braical equations of the respective forms

$$\phi_1(\xi_1, M, x) = 0, \quad \phi_2(\psi_1, M, x) = 0, \quad \phi_3(\xi_2, M, x) = 0,$$

and
$$\phi_4(\psi_2, M, x) = 0.$$

83. The connection of the results of this paper with the Riccatian

$$z'' + x^{1/i-2} z = 0,$$

where i is any integer, is made in § V. The theory of Differential Resolvents only makes its appearance collaterally, otherwise many remarks would arise upon the above forms, which indeed determine those of the roots of the algebraical equations.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics. *By* R. A. Roberts.

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1. In a paper recently published in the *Proceedings of the London Mathematical Society* (Vol. xvi., p. 242), I noticed the possibility of the inscription of a doubly infinite number of polygons in a quadric such that the sides touch two confocal quadrics, provided these three quadrics are connected by a certain pair of relations depending on the number of sides of the polygon. This result was arrived at by making use of Liouville's form of the differential equations of the system of lines touching two confocal quadrics (*Journal de Mathématiques*, t. xii., p. 418), and was seen to depend upon the finding of the $2n^{\text{th}}$ parts of the complete values of the hyperelliptic integrals $L(x)$, $M(x)$.

2. The object of the present paper is to show how to obtain the actual relations connecting the quadrics corresponding to a given number of sides of the polygon. For this purpose I employ, for the sake of symmetry, Professor Cayley's form of the equation of a system of confocal quadrics, namely,

$$\frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} - 1 = 0 \quad \dots\dots\dots\dots\dots\dots (1),$$

and take p, q, r as the roots of this equation in p. Now, it is shown in works on geometry of three dimensions that the tangent cone to the surface $p = p_1$ drawn from the point p, q, r, and referred to the normals to the three latter confocals, is

$$\frac{x^2}{p-p_1}+\frac{y^2}{q-p_1}+\frac{z^2}{r-p_1}=0.$$

Hence, taking two confocals $p = p_1$, $p = p_2$, and solving between the two equations, we have for a common tangent passing through the vertex of the cone, after having transformed to polar coordinates,

$$\cos\alpha=\sqrt{\left\{\frac{(p-p_1)(p-p_2)}{(p-q)(p-r)}\right\}},\quad \cos\beta=\sqrt{\left\{\frac{(q-p_1)(q-p_2)}{(q-p)(q-r)}\right\}},$$

$$\cos\gamma=\sqrt{\left\{\frac{(r-p_1)(r-p_2)}{(r-p)(r-q)}\right\}}\quad\text{..................}(2).$$

But if $d\sigma$ is an element of the line, and $d\sigma_1$, $d\sigma_2$, $d\sigma_3$ the elements of the lines of curvature of the confocals, we have

$$\cos\alpha=\frac{d\sigma_1}{d\sigma},\quad \cos\beta=\frac{d\sigma_2}{d\sigma},\quad \cos\gamma=\frac{d\sigma_3}{d\sigma},$$

and we know that

$$d\sigma_1=\tfrac{1}{2}\sqrt{\left\{\frac{(p-q)(p-r)}{(p+a)(p+b)(p+c)}\right\}}\,dp,\ d\sigma_2=\&\text{c.},\ d\sigma_3=\&\text{c......}(3).$$

Hence, from (2) and (3), we easily find

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}+\frac{dr}{\sqrt{R}}=0,\quad \frac{p\,dp}{\sqrt{P}}+\frac{q\,dq}{\sqrt{Q}}+\frac{r\,dr}{\sqrt{R}}=0,$$

where
$$P=(p-p_1)(p-p_2)(p+a)(p+b)(p+c),$$
$$Q=(q-p_1)(q-p_2)(q+a)(q+b)(q+c),$$
$$R=(r-p_1)(r-p_2)(r+a)(r+b)(r+c).$$

These are Liouville's equations, and by their integration we obtain

$$\left.\begin{array}{l} L(p)\pm L(q)\pm L(r)=c_1 \\ M(p)\pm M(q)\pm M(r)=c_2 \end{array}\right\}\text{......................}(4),$$

where
$$L(p)=\int\frac{dp}{\sqrt{P}},\quad M(p)=\int\frac{p\,dp}{\sqrt{P}},$$

and c_1, c_2 are constants. Putting now $L(q)\pm L(r)=u$, $M(q)\pm M(r)=v$, we have, for the two points where the line intersects the quadric p = a constant,

$$u_1 - u_2 = 2L(p), \quad v_1 - v_2 = 2M(p) \quad \text{.................} (5).$$

Hence we see that the lines of the system will form a doubly infinite number of polygons of n sides inscribed in the quadric p, if we have

$$2nL(p) = \Omega, \quad 2nM(p) = \Omega' \quad \text{....................} (6),$$

where Ω, Ω' are the periodic values of the integrals $L(p)$, $M(p)$, respectively.

