Y \\ = 4\,(x-y)\,MY+(x-y)^2\,MY'+2e\,(x-y)^3\,Y',\end{aligned}$$

which, throwing out $(x-y)$, is

$$\begin{aligned}0 = 2M\,(-J+2Y)+(x-y)\,M\,(J'+Y') \\ +2e\,(x-y)(-J+2Y)+2e\,(x-y)^2\,Y'.\end{aligned}$$

Here $-J+2Y, = -(X-Y)+(x-y)^2\Lambda$ divides by $(x-y)$: hence, throwing out the factor $x-y$, the equation is

$$0=M\{-2b-2c(x+y)-2d(x^2+xy+y^2)-2e(x^3+x^2y+xy^2+y^3)\}$$
$$+M(J'+Y')+2M(x-y)\Lambda+2e(-J+2Y)+2e(x-y)Y'.$$

In the first and second terms the factor which multiplies M is

$$c(-2x+2y)+d(-2x^2+2y^2)+e(-2x^3+4x^2y-6xy^2+4y^3),$$

which divides by $x-y$; also $-J+2Y$, $=-(X-Y)+(x-y)^2\Lambda$, divides by $(x-y)$: hence, throwing this factor out, the equation is

$$0=M\{-2c+d(-2x-2y)+e(-2x^2+2xy-4y^2)\}+2M\Lambda$$
$$+2e\{-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)\}$$
$$+2e(x-y)\Lambda+2eY'.$$

Here in the top line the coefficient of M is $=e(2xy-2y^2)$: hence, throwing out the constant factor $2e$, the equation is

$$0=-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)+Y'$$
$$+(x-y)yM+(x-y)\Lambda.$$

The top line is

$$=c(-x+y)+d(-x^2-xy+2y^2)+e(-x^3-x^2y-xy^2+3y^3),$$

which divides by $x-y$; throwing out this factor, the equation is

$$0=-c-d(x+2y)-e(x^2+2xy+3y^2)+\Lambda+yM,$$

which is right.

Verification of 𝕮.—We have

$$-2X\{2(x-y)M+2e(x-y)^2\}$$
$$+(x-y)^2X'M+4e(x-y)^3X-2e(x-y)^4X'$$
$$=-(x-y)M\{4X-(x-y)X'\}-2e(x-y)^4X',$$

which is in fact an identity.

Verification of 𝕯.—The equation may be written

$$4X+4Y+4e(x-y)^4$$
$$=2X+2Y-2(x-y)^2\Lambda$$
$$+4X-2x(x-y)^2M$$
$$+2(x-y)\{2(x-y)xM+2ex(x-y)^2-M(x-y)^2-J'\},$$

viz., this is

$$0=2X-2Y-4e(x-y)^4-2(x-y)^2\Lambda+2x(x-y)^2M$$
$$+4ex(x-y)^3-2M(x-y)^3-2(x-y)J'.$$

The first term $2(X-Y)$ divides by $2(x-y)$, or, throwing this factor out, the equation becomes

$$0=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3)-J'$$
$$-2e(x-y)^3-(x-y)\Lambda+x(x-y)M+2ex(x-y)^2-M(x-y)^2.$$

Substituting for J′ its value, the first line becomes

$$c(x-y)+d(x^2-xy)+e(x^3-5x^2y+5xy^2-y^3),$$

which divides by $(x-y)$; hence, throwing out this factor, the equation is

$$0 = c+dx+e(x^2-4xy+y^2)-\Lambda+xM$$
$$-2e(x-y)^2+2ex(x-y)-M(x-y),$$

where the top line is $= d(x-y)+e(2x^2-2xy)$: hence, again throwing out the factor $x-y$, the equation becomes

$$0 = d+2ex-2e(x-y)+2ex-M,$$

which is right.

