Some Applications of Fourier's Theorem. By H. M. MACDONALD.
Received and Read 11th December, 1902.

Fourier's theorem may be written in the form

$$f(x) = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{xy}\,dy \int_b^a e^{-zy} f(z)\,dz,$$

where a, b are two real quantities, the path of integration with respect to z is the part of the real axis in the z plane lying between $z=b$ and $z=a$, c is a real positive quantity, and the path with respect to y lies in the first and fourth quadrants of the y plane, being terminated as indicated. From this it follows that, if

$$\int_b^a e^{-zy} f(z)\,dz = \phi(y),$$

then
$$\frac{1}{2\pi\iota}\int e \quad \phi(y)\,dy = f(x), \qquad \text{(A)}$$

where the path of integration is any curve into which the path $(c-\infty\iota,\ c+\infty\iota)$ can be deformed without passing over a singularity of $\phi(y)$.

This suggests that there may be other paths of integration such that
$$f(x) = A\int e^{xy}\int e^{yz} f(z)\,dy\,dz,$$

and then corresponding relations between integrals would follow. For example, Cauchy's relation

$$f(x) = \frac{1}{2\pi\iota}\int \frac{f(z)\,dz}{z-x}$$

may be written

$$f(x) = \frac{1}{2\pi\iota}\int_0^\infty e^{-xy}\,dy \int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} f(z)\,dz,$$

where α and β are such that $\cos\alpha$ and $\cos\beta$ are negative. If the value of the integral

$$\frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} f(z)\,dz = \phi(y)$$

is known, the value of the integral

$$\int_0^\infty e^{-xy}\,\phi(y)\,dy \qquad \text{(B)}$$

is $f(x)$.

In what follows these two relations will be applied to obtain the values of certain integrals.

It is known that the integral

$$\int_0^\infty e^{-yz} J_n(z)\, z^{-m}\, dz$$

can be expressed as a hypergeometric function; it therefore follows from the above that $J^n(x)\, x^{-m}$ can be expressed as an integral involving a hypergeometric function. Writing

$$\int_0 e^{-yz} J_n(z)\, z^{-m}\, dz = \phi(y),$$

then
$$\phi(y) = \sum_{s=0}^{\infty} \frac{(-)^s}{2^{n+2s}\,\Pi(n+s)\,\Pi(s)} \int_0^\infty e^{-yz} z^{n+2s-m}\, dz,$$

that is,
$$\phi(y) = \sum_0^\infty \frac{(-)^s\,\Pi(n-m+2s)}{2^{n+2s}\,\Pi(n+s)\,\Pi(s)}\, \frac{1}{y^{n-m+2s+1}};$$

therefore
$$\phi(y) = \sum_0^\infty \frac{(-)^s\,\Pi\left(\frac{n-m}{2}+s\right)\Pi\left(\frac{n-m-1}{2}+s\right)}{2^m\,\Pi(n+s)\,\Pi(s)\,\Pi(-\frac{1}{2})}\, \frac{1}{y^{n-m+2s+1}};$$

whence
$$\phi(y) = \frac{1}{2^m\, y^{n-m+1}}\, \frac{\Pi\left(\frac{n-m}{2}\right)\Pi\left(\frac{n-m-1}{2}\right)}{\Pi(-\frac{1}{2})\,\Pi(n)}$$
$$\times F\left(\frac{n-m+1}{2},\ \frac{n-m+2}{2},\ n+1,\ -\frac{1}{y^2}\right),$$

using the usual notation for a hypergeometric function. The relation holds if the real part of $n-m$ is greater than -1 and the series converges. The hypergeometric function that occurs in the above expression can be expressed by means of a spherical harmonic, and the relation can be written

$$\int_0^\infty e^{-yz} J_n(z)\, z^{-m}\, dz = \sqrt{\frac{2}{\pi}}\, \iota^{n-m+1} e^{(m-\frac{1}{2})\pi\iota} (-y^2-1)^{\frac{1}{2}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\iota y). \qquad (1)$$

Hence, from the above,

$$J_n(x)\, x^{-m} = \frac{1}{2\pi\iota}\sqrt{\frac{2}{\pi}}\, \iota^{n-m+1} e^{(m-\frac{1}{2})\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{xy} (-y^2-1)^{\frac{1}{2}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\iota y)\, dy,$$

that is,

$$J_n(x)\, x^{-m} = \sqrt{\frac{2}{\pi}}\, \iota^{n-m-1} e^{(m-\frac{1}{2})\pi\iota} \int_{\iota c+\infty}^{\iota c-\infty} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{2}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\, d\mu.$$

The path of integration may be deformed so that it becomes a curve, beginning at $-\infty-\iota c$, ending at $-\infty+\iota c$, and crossing the real axis to the right of the point $\mu=1$. Therefore, provided a quantity a, less than unity, can be found such that

$$(\mu^2-1)^{\frac{1}{2}(2m-1)+a}\,Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)$$

does not tend to become infinite as μ approaches unity, the above relation can be written

$$J_n(x)\,x^{-m}=\sqrt{\frac{2}{\pi}}\,e^{(m-\frac{1}{2})\pi\iota}\,\frac{\iota^{n-m-1}}{2\pi}\int_{-\infty}^{1}e^{-\iota x\mu}\Big[\{(\mu-0\iota)^2-1\}^{\frac{1}{2}(2m-1)}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu-0\iota)$$
$$-\{(\mu+0\iota)^2-1\}^{\frac{1}{2}(2m-1)}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu+0\iota)\Big]\,d\mu,$$