3. What we want to do is, then, to express these relations in an algebraic form; and for this purpose I make use of the following particular case of Abel's theorem: Let ϕ, ψ, X be polynomials in x, of the n^{th}, $(n-3)^{\text{th}}$, and sixth degrees, respectively; then, if

$$\phi^2 - \psi^2 X = f(x) \quad \text{.............................} (7),$$

there is

$$\Sigma \int \frac{dx}{\sqrt{X}} = \Omega, \quad \Sigma \int \frac{x\,dx}{\sqrt{X}} = \Omega' \quad \text{................} (8),$$

where the summation refers to all the roots of $f(x)$, and the origin of the integrals is one of the roots of X. If we now suppose all the roots of $f(x)$ to coincide, we come upon the equations (6) obtained above.

4. Let us consider the case of the triangle, that is, $n=3$; then ϕ is a cubic expression, ψ a constant, and $f(x) = m^2(x-z)^6$, say, and (7) may be written $\{\phi - m(x-z)^3\}\{\phi + m(x-z)^3\} = X$. Hence, if U_1, U_2 are two cubic factors of X, we must have $\phi - m(x-z)^3 = kU_1$, $\phi + m(x-z)^3 = k^{-1}U_2$, from which, eliminating ϕ, we get

$$2mk(x-z)^3 = U_2 - k^2 U_1.$$

We see thus that X breaks up into two cubics, U_1, U_2, whose combinantive invariant Q vanishes; that is, they are such that $lU_1 + mU_2$ can be made a perfect cube, the root of the cube determining the parameter x corresponding to (6), when $n = 3$. Applying this result to (4), we have one of the roots of P infinite, and if we take for the two cubics those whose roots are ∞, p_1, p_2 and $-a$, $-b$, $-c$, respectively, we may write

$$(x+a)(x+b)(x+c) + \lambda(x-p_1)(x-p_2) = \mu(x-p)^3 \quad \text{..........} (9).$$

Substituting, then, in this identity, $-a$, $-b$, $-c$ successively for x, and eliminating λ, μ, we obtain

$$\frac{(p+a)^3}{(p_1+a)(p_2+a)} = \frac{(p+b)^3}{(p_1+b)(p_2+b)} = \frac{(p+c)^3}{(p_1+c)(p_2+c)} \quad \ldots (10).$$

From this result it appears that, given the quadric p, the quadrics p_1, p_2 are uniquely determined, their parameters then being connected by the equation

$$(b-c)\sqrt[3]{\{(a+p_1)(a+p_2)\}} + (c-a)\sqrt[3]{\{(b+p_1)(b+p_2)\}} + (a-b)\sqrt[3]{\{(c+p_1)(c+p_2)\}} = 0\,;$$

but, given one of the inscribed quadrics, there are three corresponding circumscribing quadrics, only one of which is, however, real, as is evident from the equation

$$p = p_1 - \sqrt[3]{\{(p_1+a)(p_1+b)(p_1+c)\}},$$

which may be derived directly from (9) by putting $x = p_1$; for from the coefficients of x^3 we see that $\mu = 1$.

5. Again, taking ∞, $-a$, $-b$, and p_1, p_2, $-c$, as the roots of the two cubics respectively, we have

$$\lambda(x+a)(x+b) + (x+c)(x-p_1)(x-p_2) = \mu(x-p)^3 \quad \ldots (11),$$

from which, by comparing the coefficients of x^3, we get $\mu = 1$; and then, putting $-a$, $-b$ successively for x, we find

$$(a-c)(a+p_1)(a+p_2) = (a+p)^3, \quad (b-c)(b+p_1)(b+p_2) = (b+p)^3 \ldots (12).$$

In this case then, as in the foregoing, being given the circumscribing quadric, the two inscribed are uniquely determined, and, being given one of the inscribed, there is one real circumscribed quadric.