Recapitulating, we have for the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, or for a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$,

$$z = \frac{J-2\sqrt{e}(x-y)y\sqrt{X}+2\sqrt{e}(x-y)x\sqrt{Y}-2\sqrt{XY}}{(x-y)^2M-2\sqrt{e}(x-y)\sqrt{X}+2\sqrt{e}(x-y)\sqrt{Y}},$$

the corresponding value of $\sqrt{Z}$ being

$$\sqrt{Z} = \frac{\begin{array}{l}\sqrt{e}[-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\}]\\ +[\{4Y+(x-y)Y'\}M+2e(x-y)^2Y'](x-y)\sqrt{X}\\ -[\{4X-(x-y)X'\}M+2e(x-y)^2X'](x-y)\sqrt{Y}\\ +[4(X+Y)+4e(x-y)^4]\sqrt{XY}\end{array}}{\{(x-y)^2M-2\sqrt{e}(x-y)\sqrt{X}+2\sqrt{e}(x-y)\sqrt{Y}\}^2},$$

where, as before,

$$M = d+2e(x+y),$$
$$\Lambda = c+d(x+y)+e(x^2+y^2),$$
$$J = 2a+b(x+y)+2cxy+dxy(x+y)+exy(x^2-xy+y^2).$$

X is the general quartic function $a+bx+cx^2+dx^3+ex^4$, and Y, Z are the same functions of y, z respectively.

In connexion with what precedes, I give some investigations relating to the more simple form $\Theta = a+c\theta^2+e\theta^4$, or, as it will be convenient to write it, $\Theta = 1-l\theta^2+\theta^4$.

We have

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y}\end{vmatrix} = 0, \text{ a particular integral} \left.\right\} \text{ of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z}\end{vmatrix} = 0 \text{ the general integral (of the above); a particular integral} \left.\right\} \text{ of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0,$$

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W}\end{vmatrix} = 0 \text{ the general integral (of the above); a particular integral} \left.\right\} \text{ of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

$\ldots \quad \ldots$

and so on; viz., in taking

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0 \text{ as the general integral of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

we consider z as the constant of integration: and so in other cases.

It is to be remarked that it is an essentially different problem to verify a particular integral and to verify a general integral, and that the former is the more difficult one. In fact, if $U=0$ is a particular integral of the differential equation $M dx+N dy=0$, then we must have $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, not identically but in virtue of the relation $U=0$, or we have to consider whether two given relations between x and y are in fact one and the same relation. In the case of a general solution, this is theoretically reducible to the form $c=U$, c being the constant of integration, and we have then the equation $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, satisfied identically, or, what is the same thing, U a solution of this partial differential equation.

Hence it is theoretically easier to verify that

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0$$

is a general solution, than to verify that

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix} = 0$$

is a particular solution of the differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$. Moreover, writing the first equation in the before mentioned form

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}},$$

and writing therein $z=\infty$, we see that the second equation $\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix}=0$ is in fact a particular case of the first equation, so that we only require to verify the first equation; or, what is the same thing, to verify that

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}}$$

is the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}$.

To verify this we have to show that $dz=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}\right)$, viz., that

$\sqrt{X}\dfrac{dz}{dx}=\Omega$, a symmetrical function of (x, y); for then $\sqrt{Y}\dfrac{dz}{dy}=\Omega$, and we have the relation in question.

We have

$$(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}$$
$$=\sqrt{X}\left\{(x^2-y^2)\left(\sqrt{Y}-\frac{yX'}{2\sqrt{X}}\right)-2x\,(x\sqrt{Y}-y\sqrt{X})\right\}$$
$$=\sqrt{X}\left\{(x^2-y^2-2x^2)\sqrt{Y}-\frac{(x^2-y^2)\,yX'}{2\sqrt{X}}+2xy\sqrt{X}\right\}$$
$$=-(x^2+y^2)\sqrt{XY}+2xyX-\tfrac{1}{2}(x^2-y^2)\,yX'.$$

Writing here $X=1-lx^2+x^4$, then $X'=-2lx+4x^3$, and we have

$$2xy\,(1-lx^2+x^4)+(x^2-y^2)\,xy\,(l-2x^2),$$
$$=\;xy\,\{2-2lx^2+2x^4+(x^2-y^2)(l-2x^2)\},$$
$$=\;xy\,\{2-l\,(x^2+y^2)+2x^2y^2\}.$$