that is,

$$J_n(x)\,x^{-m}=\sqrt{\frac{2}{\pi}}\,e^{(m-\frac{1}{2})\pi\iota}\,\frac{\iota^{n-m-1}}{2\pi}\Big[\int_{-\infty}^{-1}e^{-\iota x\mu}(\mu^2-1)^{\frac{1}{2}(2m-1)}\{e^{-(m-\frac{1}{2})\pi\iota}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu-0\iota)$$
$$-e^{(m-\frac{1}{2})\pi\iota}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu+0\iota)\}\,d\mu+\int_{-1}^{1}e^{-\iota x\mu}(\mu^2-1)^{\frac{1}{2}(2m-1)}\{e^{-\frac{1}{2}(m-\frac{1}{2})\pi\iota}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu-0\iota)$$
$$-e^{\frac{1}{2}(m-\frac{1}{2})\pi\iota}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu+0\iota)\}\,d\mu\Big];$$

whence

$$J_n(x)\,x^{-m}=\sqrt{\frac{2}{\pi}}\,\iota^{n-m-1}e^{(m-\frac{1}{2})\pi\iota}\,\frac{1}{2\pi}\Big[\int_{-\infty}^{-1}e^{-\iota x\mu}(\mu^2-1)^{\frac{1}{2}(2m-1)}\{e^{(n-m)\pi\iota}$$
$$+e^{-(n-m)\pi\iota}\}\,Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu$$
$$+\int_{-1}^{1}e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)}\,\iota\pi e^{(\frac{1}{2}-m)\pi\iota}P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu\Big],$$

where the harmonics are now harmonics of a real quantity. This becomes

$$J_n(x)\,x^{-m}$$
$$=\sqrt{\frac{2}{\pi}}\,\iota^{n-m}e^{(m-\frac{1}{2})\pi\iota}\,\frac{1}{2\pi}\Big[-\int_{-\infty}^{-1}e^{-\iota x\mu}(\mu^2-1)^{\frac{1}{2}(2m-1)}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu\,\sin(n-m)\pi$$
$$+\int_{-1}^{1}e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)}P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu\;\pi e^{(\frac{1}{2}-m)\pi\iota}\Big],$$

that is,

$$J_n(x)\,x^{-m}=\frac{\iota^{n-m}}{\sqrt{2\pi}}\int_{-1}^{1}e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)}P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu$$
$$-\sqrt{\frac{2}{\pi}}\,\iota^{n-m}\frac{\sin(n-m)\pi}{\pi}\int_{1}^{\infty}e^{(m-\frac{1}{2})\pi\iota}e^{\iota x\mu}(\mu^2-1)^{\frac{1}{2}(2m-1)}Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu.\qquad(2)$$

When $n-m$ is an integer the relation becomes

$$J_n(x)x^{-m} = \frac{\iota^{n-m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\,d\mu.$$

Various known results can be obtained from the above as particular cases. Putting $m = n$, and remembering that

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-n}(\mu) = \frac{1}{2^{n-\frac{1}{2}}\Pi(n-\frac{1}{2})}(1-\mu^2)^{\frac{1}{2}(2n-1)},$$

it becomes

$$J_n(x)\,x^{-n} = \frac{1}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}\int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{n-\frac{1}{2}}\,d\mu, \tag{3}$$

where the real part of n is greater than $-\frac{1}{2}$.

Putting $m = 0$, and using the relations

$$P_{n-\frac{1}{2}}^{\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi\sin\theta}}\cos n\theta,$$

$$Q_{n-\frac{1}{2}}^{\frac{1}{2}}(\cos\psi) = \sqrt{\frac{\pi}{2\sin h\psi}}\,\iota e^{-n\psi},$$

the equation (2) becomes

$$J_n(x) = \frac{\iota^n}{\pi}\left\{\int_0^{\pi} e^{-\iota x\cos\theta}\cos n\theta\,d\theta - \sin n\pi\int_0^{\infty} e^{\iota x\cosh\psi - n\psi}\,d\psi\right\}, \tag{4}$$

which holds if the imaginary part of x is greater than zero. The corresponding relation, when the imaginary part of x is less than zero, is

$$J_n(x) = \frac{(-\iota)^n}{\pi}\left\{\int_0^{\pi} e^{\iota x\cos\theta}\cos n\theta\,d\theta - \sin n\pi\int_0^{\infty} e^{-\iota x\cosh\psi - n\psi}\,d\psi\right\}.*$$

When $n-m$ is an integer, it may be shown that

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu) = \frac{2^{m-\frac{1}{2}}\Pi(m-1)\,\Pi(n-m)}{\Pi(n+m-1)\,\Pi(-\frac{1}{2})}(1-\mu^2)^{\frac{1}{2}(2m-1)}C_{n-m}^{m}(\mu),$$

where $C_{n-m}^{m}(\mu)$ is the coefficient of x^{n-m} in the expansion of $(1-2\mu x+x^2)^{-m}$ in powers of x.