By interchanging the constants a, b, c, in (12), we have two further pairs of conditions, namely,

$$(a-b)(a+p_1)(a+p_2) = (a+p)^3, \quad (c-b)(c+p_1)(c+p_2) = (c+p)^3\,;$$

$$(b-a)(b+p_1)(b+p_2) = (b+p)^3, \quad (c-a)(c+p_1)(c+p_2) = (c+p)^3.$$

6. Let us consider now the two cubics whose roots are ∞, p_1, $-c$, and p_2, $-a$, $-b$, respectively; then we have

$$\lambda(x-p_1)(x+c) + (x-p_2)(x+a)(x+b) = \mu(x-p)^3 \quad \ldots\ldots (13),$$

whence we get $\mu = 1$; and, by putting p_1 and $-c$ successively for x, there results

$$(p_1-p_2)(p_1+a)(p_1+b)=(p_1-p)^2,\quad (p_2+c)(a-c)(b-c)=(p+c)^2 \qquad \text{......... (14)}.$$

Also, putting $-a$, $-b$ successively for x in (13), and eliminating λ, we obtain

$$(a-c)(a+p_1)(b+p)^2-(b-c)(b+p_1)(a+p)^2=0 \qquad \text{...... (15)},$$

which relation must of course be implied in the other two.

Given the surface p, then we see, from (14) and (15), that the inscribed quadrics are uniquely determined; but, when one of the inscribed surfaces is given, there are two real circumscribing surfaces, arising from the fact that the connecting relations are not symmetrical in p_1, p_2.

There are evidently two other pairs of relations to be obtained by interchanging a, b, c in (14) and (15). We have thus, altogether, seven distinct pairs of relations connecting the quadrics in the case of the triangle.

7. We may now proceed to consider the quadrilateral. In this case, ϕ is of the fourth degree, and ψ linear in x; also $f(x)=m^2(x-z)^8$;

hence we have $$\phi^2-m^2(x-z)^8=\psi^2 U_2 U_4 \qquad \text{..................... (16)},$$

where U_2 is a quadratic, and U_4 a quartic factor of X.

Thus, in the same manner as before, we obtain

$$\phi+m(x-z)^4=U_4,\quad \phi-m(x-z)^4=\psi^2 U_2;$$

therefore $$2m(x-z)^4=U_4-\psi^2 U_2 \qquad \text{..................... (17)},$$

from which equation all the required results are obtainable.

Taking p_1, p_2 as the roots of U_2, and ∞, $-a$, $-b$, $-c$ as those of U_4, we have

$$\lambda(x+a)(x+b)(x+c)+(sx+t)^2(x-p_1)(x-p_2)=(x-p)^4 \qquad \text{... (18)},$$

if we suppose ψ equal to $sx+t$.

From the coefficients of x^4 we have then $s=1$, and, by putting $-a$, $-b$, $-c$ successively for x, we find

$$\frac{(a+p)^2}{\sqrt{\{(a+p_1)(a+p_2)\}}}=t-a,\quad \frac{(b+p)^2}{\sqrt{\{(b+p_1)(b+p_2)\}}}=t-b,$$

$$\frac{(c+p)^2}{\sqrt{\{(c+p_1)(c+p_2)\}}}=t-c \qquad \text{................. (19)},$$

whence, by the elimination of t, we get two relations connecting p_1, p_2, p.

Eliminating p_1, p_3 between these equations, we obtain

$$\frac{4}{t+p}=\frac{1}{t-a}+\frac{1}{t-b}+\frac{1}{t-c},$$

which gives three values of t for a given value of p. Hence, being given the surface p, there are three pairs of inscribed surfaces. Also, given p_1, to find p we have from (18) the discriminant of

$$(t+p)^4+\lambda(t-a)(t-b)(t-c),$$

with respect to t equal to zero; and then, putting p_1 for x in the same equation, there results

$$\lambda(p_1+a)(p_1+b)(p_1+c)=(p-p_1)^4.$$

We thus obtain an equation of the tenth degree in p_1, after having divided out by $(p-p_1)^6$.