Hence the equation is

$$(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}=-(x^2+y^2)\sqrt{XY}+xy\,\{2-l\,(x^2+y^2)+2x^2y^2\},$$

or we have

$$\Omega=\frac{1}{(x\sqrt{Y}-y\sqrt{X})^2}\{-(x^2+y^2)\sqrt{XY}+xy\,(2-l\,(x^2+y^2)+2x^2y^2)\},$$

which is symmetrical in (x, y), as it should be. And observe, further, that since the equation is a particular solution of $\dfrac{dx}{\sqrt{X}}+\dfrac{dy}{\sqrt{Y}}+\dfrac{dz}{\sqrt{Z}}=0$, we must have $\Omega=-\sqrt{Z}$; viz., we have

$$\sqrt{Z}\,(x\sqrt{Y}-y\sqrt{X})^2=-(x^2+y^2)\sqrt{XY}+xy\,\{2-l\,(x^2+y^2)+2x^2y^2\}.$$

Proceeding to the next case, where we have between x, y, z, w a relation which may be written

$$(x^3,\; x,\; x^2\sqrt{X},\; \sqrt{X})=0,$$

then here a, b, c, d can be determined so that

$$(c\theta^2+d)^2(1+\beta\theta^2+\gamma\theta^4)-(a\theta^3+b\theta)^2$$
$$=c^2\gamma\,(\theta^2-x^2)(\theta^2-y^2)(\theta^2-z^2)(\theta^2-w^2),$$

viz., we have $d^2=c^2\gamma\, x^2y^2z^2w^2$, or say $d=c\sqrt{\gamma}\,xyzw$.

And, supposing the ratios of a, b, c, d determined by the three equations which contain (x, y, z) respectively, we have

$$a:b:c:d$$
$$=(x,\, x^2\sqrt{X},\, \sqrt{X}):-(x^3,\, x^2\sqrt{X},\, \sqrt{X}):(x^3,\, x,\, \sqrt{X}):-(x^3,\, x,\, x^2\sqrt{X}),$$

or in particular

$$\frac{d}{c}=\frac{-(x^3,\,x,\,x^2\sqrt{X})}{(x^3,\,x,\,\sqrt{X})},\quad =\frac{-xyz\,.\,(x^2,\,1,\,x\sqrt{X})}{(x^3,\,x,\,\sqrt{X})};$$

whence we have
$$w=-\frac{(x^2,\,1,\,x\sqrt{X})}{(x^3,\,x,\,\sqrt{X})}$$

as a new form of the integral equation; viz., written at full length, this is

$$-w=\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix};$$

and taking $w=0$ and $=\infty$ respectively, we thus see how

$$\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix}=0,\qquad \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0,$$

are each of them a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0.$

Reverting to the general form

$$w=-\frac{(x^2,\,1,\,x\sqrt{X})}{(x^3,\,x,\,\sqrt{X})},$$

this will be a general integral if only

$$dw=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}\right),$$

viz., if we have

$$-\sqrt{X}\,\frac{d}{dx}\,\frac{(x^2,\,1,\,x\sqrt{X})}{(x^3,\,x,\,\sqrt{X})}=\Omega,\ \text{a symmetrical function of } (x,y,z).$$

The expression is

$$\Omega=\frac{1}{(x^3,\,x,\,\sqrt{X})^2}\left\{(x^2,\,1,\,x\sqrt{X})\sqrt{X}\frac{d}{dx}(x^3,\,x,\,\sqrt{X}) - (x^3,\,x,\,\sqrt{X})\sqrt{X}\frac{d}{dx}(x^2,\,1,\,x\sqrt{X})\right\},$$

or, writing for shortness

$$\begin{aligned} \alpha &= x\,(y^2-z^2), & a &= yz\,(y^2-z^2),\\ \beta &= y\,(z^2-x^2), & b &= zx\,(z^2-x^2),\\ \gamma &= z\,(x^2-y^2), & c &= xy\,(x^2-y^2), \end{aligned}$$