Hence, when $n-m$ is an integer,

$$J_n(x)x^{-m} = \frac{2^{m-1}\Pi(m-1)\,\Pi(n-m)}{\Pi(n+m-1)}\frac{\iota^{n-m}}{\pi}\int_{-1}^{1} e^{-ix\mu}(1-\mu^2)^{m-\frac{1}{2}}C_{n-m}^{m}(\mu)\,d\mu;*$$

* *Cf.* Sonine, *Math. Ann.*, Vol. XVI.

when $m = -n$,

$$Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu) = \frac{\Pi(2n)\,\Pi(-\frac{1}{2})}{2^{n+\frac{1}{2}}\,\Pi(n)}\,\frac{e^{(n+\frac{1}{2})\pi\iota}}{(\mu^2-1)^{\frac{1}{2}(2n+1)}},$$

and relation (1) becomes

$$\int_0^\infty e^{-yz} J_n(z)\, z^n\, dz = \sqrt{\frac{2}{\pi}}\,\frac{\Pi(2n)\,(\Pi-\frac{1}{2})}{2^{n+\frac{1}{2}}\,\Pi(n)}\,\frac{e^{(n+\frac{1}{2})\pi\iota}}{(-y^2-1)^{n+\frac{1}{2}}},$$

that is,
$$\int_0^\infty e^{-yz} J_n(z)\, z^n\, dz = \frac{2^n}{\sqrt{\pi}}\,\frac{\Pi(n-\frac{1}{2})}{(1+y^2)^{n+\frac{1}{2}}},* \tag{5}$$

where the real part of n is greater than $-\frac{1}{2}$.

The relation (2) becomes in this case

$$J_n(z)\, z^n = \frac{2^n\,\Pi(n-\frac{1}{2})}{\sqrt{\pi}}\,\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota}\frac{e^{zy}\,dy}{(1+y^2)^{n+\frac{1}{2}}},$$

that is,

$$J_n(z)\, z^n = \frac{2^{n-1}\,\Pi(n-\frac{1}{2})}{\Pi\sqrt{\pi}}\int_{-\infty-\iota c}^{\infty-\iota c}\frac{e^{\iota zy}\,dy}{(1-y^2)^{n+\frac{1}{2}}}, \tag{6}$$

where the real part of n is greater than $-\frac{1}{2}$.

This integral can be transformed into one along the real axis when the real part of n is less than $\frac{1}{2}$, and then

$$J^n(z)\, z^n = \frac{2^{n+1}}{\sqrt{\pi}\,\Pi(-n-\frac{1}{2})}\int_1^\infty \frac{\sin xy\, dy}{(y^2-1)^{n+\frac{1}{2}}} \quad (\tfrac{1}{2} > R(n) > -\tfrac{1}{2}), \tag{7}$$

a result given by Sonine. When in relation (2) $m = \frac{1}{2}$, the expression for $J_n(x)\,x^{-\frac{1}{2}}$ in terms of Legendre functions is obtained.

Taking the expression for a Bessel function

$$J_n(z) = \frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{\frac{1}{2}z(s-1/s)}\,\frac{ds}{s^{n+1}},$$

it may be written

$$J_n(a\sqrt{x})\, x^{\frac{1}{2}n} = \frac{1}{2\pi\iota}\left(\frac{2}{a}\right)^n \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{4}(a^2xu-1/u)}\,\frac{du}{u^{n+1}},$$

when the real part of n is greater than -1, which is

$$J_n(a\sqrt{x})\, x^{\frac{1}{2}n} = \frac{1}{2\pi\iota}\left(\frac{a}{2}\right)^n \int_{c-\infty\iota}^{c+\infty\iota} e^{xv-a^2/4v}\,\frac{dv}{v^{n+1}}.$$

The application of relation (B) gives the result

$$\int_0^\infty e^{-xy} J_n(a\sqrt{x})\, x^{\frac{1}{2}n}\, dx = \left(\frac{a}{2}\right)^n \frac{e^{-a^2/4y}}{y^{n+1}},$$

---

* Sonine, *l.c.*

that is,
$$\int_0^\infty e^{-yx^2} J_n(ax)\, x^{n+1}\, dx = \frac{a^n e^{-a^2/4y}}{(2y)^{n+1}}, \tag{8}$$
when the real part of n is greater than -1, a result given by Weber* and by Hankel.†

An expression for the second solution of Bessel's equation is

$$\frac{\pi}{\sin n\pi}\left[\iota^n J_{-n}(\iota z) - \iota^{-n} J_n(\iota z)\right] = \int_0^\infty e^{-\frac{1}{2}z(s+1/s)} s^{n-1}\, ds,$$

which may be written

$$\frac{\pi}{\sin n\pi}\left[\iota^n J_{-n}(\iota a x^{\frac{1}{2}}) - \iota^{-n} J_n(\iota a x^{\frac{1}{2}})\right] = \frac{2^n x^{\frac{1}{2}n}}{a^n} \int_0^\infty e^{-x\sigma - a^2/4\sigma} \sigma^{n-1}\, d\sigma;$$

therefore, by relation (A),

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{xy}\left[\iota^n J_{-n}(\iota a x^{\frac{1}{2}}) - \iota^{-n} J_n(\iota a x^{\frac{1}{2}})\right] \frac{\pi}{\sin n\pi} \frac{dx}{x^{\frac{1}{2}n}} = \frac{2^n y^{n-1}}{a^n} e^{-a^2/4y},$$

that is,

$$\frac{1}{2\pi\iota}\int_{\infty+\iota c'}^{\infty+\iota c'} e^{-yx^2} \frac{\pi}{\sin n\pi}\left[\iota^{2n} J_{-n}(ax) - J_n(ax)\right] \frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n} e^{-a^2/4y}.$$