8. Again, let us take $-a$, $-b$, $-c$, p_1 as the roots of U_4, and ∞, p_3 as those of U_3; we have then, from (17),

$$\lambda(x-p_1)(x+a)(x+b)(x+c)-\mu(x+t)^2(x-p_3)=(x-p)^4 \ldots (20),$$

in which we find $\lambda=1$, and then readily obtain

$$\frac{(b-c)(a+p)^2}{\sqrt{(a+p_3)}}+\frac{(c-a)(b+p)^2}{\sqrt{(b+p_3)}}+\frac{(a-b)(c+p)^2}{\sqrt{(c+p_3)}}=0 \quad \ldots (21),$$

$$(p_3-p_1)(a+p_3)(b+p_3)(c+p_3)-(p-p_3)^4=0 \ldots\ldots\ldots (22).$$

Hence, being given the surface p, we have from (21) four surfaces such as p_3, and then from (22) there are four corresponding surfaces such as p_1.

9. Furthermore, let $-a$, $-b$, ∞, p_1 be the roots of U_4, and $-c$, p_3 those of U_3; then we have

$$\lambda(x+a)(x+b)(x-p_1)+(x+c)(x-p_3)(sx+t)^2=(x-p)^4 \ldots (23),$$

whence we obtain $s=1$, and

$$\left.\begin{array}{l}\dfrac{(a+p)^2}{\sqrt{\{(a-c)(a+p_3)\}}}-\dfrac{(b+p)^2}{\sqrt{\{(b-c)(b+p_3)\}}}=a-b \\ (a-c)(b-c)(p_1+c)(p-p_3)^4-(a+p_3)(b+p_3)(p_1-p_3)(p+c)^4=0\end{array}\right\}(24).$$

From the first of these equations we have four values of p_3 corresponding to a given value of p, and from the second there is then a single value of p_1 for each value of p_3. In this case, there are of course three pairs of equations such as (24) altogether, which are found by interchanging a, b, c.

10. Again, let us take $-a$, $-b$, p_1, p_2 as the roots of U_4, and c, ∞ as those of U_2; we have then

$$(x+c)(sx+t)^2+\lambda(x+a)(x+b)(x-p_1)(x-p_2) = (x-p)^4 \ldots \quad (25),$$

whence $\lambda = 1$, and

$$(a-c)(b-c)(c+p_1)(c+p_2) = (c+p)^4 \ldots\ldots\ldots\ldots \quad (26),$$

$$\left.\begin{aligned} \frac{(p_1+b)(a+p)^2}{\sqrt{(c-a)}} - \frac{(p_1+a)(b+p)^2}{\sqrt{(a-b)}} = \frac{(a-b)(p-p_1)^2}{\sqrt{(p_1+c)}} \\ \frac{(p_2+b)(a+p)^2}{\sqrt{(c-a)}} - \frac{(p_2+a)(b+p)^2}{\sqrt{(c-b)}} = \frac{(a-b)(p-p_2)^2}{\sqrt{(p_2+c)}} \end{aligned}\right\} \ldots \quad (27),$$

which are of course only equivalent to two equations. From these equations it appears that, when we are given p, there are three pairs of inscribed surfaces. Also, if we are given one of the inscribed surfaces, there are eight corresponding circumscribed surfaces.

In the preceding case there are three different sets of conditions found by interchanging a, b, c; and we thus have altogether, in the case of the quadrilateral, eight distinct pairs of conditions.

11. We may now proceed to consider the case of the pentagon.

We have then either $\quad (x-z)^5 = \psi_1^2 U + \psi_2^2 V \ldots\ldots\ldots\ldots\ldots\ldots \quad (28),$

or $\quad (x-z)^5 = \psi^2 U_1 + U_5 \ldots\ldots\ldots\ldots\ldots\ldots \quad (29),$

where ψ_1, ψ_2 are the factors of the quadratic ψ, U, V are cubic factors of X, and U_1 is a linear factor and U_5 the remaining factor of the fifth degree.