we have
$$\begin{aligned} (x^2,\,1,\,x\sqrt{X}) &= \alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z},\\ (x^3,\,x,\,\sqrt{X}) &= a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z}; \end{aligned}$$

and the formula is

$$
\begin{aligned}
&(x^2,\ x,\ \sqrt{\mathrm{X}})^2\Omega \\
&\quad = (\alpha\sqrt{\mathrm{X}}+\beta\sqrt{\mathrm{Y}}+\gamma\sqrt{\mathrm{Z}})\,\{(y^3z-yz^3)\,\tfrac{1}{2}\mathrm{X}' \\
&\qquad\qquad +(-3x^2z+z^3)\sqrt{\mathrm{XY}}+(3x^2y-y^3)\sqrt{\mathrm{XZ}}\} \\
&\qquad -(a\sqrt{\mathrm{X}}+b\sqrt{\mathrm{Y}}+c\sqrt{\mathrm{Z}})\,\{(y^2-z^2)(\mathrm{X}+\tfrac{1}{2}\mathrm{X}'x) \\
&\qquad\qquad -2xy\sqrt{\mathrm{XY}}-2xz\sqrt{\mathrm{XZ}}\} \\
&\quad = (\alpha\sqrt{\mathrm{X}}+\beta\sqrt{\mathrm{Y}}+\gamma\sqrt{\mathrm{Z}})(\mathrm{L}+\mathrm{M}\sqrt{\mathrm{XY}}+\mathrm{N}\sqrt{\mathrm{XZ}}) \\
&\qquad -(a\sqrt{\mathrm{X}}+b\sqrt{\mathrm{Y}}+c\sqrt{\mathrm{Z}})(\mathrm{P}+\mathrm{Q}\sqrt{\mathrm{XY}}+\mathrm{R}\sqrt{\mathrm{XZ}}), \text{ suppose,}
\end{aligned}
$$

=

$\sqrt{\mathrm{X}}$	$+\sqrt{\mathrm{Y}}$	$+\sqrt{\mathrm{Z}}$	$+\sqrt{\mathrm{XYZ}}$
$\alpha\mathrm{L}$	$+\alpha\mathrm{MX}$	$+\alpha\mathrm{NX}$	
$+\beta\mathrm{MY}$	$+\beta\mathrm{L}$		$+\beta\mathrm{N}$
$+\gamma\mathrm{NZ}$		$+\gamma\mathrm{L}$	$+\gamma\mathrm{M}$
$-a\mathrm{P}$	$-a\mathrm{QX}$	$-a\mathrm{RX}$	
$-b\mathrm{QY}$	$-b\mathrm{P}$		$-b\mathrm{R}$
$-c\mathrm{RZ}$		$-c\mathrm{P}$	$-c\mathrm{Q}$

;

viz., this is

$$
\begin{aligned}
&= \{\alpha\mathrm{L}-a\mathrm{P}+\mathrm{Y}\,(\beta\mathrm{M}-b\mathrm{Q})+\mathrm{Z}\,(\gamma\mathrm{N}-c\mathrm{R})\}\sqrt{\mathrm{X}} \\
&\quad +\{\mathrm{X}\,(\alpha\mathrm{M}-a\mathrm{Q})+\beta\mathrm{L}-b\mathrm{P}\}\sqrt{\mathrm{Y}} \\
&\quad +\{\mathrm{X}\,(\alpha\mathrm{N}-a\mathrm{R})+\gamma\mathrm{L}-c\mathrm{P}\}\sqrt{\mathrm{Z}} \\
&\quad +(\beta\mathrm{N}+\gamma\mathrm{M}-b\mathrm{R}-c\mathrm{Q})\sqrt{\mathrm{XYZ}}.
\end{aligned}
$$

