pressure. Now, ψ involves linearly the harmonics which occur in the complementary and particular solutions of the equations of vibration, the pressure involves the same harmonics linearly, and the result is the determination of the surface-tractions applied to the inner surface in terms of these harmonics. We have already another expression for these calculated from the strain. Equating the two expressions, we should obtain sufficient equations, with those which hold at the free surface, to express all the unknown harmonics in terms of those occurring in the expression for the disturbing potential.

This completes the analysis for the bodily tides of any order in the system considered.

As an example, I have considered free vibrations of the system supposed rotating slowly, but free from gravitation, and have verified that the solution reduces to that which we already know for free vibrations when the rotation is also annulled.

The results in the general case bear out those arrived at by other methods when the disturbing function is of order 2, shewing how the effect of the rotation in altering an equilibrium solution is null, unless the period of the disturbance is long compared with that of the rotation.

On the Volume generated by a Congruency of Lines.

By R. A. ROBERTS, M.A.

[*Read Feb. 9th,* 1888.]

In this paper I propose to investigate an expression for the volume generated by a congruency of lines, and to deduce therefrom an extension of Abel's theorem to double integrals.

Taking rectangular coordinates, we may write the equation of a line in the form $x = pz + \alpha, \quad y = qz + \beta$ ....(1),

and then, if the line belongs to a congruency, that is, if it varies subject to two conditions, we may suppose that α, β are given as functions of p, q. Thus for any point in space, drawing through it a line of the system, we may regard x, y as functions of p, q, and z; so that, if the element of volume $dx\,dy\,dz$ be expressed in terms of the latter variables, we have

$$dV = \left(\frac{dx}{dp}\frac{dy}{dq} - \frac{dx}{dq}\frac{dy}{dp}\right) dp\,dq\,dz$$

$$= \left\{ \left(z + \frac{d\alpha}{dp}\right)\left(z + \frac{d\beta}{dq}\right) - \frac{d\alpha}{dq}\frac{d\beta}{dp} \right\} dp\,dq\,dz$$

from (1), by the ordinary transformation for the element of a multiple integral. Now the coefficient of $dp\,dq\,dz$, being equated to zero, is evidently the result which we should obtain if we differentiated (1) with regard to p, q, and eliminated dp/dq; and this gives the two points in which a line of the system is intersected by two other consecutive lines, or is, in general, bitangent to the focal surface. Hence, if z_1, z_2 belong to the points just mentioned, we have

$$dV = (z - z_1)\,(z - z_2)\,dp\,dq\,dz.$$

But, if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the line (1), we have

$$p = \frac{\cos\alpha}{\cos\gamma}, \quad q = \frac{\cos\beta}{\cos\gamma},$$

whence $$p = \tan\theta\cos\phi, \quad q = \tan\theta\sin\phi,$$

if we put $$\cos\alpha = \sin\theta\cos\phi, \quad \cos\beta = \sin\theta\sin\phi, \quad \gamma = \theta;$$

and therefore $dp\,dq$ is replaced by $\tan\theta\sec^2\theta\,d\theta\,d\phi$. Also, if ρ, ρ_1, ρ_2 are distances measured along the line corresponding to z, z_1, z_2, respectively, we have

$$z - z_1 = (\rho - \rho_1)\cos\theta, \quad z - z_2 = (\rho - \rho_2)\cos\theta,$$

$$dz = d\rho\cos\theta.$$

We thus find

$$dV = (\rho - \rho_1)\,(\rho - \rho_2)\sin\theta\,d\theta\,d\phi\,d\rho = (\rho - \rho_1)\,(\rho - \rho_2)\,d\rho\,d\omega \ldots\ldots(2),$$

if $d\omega$ denotes the solid angle of the elementary cone formed by drawing parallels to the lines of the system through a point. Hence, integrating, we get

$$V = \iiint (\rho - \rho_1)(\rho - \rho_2)\,d\rho\,d\omega = \iint \left(\tfrac{1}{3}\rho^3 - \tfrac{1}{2}\rho^2(\rho_1 + \rho_2) + \rho\rho_1\rho_2\right) d\omega \ldots\ldots(3),$$

if the integration with regard to ρ is effected between the limits ρ and 0.