Let α, β, γ be the roots of U, and α', β', γ' those of V; then, from (28), we easily find

$$(\beta'-\gamma')\sqrt{\left\{\frac{(\alpha'-z)^5}{(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma)}\right\}}$$

$$+(\gamma'-\alpha')\sqrt{\left\{\frac{(\beta'-z)^5}{(\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma)}\right\}}$$

$$+(\alpha'-\beta)\sqrt{\left\{\frac{(\gamma'-z)^5}{(\gamma'-\alpha)(\gamma'-\beta)(\gamma'-\gamma)}\right\}} = 0 \ldots\ldots\ldots \quad (30),$$

$$(\beta-\gamma)\sqrt{\left\{\frac{(\alpha-z)^5}{(\alpha-\alpha')(\alpha-\beta')(\alpha-\gamma')}\right\}}$$

$$+(\gamma-\alpha)\sqrt{\left\{\frac{(\beta-z)^5}{(\beta-\alpha')(\beta-\beta')(\beta-\gamma')}\right\}}$$

$$+(\alpha-\beta)\sqrt{\left\{\frac{(\gamma-z)^5}{(\gamma-\alpha')(\gamma-\beta')(\gamma-\gamma')}\right\}}=0\ldots\ldots\ldots (31).$$

Applying this result to the case we are considering, that is, putting $z=p$, and taking ∞, p_1, p_2, $-a$, $-b$, $-c$ as the roots of X, we get seven pairs of relations connecting p_1, p_2, p. Again, let a be the root of U_1 and a_1, a_2, a_3, a_4, a_5 the roots of U_5; then, substituting these latter quantities for x successively in (29), we get

$$\Sigma'_5 \frac{1}{\chi'(a_i)}\sqrt{\left(\frac{(a_i-z)^5}{a_i-a}\right)}=0,\quad \Sigma'_5 \frac{x}{\chi'(a_i)}\sqrt{\left(\frac{(a_i-z)^5}{a_i-a}\right)}=0\ldots\ldots(32),$$

where $\chi(x)\equiv U_5$, and the summation refers to the five roots of U_5. From these conditions we get five pairs of relations connecting p_1, p_2, p. We have thus altogether twelve distinct pairs of relations corresponding to the case of the pentagon.

12. Proceeding to the case of the Hexagon, we find, from (7),

$$(x-z)^6=\psi_1^2 U_4+\psi_2^2 U_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

where ψ_1 is a linear and ψ_2 the corresponding quadratic factor of ψ, and U_4 is a biquadratic and U_2 the corresponding quadratic factor of X. If, then, a_1, a_2, a_3, a_4 are the roots of $U_4\equiv\chi(x)$, say, and β_1, β_2 those of U_2, we easily find

$$\Sigma'_4 \frac{(a_i-z)^3}{\chi'(a_i)\sqrt{\{(a_i-\beta_1)(a_i-\beta_2)\}}}=0\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

where the summation refers to all the roots of $\chi(x)$. The other relation is of a more complicated nature. A particular form of it can be found thus:

Putting $\quad \psi_2=\lambda+\mu x+\nu x^2$ and $\psi_1=\rho+\sigma x$,

we can find expressions for λ, μ, ν, ρ, σ by the substitution of the roots of χ successively in (33); and then, by the comparison of any of the coefficients of the powers of x, we can get another relation; as, for instance, from the coefficients of x^6 we have

$$\sigma^2+\nu^2=1.$$

We thus find eight distinct pairs of equations altogether in the case of the hexagon.

13. For polygons of a greater number of sides we can find, in the same way, the necessary conditions; but they are not capable of being expressed in the same simple form by means of radicals as in the cases we have considered. It may be observed, however, that the general conditions can be expressed by means of certain determinants. We may suppose $p = 0$, without loss of generality; and then, since

$$\phi - \psi\sqrt{X} \qquad (35)$$

is to be satisfied by $2n$ equal values of x, we may differentiate this expression $2n-1$ times, and then put $x = 0$. Differentiating (35) $n+1$ times with regard to x, we eliminate all the constants in ϕ, and there is then

$$\frac{d^{n+1}}{dx^{n+1}}(\psi\sqrt{X}) = 0 \qquad (36).$$

Now let us suppose $\sqrt{X}$ expanded in the form

$$A_0 + A_1 x + A_2 x^2 + \&c.,$$

and let
$$\psi = \alpha + \beta x + \gamma x^2 + \ldots + \lambda x^{n-3};$$

then, putting $x = 0$, in (36), and its successive $(n-2)^{\text{th}}$ differential coefficients, we get

$$\begin{aligned}
\alpha A_{n-2} + \beta A_{n-1} + \gamma A_n \;\; + \ldots &= 0,\\
\alpha A_{n-3} + \beta A_{n-2} + \gamma A_{n-1} + \ldots &= 0,\\
\alpha A_{n-4} + \beta A_{n-3} + \gamma A_{n-2} + \ldots &= 0,\\
+ \ldots &= 0.
\end{aligned}$$