The coefficient of $\sqrt{\mathrm{XYZ}}$ is here

$$
\begin{aligned}
= \ & y\,(z^2-x^2)(3x^2y-y^3) \quad , & = \ & y^2\,(z^2-x^2)(3x^2-y^2) \\
& +z\,(x^2-y^2)(-3x^2z+z^3) & & +z^2\,(x^2-y^2)(-3x^2+z^2) \\
& -zx\,(z^2-x^2)(2xz) & & -2x^2z^2\,(z^2-x^2) \\
& -xy\,(x^2-y^2)(-2xy) & & +2x^2y^2\,(x^2-y^2);
\end{aligned}
$$

which is $= 6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^4y^2-x^2y^4.$

The coefficient of $\sqrt{\mathrm{Y}}$ is

$$
\begin{aligned}
&= [x\,(y^2-z^2)(-3x^2z+z^3)+yz\,(y^2-z^2)\,2xy]\,\mathrm{X} \\
&\quad +y\,(z^2-x^2)\,\tfrac{1}{2}\mathrm{X}'.\,(y^3z-yz^3)-zx\,(z^2-x^2)(y^2-z^2)(\mathrm{X}+\tfrac{1}{2}\mathrm{X}'x) \\
&= -2xz\,(x^2-y^2)(y^2-z^2)\,\mathrm{X}-z\,(x^2-y^2)(y^2-z^2)(z^2-x^2)\,\tfrac{1}{2}\mathrm{X}' \\
&= -(x^2-y^2)(y^2-z^2)\,z\,\{2x\mathrm{X}+\tfrac{1}{2}\,(z^2-x^2)\,\mathrm{X}'\},
\end{aligned}
$$

where the term in { } is

$$
\begin{aligned}
&= 2x\,(1-lx^2+x^4)+(z^2-x^2)(-lx+2x^3), \\
&= x\,\{2-l\,(z^2+x^2)+2z^2x^2\},
\end{aligned}
$$

or the whole coefficient is

$$= -(x^2-y^2)(y^2-z^2)\,zx\,\{2-l\,(z^2+x^2)+2z^2x^2\}.$$

We obtain in like manner the coefficient of $\sqrt{Z}$, and with a little more trouble that of $\sqrt{X}$; and the final result is

$$\begin{aligned}\Omega\,(x^3, x, \sqrt{X})^2 = &-(z^2-x^2)(x^2-y^2)\,yz\,\{2-l\,(y^2+z^2)+2y^2z^2\}\,\sqrt{X}\\ &-(x^2-y^2)(y^2-z^2)\,zx\,\{2-l\,(z^2+x^2)+2z^2x^2\}\,\sqrt{Y}\\ &-(y^2-z^2)(z^2-x^2)\,xy\,\{2-l\,(x^2+y^2)+2x^2y^2\}\,\sqrt{Z}\\ &+(6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^2y^4-x^4y^2)\,\sqrt{XYZ}.\end{aligned}$$

And inasmuch as the equation is a solution of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

it follows that $\Omega=-\sqrt{W}$, viz., that $\sqrt{W}$ is by the foregoing equation expressed as a function of x, y, z.

The equation $(x^3, x, x^2\sqrt{X}, \sqrt{X})=0$, that is,

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X}\\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y}\\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z}\\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W}\end{vmatrix}=0,$$

gives

$$w=\frac{(x^2, 1, x\sqrt{X})}{(x^3, x, \sqrt{X})},$$

where the numerator and denominator are determinants formed with the x, y, z.

Writing $\frac{1}{w}$ for w, it follows that the equation

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X}\\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y}\\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z}\\ w, & w^3, & \sqrt{W}, & w^2\sqrt{W}\end{vmatrix}=0$$

gives

$$w=\frac{(x^3, x, \sqrt{X})}{(x^2, 1, x\sqrt{X})},$$

which last is a transformation of

$$\begin{vmatrix} x^4, & x^2, & 1, & x\sqrt{X}\\ y^4, & y^2, & 1, & y\sqrt{Y}\\ z^4, & z^2, & 1, & z\sqrt{Z}\\ w^4, & w^2, & 1, & w\sqrt{W}\end{vmatrix}=0.$$

The two equations involving these determinants of the order 4 are consequently equivalent equations.