Now
$$\int_{\infty+\iota c'}^{\infty+\iota c'} J_n(ax)\, \frac{dx}{x^{n-1}} = 0;$$

therefore
$$\frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_{-\infty+\iota c'}^{\infty+\iota c'} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n} e^{-a^2/4y}.$$

This can be transformed into integrals along real paths as follows: the relation may be written

$$\frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_\rho^\infty e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}}(1 - e^{-2n\pi\iota}) + \frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_{-\rho}^{\rho} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}}$$
$$= \frac{(2y)^{n-1}}{a^n} e^{-a^2/4y}.$$

Writing
$$I = \frac{e^{n\pi i}}{2\iota \sin n\pi}\int_{-\rho}^{\rho} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}},$$
$$x = \rho e^{\phi\iota},$$

then
$$I = \frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_\pi^0 e^{-y\rho^2 e^{2\phi i}} J_{-n}(a\rho e^{\phi\iota})\, \frac{\iota\, d\phi}{\rho^{n-2} e^{(n-2)\phi\iota}},$$

---

* *Crelle*, Vol. LXIX. † *Math. Ann.*, Vol. VIII.

that is,

$$I=-\frac{e^{n\pi\iota}}{2\rho^{n-2}\sin n\pi}\sum_0^\infty\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^\pi e^{-y\rho^2e^{2\phi\iota}-2(n-p-1)\phi\iota}\,d\phi$$

or

$$I=-\frac{e^{n\pi\iota}}{4\rho^{n-2}\sin n\pi}\sum_0^\infty\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^{2\pi} e^{-y\rho^2e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta.$$

Now $\int_0^{2\pi} e^{-y\rho^2e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$

$$=2e^{(p+1-n)\pi\iota}\int_0^\pi e^{y\rho^2\cos\theta}\cos\{y\rho^2\sin\theta-(n-p-1)\theta\}\,d\theta,$$

that is,

$$\int_0^{2\pi} e^{-y\rho^2e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$$

$$=2e^{(p+1-n)\pi\iota}\sin(p+1-n)\pi\int_1^\infty e^{-y\rho^2/\sigma}\frac{d\sigma}{\sigma^{p-n+2}}$$

when $p>n+1$, and

$$\int_0^{2\pi} e^{-y\rho^2e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$$

$$=2e^{(p+1-n)\pi\iota}\left[\sin(n-p-1)\pi\int_1^\infty e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}}+\frac{\pi(y\rho^2)^{n-p-1}}{\Pi(n-p-1)}\right]$$

when $p<n+1$; therefore

$$I=\frac{\pi}{2\sin n\pi}\sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}}{2^{-n+2p}\Pi(p)\Pi(-n+p)\Pi(n-p-1)}$$

$$-\tfrac{1}{2}\sum_0^{(n-1)}\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\rho^{n-2}\Pi(p)\Pi(-n+p)}\int_1^\infty e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}}$$

$$+\tfrac{1}{2}\sum_{(n)}^\infty\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\rho^{n-2}\Pi(p)\Pi(-n+p)}\int_0^1 e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}},$$

where (n) denotes the greatest integer less than n, and this is equivalent to

$$I=\sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}(-)^p}{2^{2p-n+1}\Pi(p)}-\sum_0^{(n-1)}\frac{a^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_\rho^\infty e^{-yx^2}x^{2p-2n+1}\,dx$$

$$+\sum_{(n)}^\infty\frac{a^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^\rho e^{-yx^2}x^{2p-2n+1}\,dx.$$

Hence $\int_0^\infty e^{-yx^2}\left[J_{-n}(ax)-\sum_0^{(n-1)}\frac{a^{-n+2p}(-)^px^{-n+2p}}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\right]\frac{dx}{x^{n-1}}$

$$=\frac{(2y)^{n-1}}{a^n}e^{-a^2/4y}-\sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}(-)^p}{2^{2p-n+1}\Pi(p)},$$

or, denoting the part of the series for $J_{-n}(ax)$ which begins at the $[(n)+1]$-th term by $J_{-n}^{(n)}(ax)$,

$$\int_0^\infty e^{-yx^2} J_{-n}^{(n)}(ax)\,\frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n}\,e^{-a^2/4y} - \sum_0^{(n-1)} \frac{a^{-n+2p}y^{n-p-1}(-)^p}{2^{2p-n+1}\,\Pi(p)}. \quad (9)$$

When n is an integer this becomes

$$\int_0^\infty e^{-yx^2} J_n(ax)\,\frac{dx}{x^{n-1}} = \sum_0^\infty \frac{a^{n+2p}y^{-p-1}(-)^p}{2^{n+2p+1}\,\Pi(n+p)}, \quad (10$$

which can easily be shown to be true for all values of n.

