SCIENCE:

PUBLISHED BY N. D. C. HODGES, 874 BROADWAY, NEW YORK.

SUBSCRIPTIONS.—United States and Canada.................$3.50 a year.

Great Britain and Europe.................4.50 a year.

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NON-EUCLIDEAN GEOMETRY.

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EUCLID'S elementary geometry was written about three centuries before the Christian era. We must conclude that it was much superior to all preceding works on this subject. Proclus, who wrote a commentary on Euclid's Elements in the fifth century of our era, represents it such, and his statements are corroborated by the facts that all similar works of Euclid's predecessors have ceased to exist, and, if any elementary geometry was written by a Greek after Euclid, there is no mention made of this anywhere.[1]

The facts that Euclid's Elements are still used as a text-book — especially in England — and that the works used in its place are generally based upon it, are perhaps still stronger evidences of its excellence.

No geometry can be written without making some assumptions with respect to the space with which it deals. These are generally of such a nature as to commend themselves to our full confidence by their mere mention, and are commonly called axioms. It is the duty of the geometer to demonstrate properties and relations of magnitudes by non-contradictory statements which rest ultimately upon these axioms. It is evident that the axioms should be as few and as clear as possible. Upon essentially different axioms essentially different geometries may be established.

Among the axioms of Euclid there is at least one which is not axiomatic.[2] This is the axiom of parallels, which reads as follows: —

"If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right-angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right-angles."

All the popular text-books on elementary geometry employ this axiom either in this form or in some shorter form, such as, "Through a point without a line only one line can be drawn parallel to the given line."

Many efforts have been made to demonstrate this axiom. Since it does not depend upon more elementary axioms, such attempts must be futile. If we assume it to be true, it follows directly that the sum of the three angles of a plane triangle is two right-angles; and, conversely, if we should assume that the sum of the internal angles of a plane triangle is two right-angles, this axiom would follow.[3]

As the geometers who do not adopt all the axioms of Euclid deny this, non-Euclidean geometry is sometimes defined as the geometry which does not assume that the sum of the three angles of a plane triangle is two right-angles. A more satisfactory definition is, non-Euclidean geometry is a geometry which assumes other properties of space in place of the following properties of Euclidean space: —

The sum of the three angles of a plane triangle is two right-angles, space is an infinite continuity of three dimensions, and rigid bodies may be moved in every way in space without change of form.

Just one hundred years ago (1792) the famous mathematician Gauss began the study of a geometry free from the first of these assumptions. He did not publish the results of his study. We may infer something in regard to them from his letters.[4] It was not until 1840 that a geometry was published in which Euclid's axiom of parallels was replaced by another, and the sum of the angles of a plane finite triangle was thus shown to be less than two right-angles. The work was written by a Russian mathematician named Lobatschewsky. It contains only sixty-one pages and bears the title "Geometrische Untersuchungen zur Theorie der Parallellinien." He began his treatment of parallels by observations, in substance, as follows: —

Given a fixed line (L) and a fixed point (A) not on this line. The lines through A lying in the plane determined by A and L may be divided with respect to L into two classes — (1) those intersecting L, and (2) those not intersecting L. The assumption that the second class consists of the single line which is at right-angles with the perpendicular from A to L is the foundation of a great part of the ordinary geometry and plane trigonometry. While the assumption that the second class consists of more than one line leads to a newer geometry, whose results are also free from contradictions.[5] This newer geometry was called non-Euclidean geometry by Gauss, imaginary geometry by Lobatschewsky, and absolute geometry by Johann Bolyai.[6]

It is certainly of interest to learn what some of the foremost mathematicians have said with respect to this geometry. Professor Sylvester said in regard to Lobatschewsky's work: —

"In quaternions the example has been given of algebra released from the yoke of the commutative principle of multiplication — an emancipation somewhat akin to Lobatschewsky's of geometry from Euclid's noted empirical axiom."

Professor Cayley said: —

"It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was not assumed to hold good, or, say, a system of non-Euclidean plane geometry."

Another very eminent mathematician, Professor Clifford, in speaking about the same work, said: —

"What Vesalius was to Galen, what Copernicus was to Ptolemy, that was Lobatschewsky to Euclid."

Something of the nature of this geometry may be inferred from a few of its theorems which differ from the corresponding theorems of the ordinary geometry. In addition to the important theorem that the sum of the internal angles of a plane finite triangle is less than two right-angles, it is proved that if we have given a line (L) and a perpendicular (B) to L, the parallels to L through points on B will make angles with B varying from $\frac{\pi}{2}$ to 0; so that we can draw through B a parallel to L making any given angle with B.[7]

The locus of a point at a constant distance from a straight line is a curved line.[8]

The areas of two plane triangles are to each other in the ratio of the excesses of two right-angles over the sums of their angles.[9]

We proceed now to some observations on the second property of Euclidean space mentioned above, viz., that space is an infinite continuity of three dimensions. We shall not take up the question of the infinitude of space nor Riemann's distinction between

1 Cantor's Vorlesungen über Geschichte der Mathematik, Vol. I., p. 224.

2 Encyclopædia Britannica, Vol. VIII., p. 657.

3 Frischauf's Absolute Geometrie, pp. 14, 15.

4 Briefwechsel zwischen Gauss und Schumacher,— especially Vol. II., pp. 268-271.

5 Lobatschewsky's Theorie der Parallellinien, Art. 22.

6 Frischauf's Absolute Geometrie, Art. 13.

7 Lobatschewsky's Theorie der Parallellinien, Art. 23.

8 Frischauf's Absolute Geometrie, p. 18.

9 Frischauf's Absolute Geometrie, p. 50.