In this expression (3) it is evident that the volume described is included between a line of length ρ one of whose extremities is situated at the distances ρ_1, ρ_2 from the points of contact with the surface; and, when the integration is effected, the boundary of the space is formed by the loci of the extremities just mentioned, and a ruled surface described by the lines of the system subject to a relation connecting p, q, or θ, ϕ.

As a particular case of (3), let us consider the volume enclosed within the lines joining the points of contact on the focal surface. Putting $\rho_1 = \delta$, $\rho_2 = 0$, $\rho = \delta$, we get from (3)

$$V = \tfrac{1}{6}\iint \delta^3\, d\omega \text{ ....} (4);$$

that is, the volume sought is equal to half the volume enclosed within a parallel cone, and the surface formed by measuring equal lengths on the parallel radii vectores through the vertex of the cone, as readily follows from the expression for the volume in polar coordinates. If we seek the volume described by either half of the line joining the foci, we find

$$V = \tfrac{1}{12}\iint \delta^3\, d\omega,$$

which is equal to half (4); that is, the locus of the middle point of the foci bisects the volume described by the entire line.

Again, suppose that the lines of the system are inflexional tangents of a surface; then, putting $\rho_1 = \rho_2 = 0$, $\rho = \delta$, we have from (3)

$$V = \tfrac{1}{3}\iint \delta^3\, d\omega \text{ ....} (5).$$

Hence the volume described by the inflexional tangents of a surface measured from the points of contact is equal to the volume enclosed within the extremities of parallel lines of equal length drawn through a point. Applying (3) to the case of the normals to a surface, we get

$$V = \iiint (r-R_1)(r-R_2)\, dr\, d\omega = \iint \left\{\tfrac{1}{3}r^3 - \tfrac{1}{2}r^2(R_1+R_2) + rR_1R_2\right\} d\omega,$$

where R_1, R_2 are the principal radii of curvature, and r is the length measured out on the normal.

Hence for the volume enclosed within a surface, a system of normals, and a parallel surface, we get

$$V = \tfrac{1}{3}r^3\iint d\omega - \tfrac{1}{2}r^2\iint\left(\frac{1}{R_1}+\frac{1}{R_2}\right) dS + rS,$$

where dS is an element of area.

We now proceed to apply the formula (3) so as to obtain a result which may be considered as an extension of Abel's theorem concerning transcendents to double integrals. Suppose a line to meet the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-3} = 0 \quad \text{.......................(6)},$$

where ϕ_n, ϕ_{n-3} are general surfaces of the degrees n and $n-3$ respectively; then for the points of intersection, transforming to polar coordinates at a point on the line, and writing

$$\phi_n = A_0 r^n + A_1 r^{n-1} + \ldots + A_n = 0,$$

$$\phi_{n-3} = B_0 r^{n-3} + B_1 r^{n-4} + \ldots + B_{n-3} = 0,$$

we evidently get, by the theory of algebraic equations,

$$\Sigma r = -\frac{A_1}{A_0} = \Sigma r',$$

$$\Sigma r^2 = \frac{A_1^2 - 2A_0 A_2}{A_0^2} = \Sigma r'^2,$$

$$\Sigma r^3 = -\frac{A_1^3 + 3A_1 A_2 A_0 - 3A_3 A_0^2}{A_0^3},$$

$$\Sigma r'^3 = -\frac{A_1^3 + 3A_1 A_2 A_0 - 3A_3 A_0^2}{A_0^3} - \frac{3kB_0}{A_0},$$

where the unaccented and accented letters refer to the two surfaces (6), respectively.

Hence
$$\Sigma r - \Sigma r' = 0, \quad \Sigma r^2 - \Sigma r'^2 = 0,$$

$$\Sigma r^3 - \Sigma r'^3 = \frac{3kB_0}{A_0},$$

so that, if a congruency of lines intersect the surfaces (6), we get from (3)

$$\Sigma V - \Sigma V' = \iint \left\{ \tfrac{1}{3}(\Sigma r^3 - \Sigma r'^3) - \tfrac{1}{2}(\rho_1 + \rho_2)(\Sigma r^2 - \Sigma r'^2) + \rho_1\rho_2(\Sigma r - \Sigma r') \right\} d\omega$$

$$= k \iint \frac{B_0}{A_0}\, d\omega \quad \text{..(7)}.$$

Now, since A_0 and B_0 are functions of the direction of a line of the system, that is, of θ and ϕ, we see that the integration can be effected in (7). This result thus shows that the algebraic sum of the volumes intercepted between the surfaces (6) by a congruency of lines, can be expressed by a double definite integral which depends upon the direction of the lines only.