The relation (2) when $n-m$ is a positive integer is

$$J_n(x)\,x^{-m} = \frac{\iota^{n-m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\,d\mu,$$

and this may be written

$$J_n(-\iota x)\,x^{-m} = \frac{\iota^{n-2m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x\mu}(1-\mu^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\,d\mu;$$

hence by relation (A)

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{xy} J_n(-\iota x)\,x^{-m}\,dx = \frac{\iota^{n-2m}}{\sqrt{2\pi}}\,(1-y^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(y),$$

that is,

$$\frac{1}{2}\int_{-\infty-\iota c}^{\infty-\iota c} e^{\iota xy} J_n(x)\,x^{-m}\,dx = \sqrt{\frac{\pi}{2}}\,\iota^{n-m}(1-y^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(y);$$

therefore

$$\int_0^\infty J_n(x)\,x^{-m}\cos\left(xy+\frac{n-m}{2}\pi\right)dx = \sqrt{\frac{\pi}{2}}\,(1-y^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(y)$$

when $n > -\frac{1}{2}$ and $|y| > 1$, and vanishes when $|y| > 1$, or

$$\int_0^\infty J_n(ax)\,x^{-m}\cos\left(bx+\frac{n-m}{2}\pi\right)dx = \sqrt{\frac{\pi}{2a}}\,(a^2-b^2)^{\frac{1}{2}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}\left(\frac{b}{a}\right) \quad (11)$$

when $a > b$, and vanishes when $a < b$, n being greater than $-\frac{1}{2}$. When $n = m$, this becomes

$$\left.\begin{aligned}\int_0^\infty J_n(ax)\,x^{-n}\cos bx\,dx &= \frac{\sqrt{\pi}\,(a^2-b^2)^{n-\frac{1}{2}}}{(2a)^n\,\Pi(n-\frac{1}{2})} \quad (a > b)\\ &= 0 \quad (a < b)\end{aligned}\right\}, \quad (12)$$

a known result.* When $m = \frac{1}{2}$, and n is a positive integer, the relation (11) becomes

$$\left.\begin{aligned}\int_0^\infty \cos bx\, J_{n+\frac{1}{2}}(ax)\, x^{-\frac{1}{2}} dx &= \sqrt{\frac{\pi}{2a}}\, P_n\left(\frac{b}{a}\right) \quad (a > b)\\ &= 0 \quad (a < b)\end{aligned}\right\}. \tag{13}$$

Similarly from the relation (7) it can be deduced that

$$\left.\begin{aligned}\int_0^\infty \sin ax\, J_n(bx)\, x^n\, dx &= \frac{\sqrt{\pi}\,(2b)^n}{\Pi(-n-\frac{1}{2})(a^2-b^2)^{n+\frac{1}{2}}} \quad (a > b)\\ &= 0 \quad (a < b)\end{aligned}\right\} \tag{14}$$

when $\frac{1}{2} > n > -\frac{1}{2}$, a known result.

Writing the integral

$$\frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} I_n(z)\, z^m dz = \psi(y),$$

where the real parts of $ye^{\alpha\iota}$, $ye^{\beta\iota}$ are negative, then

$$\psi(y) = \sum_0^\infty \frac{1}{2^{n+2p}}\,\frac{1}{\Pi(n+p)\,\Pi(p)}\,\frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} z^{n+m+2p} dz,$$

that is, $\psi(y) = \sum_0^\infty \dfrac{y^{-2p-n-m-1}}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)\,\Pi(-2p-n-m-1)}$

when $|y| > 1$, or

$$\psi(y) = -\frac{\sin(n+m)\pi}{\pi}\sum_0^\infty \frac{\Pi(2p+n+m)}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)}\,\frac{1}{y^{2p+n+m+1}},$$

$$\text{whence } \psi(y) = -\frac{\sin(n+m)\pi}{\pi\sqrt{\pi}}\,\frac{2^m}{y^{n+m+1}}\,\frac{\Pi\left(\frac{n+m}{2}\right)\Pi\left(\frac{n+m-1}{2}\right)}{\Pi(n)}$$
$$\times F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2},\ n+1,\ \frac{1}{y^2}\right),$$

which may be written

$$\psi(y) = -\sqrt{\frac{2}{\pi}}\,\frac{\sin(n+m)\pi}{\pi}\, e^{-(m+\frac{1}{2})\pi\iota}\,(y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})}\, Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(y).$$

---

* *Cf.* Sonine, *l.c.*

If now $K_n(z)$ be the second solution of Bessel's equation with imaginary argument, where

$$K_n(z) = \frac{\pi}{2\sin n\pi}[I_{-n}(z) - I_n(z)],$$

the integral
$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz = 0$$

when $|y| < 1$, and by the above

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz$$
$$= \frac{1}{\sqrt{2\pi}}\frac{e^{-(m+\frac{1}{2})\pi\iota}}{\sin n\pi}(y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})}\left[\sin(n+m)\pi Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(y) - \sin(m-n)\pi Q_{-n-\frac{1}{2}}^{m+\frac{1}{2}}(y)\right],$$

that is,

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz = \sqrt{\frac{\pi}{2}}\frac{\cos(m+\frac{1}{2})\pi}{\sin n\pi}(y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})} P_{n-\frac{1}{2}}^{m+\frac{1}{2}}(y) \tag{15}$$

when $|y| > 1$. The application of relation (B) gives the result

$$K_n(x)\, x^m = \sqrt{\frac{\pi}{2}}\frac{\cos(m+\frac{1}{2})\pi}{\sin n\pi}\int_1^\infty e^{-xy}(y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})} P_{n-\frac{1}{2}}^{m+\frac{1}{2}}(y)\, dy \tag{16}$$

when $m+\frac{1}{2} < 1$. Putting $m = -n$, this becomes

$$K_n(x)\, x^{-n} = \sqrt{\frac{\pi}{2}}\int_1^\infty e^{-xy}(y^2-1)^{-\frac{1}{2}(\frac{1}{2}-n)} P_{n-\frac{1}{2}}^{\frac{1}{2}-n}(y)\, dy,$$

that is,
$$K_n(x)\, x^{-n} = \frac{\sqrt{\pi}}{2^n\Pi(n-\frac{1}{2})}\int_1^\infty e^{-xy}(y^2-1)^{n-\frac{1}{2}}\, dy, \tag{17}$$

a known result. Putting $m = -\frac{1}{2}$, it becomes

$$K_n(x)\, x^{-\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\frac{1}{\sin n\pi}\int_1^\infty e^{-xy} P_{n-\frac{1}{2}}(y)\, dy. \tag{18}$$