That is, we have $n-1$ linear homogeneous equations connecting the $n-2$ quantities α, β, γ, &c.; hence, by elimination, we obtain $n-1$ determinants, which, being equated to zero, represent the two relations sought.

14. In the same way we can demonstrate the possibility of circumscribing an infinite number of polygons about two confocal quadrics, so that each vertex may move on a confocal quadric. In fact, from (5), we find

$$2\Sigma'_n L(p_i) = \Omega, \quad 2\Sigma'_n M(p_i) = \Omega' \qquad (37),$$

where n is the number of the vertices of the polygon, and the summation refers to the n confocal quadrics passing through these points. Let us suppose, now, f to be the quantic whose roots are the parameters of the n circumscribing quadrics; then we have

$$\phi^2-\psi^2X=f^2, \text{ or } \phi^2-f^2=\psi^2X;$$

whence, if P, Q are two factors of ψ^2X of the n^{th} degree in x, we find

$$f=\lambda P+\mu Q,$$

from which the two required conditions can be found. For instance, for the triangle, if

$$f=(x-\varpi_1)(x-\varpi_2)(x-\varpi_3),$$

we have $$(x+a)(x+b)(x+c)+\lambda(x-p_1)(x-p_2)=kf,$$

or equations of the form

$$(x-p_1)(x-p_2)(x+c)+\lambda(x+a)(x+b)=kf,$$

$$(x-p_1)(x+a)(x+b)+\lambda(x-p_2)(x+c)=kf,$$

from which the respective pairs of conditions are easily found; and in the same way, for polygons of a greater number of sides, the equations of condition are found with no greater difficulty than in the case in which all the circumscribing surfaces coincide.

15. It may be observed that one or both of the inscribed quadrics can be supposed to degenerate into focal conics, if the sides of the polygon are not less than four in number. In this case the equations of condition are in general considerably simplified. For instance, suppose we wish to find the relations connecting four confocal surfaces, each of which passes through a vertex of a quadrilateral whose sides intersect the two real focal conics; then, putting

$$p_1+a=0, \quad p_2+b=0$$

for the focal conics, the only relevant equation is

$$(x+a)^2(x+b)^2+(sx+t)^2(x+c)=f(x) \quad\text{.............(38)},$$

where $$f(x)=(x-p_1)(x-p_2)(x-p_3)(x-p_4).$$

We have then $$f(-c)=(a-c)^2(b-c)^2,$$

and the other condition is found by dividing $(x+a)^2(x+b)^2-f(x)$ by $x+c$ and equating to zero the discriminant of the resulting quadratic.

P 2

16. In the case in which the vertices of the polygon lie on the same quadric, if the inscribed quadrics coincide, the problem we have been considering becomes—to inscribe in a quadric U polygons such that the sides touch a quadric V along the curve of contact of the common tangent planes of U and V. The two conditions can then be expressed in terms of the roots of the discriminant of $U+\lambda V$. For instance, for the case of the triangle we have, from (10),

$$\frac{a^3}{(p_1+a)^2}=\frac{b^3}{(p_1+b)^2}=\frac{c^3}{(p_1+c)^2} \quad\text{..................(39).}$$

But if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of the discriminant of $U+\lambda V$, namely,

$$\Delta\lambda^4+\Theta\lambda^3+\Phi\lambda^2+\Theta'\lambda+\Delta'=0 \quad\text{..................(40)}$$

we have

$$\lambda_1=\lambda_4\left(1+\frac{p}{a}\right),\quad \lambda_2=\lambda_4\left(1+\frac{p}{b}\right),\quad \lambda_3=\lambda_4\left(1+\frac{p}{c}\right),$$

whence, from (39), we get $\lambda_1+\lambda_2+\lambda_3=\lambda_4$, and $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=0$, and then, from (40), $\Theta^2=4\Delta\Phi$, $\Theta\Theta'=2\Delta\Delta'$. For the quadrilateral we find

$$\lambda_1+\lambda_2-\lambda_3-\lambda_4=0,\quad \lambda_1\lambda_2+\lambda_3\lambda_4=0,$$

from which we get $\quad \Theta^2-4\Delta\Phi=0,\quad \Theta'=0$;

or $\quad (\lambda_1-\lambda_2)^2=\lambda_3^2+\lambda_4^2+6\lambda_3\lambda_4,$

$$(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)=(\lambda_3-\lambda_4)^2.$$