If B_0 vanishes, the surface ϕ_{n-3} reduces to the degree $n-4$, and the double integral in (7) disappears. This result may be stated as follows: If a congruency of lines intersect the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-4} = 0,$$

the algebraic sum of the intercepted volumes is equal to zero.

By taking k indefinitely small in the preceding results, we can arrive at the relation connecting double integrals referred to above.

The volume intercepted between the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-3} = 0,$$

at any point of the former, is evidently $dp\,dS$, where dS is an element of area, and dp is the portion of the normal intercepted between the two surfaces. Now, if the point x, y, z lie on the surface $\phi_n = 0$, and the consecutive point $x+\delta x, y+\delta y, z+\delta z$ on the normal to ϕ_n satisfy the equation of the surface

$$\phi_n + k\phi_{n-3} = 0,$$

we have evidently

$$\frac{d\phi_n}{dx}\delta x + \frac{d\phi_n}{dy}\delta y + \frac{d\phi_n}{dz}\delta z + k\phi_{n-3} = 0,$$

where k is regarded as an infinitesimal of the first order.

But $$\delta x = \frac{dp}{G}\frac{d\phi_n}{dx}, \quad \delta y = \frac{dp}{G}\frac{d\phi_n}{dy}, \quad \delta z = \frac{dp}{G}\frac{d\phi_n}{dz},$$

where $$G = \sqrt{\left\{\left(\frac{d\phi_n}{dx}\right)^2 + \left(\frac{d\phi_n}{dy}\right)^2 + \left(\frac{d\phi_n}{dz}\right)^2\right\}}.$$

We thus find $$dp = -\frac{k\phi_{n-3}}{G};$$

hence, since $$dS = \frac{G\,dx\,dy}{\frac{d\phi_n}{dz}},$$

we have $$dp\,dS = -k\phi_{n-3}\,du,$$

where we have written du for $\dfrac{dx\,dy}{\frac{d\phi_n}{dz}}$. We get, therefore, substituting this value for the difference of the volumes in (7), and dividing by k,

$$\Sigma\phi_{n-3}\,du = -\frac{B_0}{A_0}\,d\omega \;\dots\dots\dots\dots (8),$$

and, therefore, $$\Sigma\iint\phi_{n-3}\,du = -\iint\frac{B_0}{A_0}\,d\omega \;\dots\dots\dots\dots (9).$$

This result is an extension of a theorem given already by me in a paper published in the *Proceedings*, Vol. XVI., p. 238. It is an exact analogue of Abel's theorem for double integrals. And it is evident that there are precisely similar results for multiple integrals which can be readily obtained by considerations of space of n dimensions. Of course, these relations connecting the multiple integrals and the algebraic systems of equations are merely consistent with each other, whereas, in the case of Abel's theorem, the algebraic conditions follow necessarily from the transcendental equations.

As a particular case, let us consider the application of (9) to the cubic surface. Denoting $\iint du_r$ by u_r, we get for the three systems of points in which a congruency meets the surface

$$u_1+u_2+u_3+\iint\frac{d\omega}{A_0}=0 \quad\text{.......................(10)},$$

where we have put $\phi_{n-3}=B_0=1.$

It may be observed that the integral u_r vanishes at any point of the surface which describes a curve. For instance, if a chord of a curve lying on the cubic meet the surface again at a point the integral corresponding to which is u, we get

$$u=-\iint\frac{d\omega}{A_0} \quad\text{.............................(11)}.$$

As *in plano* for the cubic curve, we can readily find an expression for du in the case of the cubic surface by taking rectangular axes such that the axis of z passes through a point at infinity on the surface. We may write, then,

$$U=v_1z^2+v_2z+v_3=0,$$

where v_1, v_2, v_3 are expressions in x, y of the first, second, and third degrees, respectively. Hence