When $m = 0$, the relation (15), using its first form, becomes

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, dz = \frac{e^{-\frac{1}{2}\pi\iota}}{\sqrt{2\pi}}(y^2-1)^{-\frac{1}{4}}\left[Q_{n-\frac{1}{2}}^{\frac{1}{2}}(y) + Q_{-n-\frac{1}{2}}^{\frac{1}{2}}(y)\right],$$

that is,
$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{z\cosh y} K_n(z)\, dz = \frac{\cosh ny}{\sinh y}; \tag{19}$$

and therefore, by relation (B),

$$K_n(x) = \int_0^\infty e^{-x\cosh y} \cosh ny \, dy, \tag{20}$$

a known result.

It has been proved that

$$J_n(ax)\, J_n(bx) = \frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t - [x^2(a^2+b^2)]/2t}\, I_n\left(\frac{x^2ab}{t}\right) \frac{dt}{t},$$

when the real part of n is greater than -1.*

This may be written

$$J_n(ax^{\frac{1}{2}})\, J_n(bx^{\frac{1}{2}}) = \frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{xs - (a^2+b^2)/4s}\, I_n\left(\frac{ab}{2s}\right) \frac{ds}{s},$$

whence, by relation (B),

$$\int_0^\infty e^{-yx} J_n(ax^{\frac{1}{2}})\, J_n(bx^{\frac{1}{2}})\, dx = \frac{e^{-(a^2+b^2)/4y}}{y}\, I_n\left(\frac{ab}{2y}\right),$$

that is,

$$\int_0^\infty e^{-yx^2} J_n(ax)\, J_n(bx)\, x\,dx = \frac{e^{-(a^2+b^2)/4y}}{2y}\, I_n\left(\frac{ab}{2y}\right).\dagger \tag{21}$$

From this it follows that

$$\int_{-\infty}^\infty e^{-yx^2} J_n(ax)\, J_n(bx)\, x\,dx = (1-e^{2n\pi\iota})\, \frac{e^{-(a^2+b^2)/4y}}{2y}\, I_n\left(\frac{ab}{2y}\right).$$

Now

$$\int_{-\infty}^\infty e^{-yx^2} J_n(ax)\, J_{-n}(bx)\, x\,dx = 0;$$

therefore

$$\int_{-\infty}^\infty e^{-yx^2} J_n(ax)\left[\iota^n J_{-n}(bx) - \iota^{-n} J_n(bx)\right] x\,dx$$
$$= \iota^{-n}(e^{2n\pi\iota}-1)\, \frac{e^{-(a^2+b^2)/4y}}{2y}\, I_n\left(\frac{ab}{2y}\right),$$

that is,

$$\int_{c-\infty\iota}^{c+\infty\iota} e^{y\xi^2} J_n(a\xi\iota)\left[\iota^n J_{-n}(b\xi\iota) - \iota^{-n} J_n(b\xi\iota)\right] \xi\,d\xi$$
$$= \iota^{n+1} \sin n\pi\, \frac{e^{-(a^2+b^2)/4y}}{y}\, I_n\left(\frac{ab}{2y}\right);$$

---

* *Proc. Lond. Math. Soc.*, Vol. XXXII.

† *Cf.* Weber, Hankel, *l.c.*

whence, by relation (B),

$$\frac{1}{2\pi\iota}\int_0^\infty e^{-x^2 y}\,\iota^{n+1}\sin n\pi\, e^{-(a^2+b^2)/4y}\, I_n\left(\frac{ab}{2y}\right)\frac{dy}{y}$$

$$= \tfrac{1}{2}J_n(ax\iota)\left[\iota^n J_{-n}(bx\iota)-\iota^{-n}J_n(bx\iota)\right];$$

and therefore

$$\int_0^\infty e^{-\frac{1}{2}x[s+(a^2+b^2)/s]}\, I_n\left(\frac{abx}{s}\right)\frac{ds}{s} = \frac{\pi}{\sin n\pi}J_n(ax\iota)\left[J_{-n}(bx\iota)-\iota^{-2n}J_n(bx\iota)\right], \tag{22}$$

a result equivalent to one given in the previous paper.*

Writing the integral

$$\int_0^\infty e^{-yx^2}J_n(ax)\,x^{2m-n+1}\,dx = f(y),$$

then

$$f(y) = \sum_0^\infty \frac{(-)^p a^{n+2p}}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)}\int_0^\infty e^{-yx^2}x^{2m+2p+1}\,dx,$$

that is,

$$f(y) = \sum_0^\infty \frac{(-)^p a^{n+2p}}{2^{n+2p+1}\,\Pi(n+p)\,\Pi(p)}\,\frac{\Pi(m+p)}{y^{m+p+1}}$$

or $$f(y) = \sum_0^\infty \frac{(-)^p a^{n+2p}}{2^{n+2p+1}\,\Pi(p)\,y^{m+p+1}}\,\frac{1}{\Pi(n-m-1)}\int_0^1 z^{m+p}(1-z)^{n-m-1}\,dz,$$

when $n > m > -1$; whence

$$\int_0^\infty e^{-yx^2}J_n(ax)\,x^{2m-n+1}\,dx$$

$$= \frac{1}{2^n a^n y^{m+1}\,\Pi(n-m-1)}\int_0^a e^{-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\,\xi^{2m+1}\,d\xi \quad (n > m > -1). \tag{23}$$ †

This relation may be written

$$\int_0^\infty e^{-yx}J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n}\,dx$$

$$= \frac{1}{2^{n-1}a^n y^{m+1}\,\Pi(n-m-1)}\int_0^a e^{-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\,\xi^{2m+1}\,d\xi,$$

---

* *Proc. Lond. Math. Soc.*, Vol. XXXII.