17. In the preceding results I have called the system of quadrics confocal, for the sake of convenience; but of course everything remains true if we suppose the quadrics inscribed in the same developable. If the two quadrics referred to their common self-conjugate tetrahedron

are $\quad x^2+y^2+z^2+u^2=0,\quad \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{u^2}{d}=0,$

the equation of any quadric inscribed in their circumscribed developable is

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}+\frac{u^2}{d+\lambda}=0.$$

Then, if p, q, r are the roots of this equation in λ, we have

$$Mx^2=\frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)(a-d)},\quad My^2=\text{\&c.} \quad\text{..............(41).}$$

We may now give here a purely algebraical proof of the two differential equations of the lines touching two quadrics of the system. Let L be a sextic in a variable λ, and ϕ, ψ a cubic and a linear expression, respectively, in the same variable; then, by a particular case of Abel's theorem, if $\phi^2-\psi^2L=f(\lambda)$, we have

$$\Sigma'_6\int\frac{d\lambda}{\sqrt{L}}=\Omega,\qquad \Sigma'_6\int\frac{\lambda d\lambda}{\sqrt{L}}=\Omega'\text{.....................}(42),$$

where Ω, Ω' have the same meaning as at (6), and the summation refers to the six roots of $f(\lambda)$.

Now let $L\equiv(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda+a)(\lambda+b)(\lambda+c)(\lambda+d)$,

and suppose the roots of $f(\lambda)$ to be the parameters p, q, r, p', q', r', as found above, of two points in space; then, if we have (42), the six parameters are the roots of $\phi^2-\psi^2L=0$. But, substituting the six quantities p, q, r, p', q', r' successively in $\phi-\psi\sqrt{L}=0$, and eliminating the constants in ϕ, ψ linearly, we get two equations which may be written

$$\left.\begin{aligned}\Sigma\frac{\sqrt{f(-a)}}{(a+\lambda_1)\chi'(a)}&=\frac{\sqrt{f(\lambda_1)}}{\chi(\lambda_1)}\\ \Sigma\frac{\sqrt{f(-a)}}{(a+\lambda_2)\chi'(-a)}&=\frac{\sqrt{f(\lambda_2)}}{\chi(\lambda_2)}\end{aligned}\right\}\text{..................}(43),$$

where the summation refers to the four quantities a, b, c, d, and $\chi(x)=(x+a)(x+b)(x+c)(x+d)$. Transforming now by means of (41), and making use of the relation

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}+\frac{u^2}{d+\lambda}\equiv U=\frac{(\lambda-p)(\lambda-q)(\lambda-r)}{\chi(\lambda)},$$

the equations (43) become

$$\frac{xx'}{a+\lambda_1}+\frac{yy'}{b+\lambda_1}+\frac{zz'}{c+\lambda_1}+\frac{uu'}{d+\lambda_1}=\sqrt{U_1U'_1},$$

$$\frac{xx'}{a+\lambda_2}+\frac{yy'}{b+\lambda_2}+\frac{zz'}{c+\lambda_2}+\frac{uu'}{d+\lambda_2}=\sqrt{U_2U'_2}.$$

But these are, it is easy to see, the conditions that the lines joining the points $xyzu$, $x'y'z'u'$ should touch the quadrics U_1, U_2. Hence, from (42), if we suppose one of the points to remain fixed, we see that the differential equations of the lines are

$$\frac{dp}{\sqrt{L(p)}}+\frac{dq}{\sqrt{L(q)}}+\frac{dr}{\sqrt{L(r)}}=0,$$

$$\frac{p\,dp}{\sqrt{L(p)}}+\frac{q\,dq}{\sqrt{L(q)}}+\frac{r\,dr}{\sqrt{L(r)}}=0.$$