$$du=\frac{dx\,dy}{\frac{dU}{dz}}=\frac{dx\,dy}{2v_1z+v_2}=\frac{dx\,dy}{\sqrt{(v_2^2-4v_1v_3)}},$$

and

$$u=\iint\frac{dx\,dy}{\sqrt{(v_2^2-4v_1v_3)}} \quad\text{.......................(12)},$$

where it may be observed that the expression under the radical is the most general rational integral function of the fourth degree in x, y; for $v_2^2-4v_1v_3$ being equated to zero represents a cylinder circumscribed about the surface parallel to the axis of z, that is, a circumscribed cone having its vertex on the surface; and we know that the tangent cone of a cubic surface drawn from any point of itself is the most general cone of the fourth degree.

This result gives a transformation which seems of some importance in the theory of double integrals. It shows that any double integral of the form

$$\iint \frac{dx\,dy}{\sqrt{\Phi}} \quad \text{.............................. (13)},$$

where Φ is the most general rational integral expression in x, y of the fourth degree, can be transformed so as to become the rational integral

$$\iint \frac{dp\,dq}{V} \quad \text{.............................. (14)},$$

where V is a rational integral expression in p, q of the third degree.

For, by (11) and (12), the first integral can be transformed into

$$\iint \frac{d\omega}{A_0},$$

which, by putting

$$\cos\alpha = \sin\theta\cos\phi, \quad \cos\beta = \sin\theta\sin\phi, \quad \cos\gamma = \cos\theta,$$

and then $\tan\theta\cos\phi = p, \quad \tan\theta\sin\phi = q,$

becomes of the form (14).

The simplest curves which we could take on the cubic would be two non-intersecting right lines. Hence, to transform (13) to the form (14), we describe a cubic surface S so as to be inscribed in the cylinder Φ, when V is immediately known, and the limiting values of p, q are given by the directions of the lines drawn through the limiting curve on S to intersect two non-intersecting lines lying on S.

For the four systems of points in which a congruency meets a surface of the fourth degree, we find

$$\left.\begin{aligned} \Sigma\iint du &= 0, \\ \Sigma\iint x\,du + \iint \frac{\cos\alpha\,d\omega}{A_0} &= 0 \\ \Sigma\iint y\,du + \iint \frac{\cos\beta\,d\omega}{A_0} &= 0 \\ \Sigma\iint z\,du + \iint \frac{\cos\gamma\,d\omega}{A_0} &= 0 \end{aligned}\right\} \quad \text{.................... (15)},$$

where A_0 is now a rational integral homogeneous function of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the fourth degree.

As to the geometrical meaning of the preceding results, we may notice that the integral u, taken over a portion of the surface U, is proportional to the mass of the shell formed by U and the consecutive surface $U+k$, where k is indefinitely small. It is to be observed that, for two different branches of U, the surface $U+k$ may lie on the inner and outer sides, so that, to represent the mass of the shell, u must be taken with the proper sign. As, for instance, if a quartic consist of two oval surfaces, two values of u on one oval should be taken positively, and the other two on the second oval negatively.

Again, it is evident that the relations connecting the integrals $\iint x\,du$, &c. will give theorems concerning the centres of gravity of the shells; thus, in the case of the quintic surface which is intersected by a congruency in five shells, the centre of gravity of two of the shells must coincide with that of the three others. Similarly, the relations connecting the integrals $\iint x^2\,du$ will give theorems concerning the moments of inertia of the shells.

In the case of particular surfaces, the integral u has a simple geometrical meaning. For instance, for the surface

$$z^n U = \text{a constant},$$

where U is a rational integral function of x, y, u is proportional to $\iint z\,dx\,dy$, that is, the volume of the cylinder included between the surface and the plane of xy. As a further particular case, if the surface is of the fourth order, chords of a curve lying on the surface intersect it again in two areas which are such that the volumes of the cylinders between them and the plane of xy are equal.

Again, for the surface whose equation is

$$V = \text{a constant},$$

where V is a rational integral homogeneous expression in x, y, z, we find, by transformation to polar coordinates, that u is proportional to $\iint r^3 d\omega$, namely, the volume of the cone having its vertex at the origin and standing on the boundary of a superficial area.