† *Cf.* Sonine, *l.c.*

and therefore, by relation (B),

$$J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n} = \frac{1}{2^{n-1}a^n\Pi(n-m-1)}\,\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota}\int_0^a e^{xy-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\xi^{2m+1}d\xi\,\frac{dy}{y^{m+1}},$$

that is,

$$J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n} = \frac{x^{\frac{1}{2}m}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_m(\xi x^{\frac{1}{2}})(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi$$

or
$$J_n(ax) = \frac{x^{n-m}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_m(\xi x)(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi$$

$$(n > m > -1), \quad (24)$$

a known relation.

From (24) it follows that

$$J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}}) = \frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_n(bx^{\frac{1}{2}})\,J_m(\xi x^{\frac{1}{2}})(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi,$$

that is,

$$J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}}) = \frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a (a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi \times\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[x(b^2+\xi^2)]/2t}\,I_m\left(\frac{b\xi x}{t}\right)\frac{dt}{t}$$

or
$$J_n(ax^{\frac{1}{2}})\,J_m(bx)^{\frac{1}{2}} = \frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a (a^2-\xi^2)\,\xi^{m+1}d\xi \times\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{xs-(b^2+\xi^2)/4s}\,I_m\left(\frac{b\xi}{2s}\right)\frac{ds}{s};$$

therefore, by relation (B),

$$\int_0^\infty e^{-yx}J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}})\,x^{\frac{1}{2}(m-n)}\,dx = \frac{1}{2^{n-m-1}\Pi(n-m-1)\,a^ny}\int_0^a (a^2-\xi^2)^{n-m-1}\xi^{m+1}e^{-(b^2+\xi^2)/4y}\,I_m\left(\frac{b\xi}{2y}\right)d\xi,$$

that is,

$$\int_0^\infty e^{-yx^2}J_n(ax)\,J_m(bx)\,x^{m-n+1}dx = \frac{1}{2^{n-m}\Pi(n-m-1)\,a^ny}\int_0^a (a^2-\xi^2)^{n-m-1}\xi^{m+1}e^{-(b^2+\xi^2)/4y}\,I_m\left(\frac{b\xi}{2y}\right)d\xi$$

$$(n > m > -1). \quad (25)$$

Proceeding to the limit when $y = 0$, the integrand on the left-hand side vanishes except for values of ξ in the neighbourhood of b, and, evaluating it,

$$\left.\begin{aligned}\int_0^\infty J_n(ax)\,J_m(bx)\,x^{m-n+1}\,dx &= \frac{(a^2-b^2)^{n-m-1}b^m}{2^{n-m-1}\Pi(n-m-1)\,a^n} \quad (a>b)\\ &= 0 \quad (a<b,\ n>m>-1)\end{aligned}\right\}. \tag{26}$$

Relations (25) and (26) have been given by Sonine.*

Again, from (24), writing

$$X = \frac{\pi}{2\sin m\pi}\,\frac{J_n(ax\iota)}{(x\iota)^{n-m}}\left[J_{-m}(bx\iota) - \iota^{-2m}J_m(bx\iota)\right],$$

then

$$X = \frac{\pi}{\sin m\pi\; 2^{n-m}\,\Pi(n-m-1)\,a^n}\int_0^a \xi^{m+1}(a^2-\xi^2)^{n-m-1}J_m(\xi x\iota) \times\left[J_{-m}(bx\iota) - \iota^{-2m}J_m(bx\iota)\right]d\xi,$$

that is, by (22),

$$X = \frac{1}{2^{n-m}\Pi(n-m-1)\,a^n}\int_0^a\int_0^\infty \xi^{m+1}(a^2-\xi^2)^{n-m-1}e^{-x^2s-(b^2+\xi^2)/4s}I_n\left(\frac{b\xi}{2s}\right)\frac{ds}{s}\,d\xi,$$

which, using (25), becomes

$$X = \int_0^\infty\int_0^\infty e^{-x^2s-\xi^2s}J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}\,d\xi\,ds,$$

that is, $$X = \int_0^\infty \frac{J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}}{\xi^2+x^2}\,d\xi;$$

and therefore

$$\int_0^\infty \frac{J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}\,d\xi}{\xi^2+x^2} = \frac{\pi}{2\sin m\pi}\,\frac{J_n(ax\iota)}{(x\iota)^{n-m}}\left[J_{-m}(bx\iota) - \iota^{-2m}J_m(bx\iota)\right]$$
$$(b>a,\ n>m>-1). \tag{27}$$

The relation (23) suggests the evaluation of the integral

$$\int_0^a e^{-\xi^2/4y}J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\}\,(a^2-\xi^2)^{\frac{1}{2}(n-m-1)}\xi^{2m+1}\,d\xi.$$

Writing $f(y)$ for this integral,

$$f(y) = \frac{1}{c^{n-m-1}}\,\frac{1}{2\pi\iota}\int_0^a\int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{4}tc^2(a^2-\xi^2)-1/2t-\xi^2/4y}\,\xi^{2m+1}\,d\xi\,\frac{dt}{t^{n-m}},$$

* *L.c.*

that is, performing the integration with respect to ξ,

$$f(y) = \frac{\Pi(m)}{c^{n-m-1} 4\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{4}a^2c^2t - 1/2t} \frac{1}{\left(\frac{1}{4y} + \frac{tc^2}{2}\right)^{m+1}} \frac{dt}{t^{n-m}},$$

which may be written

$$f(y) = \frac{2^{m+1}\Pi(m)\, y^{m+1}}{c^{n+m+1} 4\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{4}a^2c^2t - 1/2t} \frac{1}{\left(\frac{1}{2tc^2} + y\right)^{m+1}} \frac{dt}{t^{n+1}};$$

whence

$$f(y) = \frac{2^{m+1} y^{m+1}}{c^{n+m+1} 2\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} \int_0^\infty e^{\frac{1}{4}a^2c^2t - 1/2t - y\zeta^2 - \zeta^2/2tc^2} \zeta^{2m+1}\, d\zeta \frac{dt}{t^{n+1}},$$

that is,

$$f(y) = 2^{m+1} y^{m+1} c^{n-m-1} a^{2n} \frac{1}{2\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} \int_0^\infty e^{\frac{1}{4}s - [a^2(c^2+\zeta^2)/2s] - y\zeta^2} \zeta^{2m+1}\, d\zeta \frac{ds}{s^{n+1}};$$

and therefore

$$(y) = 2^{m+1} y^{m+1} c^{n-m-1} a^n \int_0^\infty e^{-y\zeta^2} \frac{J_n\{a\sqrt{(c^2+\zeta^2)}\}}{(c^2+\zeta^2)^{\frac{1}{2}n}} \zeta^{2m+1}\, d\zeta,$$

that is,

$$\int_0^\infty e^{-y\zeta^2} \frac{J_n\{a\sqrt{(c^2+\zeta^2)}\}}{(c^2+\zeta^2)^{\frac{1}{2}n}} \zeta^{2m+1}\, d\zeta$$

$$= \frac{1}{a^n c^{n-m-1}(2y)^{m+1}} \int_0^a e^{-\xi^2/4y} J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\} (a^2-\xi^2)^{\frac{1}{2}(n-m-1)} \xi^{2m+1}\, d\xi. \tag{28}$$

From this it follows, by applying relation (B) as in (24), that

$$\frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} = \frac{1}{a^n c^{n-m-1} x^m} \int_0^a J_m(x\xi)\, J_n\{c\sqrt{(a^2-\xi^2)}\} (a^2-\xi^2)^{\frac{1}{2}n} \xi^{m+1} d\xi$$

$$(n > m > -1). \tag{29}$$

Similarly as in (25), by (B),

$$\int_0^\infty e^{-yx^2} \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} J_m(bx)\, x^{m+1} dx$$

$$= \frac{1}{2a^n c^{n-m-1} y} \int_0^a e^{-(b^2+\xi^2)/4y} I_m\left(\frac{b\xi}{2y}\right) J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\}$$

$$\times (a^2-\xi^2)^{\frac{1}{2}(n-m-1)} \xi^{2m+1} d\xi \quad (n > m > -1). \tag{30}$$

As in (26), when $y = 0$, this becomes

$$\int_0^\infty \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} J_m(bx)\, x^{m+1}\, dx$$

$$= \frac{J_{n-m-1}\{c\sqrt{(a^2-b^2)}\}\,(a^2-b^2)^{\frac{1}{2}(n-m-1)}\, b^m}{a^n c^{n-m-1}} \quad (a > b)$$

and $$= 0 \quad (a < b)$$

$$(n > m > -1). \quad (31)$$

Again, as in (27), it may be proved that

$$\int_0^\infty \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} \frac{J_m(bx)\, x^{m+1}}{x^2+\xi^2}\, dx$$

$$= \frac{\pi}{2\sin m\pi} \frac{J_n\{a\sqrt{(c^2-\xi^2)}\}}{(c^2-\xi^2)^{\frac{1}{2}n}} (\xi\iota)^m \{J_{-m}(b\xi\iota) - \iota^{-2m} J_m(b\xi\iota)\}$$

$$(b > a,\ n > m > -1). \quad (32)$$

Relations (29) and (32) have been given by Sonine, the method of proof of (32) being much more difficult.

The preceding examples are illustrative of the results which can be obtained by using the relations (A) and (B), and are capable of being greatly extended.

Generational Relations for the Abstract Group Simply Isomorphic with the Linear Fractional Group in the GF $[2^n]$. *By* L. E. Dickson.

Received December 22nd, 1902. Read January 8th, 1903.

1. The object of this paper is the determination of two linear fractional transformations A and B having the following properties:

(i.) A and B generate the group Γ of all linear fractional transformations of determinant unity in the $GF[2^n]$ $(n > 1)$.

(ii.) A is of period 2^n+1, B of period 2, AB of period 3.

(iii.) A and B satisfy relations of the form

$$(1) \quad (BA^rBA^s)^2 = I \quad (r = 1, 2, \ldots, 2^n),$$

the integer s being uniquely determined modulo 2^n+1 by r.