The formula for the volume in the case in which the lines of the congruency are chords of a curve may be noticed here.

For any point x, y, z of the line joining the points x_1, y_1, z_1; x_2, y_2, z_2, we may obviously put

$$x = \theta x_1 + (1-\theta)\,x_2, \quad y = \theta y_1 + (1-\theta)\,y_2, \quad z = \theta z_1 + (1-\theta)\,z_2.$$

Hence, if we consider x_1, y_1, z_1 as functions of s_1, and x_2, y_2, z_2 as functions of s_2, where s is the length of the arc, x, y, z will be functions of the three variables θ, s_1, s_2, and $dx\,dy\,dz$ can be replaced by

$$\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2 \\ \dfrac{dx_1}{ds_1}, & \dfrac{dy_1}{ds_1}, & \dfrac{dz_1}{ds_1} \\ \dfrac{dx_2}{ds_2}, & \dfrac{dy_2}{ds_2}, & \dfrac{dz_2}{ds_2} \end{vmatrix} \theta\,(1-\theta)\,d\theta\,ds_1\,ds_2,$$

by the formula for the transformation of a triple integral. Now, by solid geometry, the determinant is equal to $D \sin\phi$, where D is the shortest distance between the tangents to the curve at the extremities of the chord, and ϕ is the angle between the same lines. Also

$$\theta = (\rho - r_1)/(r_2 - r_1),$$

so that we get $$dV = \frac{(\rho-r_1)(\rho-r_2)\,D \sin\phi\,ds_1\,ds_2\,d\rho}{(r_2-r_1)^3}.$$

Thursday, March 8th, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. R. W. D. Christie, Carlton School, near Selby, Yorkshire, was elected a member.

The following communications were made:—

Supplementary Remarks on the Theory of Distributions: Captain P. A. MacMahon, R.A.

Complex Multiplication Moduli: Prof. A. G. Greenhill, M.A.

Geometrical Proof of Feuerbach's Nine-Point-Circle Theorem: Prof. Genese, M.A.

Isostereans: R. Tucker, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLIII., Nos. 261 and 262.

"Educational Times," for March, 1888.

"An Elementary Treatise on the Integral Calculus," by B. Williamson, M.A., F.R.S., Fifth Edition, 8vo; London, 1888,

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Pt. III.; 1888.

"Royal Irish Academy:—Transactions," Vol. XXIX., Parts I. & II.—List of Papers published between 1786 and 1886.

"Royal Irish Academy:—Cunningham Memoirs,—Dynamics and Modern Geometry," by Sir R. S. Ball; Dublin, June, 1887.

"Royal Irish Academy:—Proceedings—Science," Vol. IV., No. 6; "Polite Literature and Antiquities," Vol. II., No. 8.

'Œuvres de Fourier," publiées par les soins de M. Gaston Darboux, Tome I., 4to; Paris, 1888.

"Bulletin des Sciences Mathématiques," Feb., 1888.

"Journal für die reine und angewandte Mathematik," Band 102, Heft IV.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 9.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," No. 51.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXXII., Heft 2 and 3.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo I., Nos. 6 and 7.

Geometrical Demonstration of Feuerbach's Theorem concerning the Nine-Point Circle. *By* Professor R. W. Genese, M.A.

[*Read March 8th*, 1888.]

Let A', B', C' be the mid-points of the sides of the triangle ABC; D, E, F the feet of the perpendiculars; O the circumcentre, I the in-centre, X, Y, Z the points of contact of the in-circle with the sides.

Let OA' meet the circumcircle at U, OC' at W. Then AU bisects $B\widehat{A}C$, and therefore also $O\widehat{A}D$.

Let P, Q, R be the mid-points of the arcs of the nine-point circle exterior to the triangle. Then the tangent at P is parallel to BXD; therefore PX produced will pass through the external centre of similitude T of the nine-point circle and in-circle, and so will RZ.

Feuerbach's theorem will be proved if we show that T is on either the in-circle or the nine-point circle.

Let $A'A''$ be the diameter of the nine-point circle through A.
Then AA'' is equal and parallel to OA';
therefore AA'' is equal and parallel to OA;