



XXVII. On the relative values of the pieces in chess

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more simply. If, namely, we put

$$U = \frac{ee'}{r},$$

$$V = \frac{ee'}{r} vv' \cos \epsilon,$$

$$= k \frac{ee'}{r} \left(\frac{dx}{dt} \frac{dx'}{dt} + \frac{dy}{dt} \frac{dy'}{dt} + \frac{dz}{dt} \frac{dz'}{dt} \right),$$

and regard U as a function of the six coordinates x, y, z, x', y', z' , and V as a function of these six coordinates and their differential-coefficients according to t , we can write

$$X_{ee'} = \frac{d(V-U)}{dx} - \frac{d}{dt} \left(\frac{dV}{d \frac{dx}{dt}} \right);$$

and in just the same manner can the other five force-components be derived from the two functions U and V by differentiation.

For the components of the force which is exerted upon a galvanic current-element ds by a current-element ds' we get from the simplified form of the fundamental equations the following expressions:—

$$cii' ds ds' \left(\frac{d \frac{1}{r}}{dx} \cos \epsilon - \frac{d \frac{1}{r}}{ds} \frac{dx'}{ds'} \right);$$

$$cii' ds ds' \left(\frac{d \frac{1}{r}}{dy} \cos \epsilon - \frac{d \frac{1}{r}}{ds} \frac{dy'}{ds'} \right);$$

$$cii' ds ds' \left(\frac{d \frac{1}{r}}{dz} \cos \epsilon - \frac{d \frac{1}{r}}{ds} \frac{dz'}{ds'} \right).$$

XXVII. *On the Relative Values of the Pieces in Chess.* By
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Cambridge*.

THE object of this paper is to ascertain the relative values of the pieces on a chessboard. If a piece be placed on a square of a chessboard, the number of squares it commands depends in general on its position. If we calculate the average number of squares which any particular piece commands

* Communicated by the Author.

when placed in succession on every square of the board, it seems fair to assume that this gives a not very inexact measure of the value of the piece.

For special reasons the above problem is stated in the following manner :—“ A king and a piece of different colours are placed at random on two squares of a chessboard of n^2 squares : it is required to find the chance that the king is in check.”

The ordinary chessboard has an even number of squares ; and as some of the results take different forms for odd and even values of n , the results are given merely for even values of n , and the results for the ordinary chessboard of 64 squares deduced from them.

As the relative values of the knight and bishop on the ordinary chessboard on this hypothesis came out in a ratio very different from the ratio that is ordinarily received by chess-players, it occurred to the author to investigate the chance that when a king and a piece of different colours were placed at random on two squares of a board, the king would be in check but unable to take the piece. This check is called *safe* check in contradistinction to a mere check, which may be safe or unsafe and which is called *simple* check.

Simple check from one rook.

A rook in any position checks $2(n-1)$ squares. The king can be placed on n^2-1 for any given position of the rook. The chance of check, therefore, is

$$\frac{2(n-1)}{n^2-1} = \frac{2}{n+1}.$$

If $n=8$,

the chance = $\frac{2}{9}$.

Safe check from one rook.

a	b	b	
b	c	c	
b	c	c	

If the rook be on a corner square, it could be taken by a king in check on two squares, and so on. The number of safe checks by a rook on the different squares is given by the following scheme :—

Rook on	Number of safe checks.	Number of such positions of the rook.
<i>a</i>	$2n-4$	4
<i>b</i>	$2n-5$	$4(n-2)$
<i>c</i>	$2n-6$	$(n-2)^2$

The chance

$$= \frac{4(2n-4) + 4(n-2)(2n-5) + (n-2)^2(2n-6)}{n^2(n^2-1)} = \frac{2(n-2)}{n(n+1)}.$$

When $n=8$,

the chance = $\frac{1}{6}$.

Simple check with one knight.

The number of squares attacked by a knight placed on any square of a chessboard is given by the following scheme :—

<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>	
<i>b</i>	<i>c</i>	<i>d</i>	<i>d</i>	
<i>c</i>	<i>d</i>	<i>e</i>	<i>e</i>	
<i>c</i>	<i>d</i>	<i>e</i>	<i>e</i>	

Knight on	Number of checks.	Number of such positions of knight.
<i>a</i>	2	4
<i>b</i>	3	8
<i>c</i>	4	$4(n-3)$
<i>d</i>	6	$4(n-4)$
<i>e</i>	8	$(n-4)^2$

The chance of check

$$\frac{2 \cdot 4 + 3 \cdot 8 + 4 \cdot 4(n-3) + 6 \cdot 4(n-4) + 8(n-4)^2}{n^2(n^2-1)} = \frac{8(n-2)}{n^2(n+1)}.$$

If $n=8$,

chance = $\frac{1}{12}$.

For a knight all checks are *safe* checks.

Simple check with one bishop.

We will at first assume that a bishop can be put on either a

white or a black square. The number of squares attacked by a bishop on any square of the chessboard is given by the following scheme :—

<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	
<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	
<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>		
<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>		

Bishop on		Number of checks.	Number of such positions of bishop.
<i>a</i>	$n-1$	$4(n-1)$
<i>b</i>	$n+1$	$4(n-3)$
<i>c</i>	$n+3$	$4(n-5)$
<i>d</i>	$n+5$	$4(n-7)$
.	.	.	.

If n be even, we have $\frac{n}{2}$ lines of this scheme all obeying the same law ; if n be odd, we have $\frac{n-1}{2}$ such lines, and another for the middle square of the board for which the number of checks $= 2(n-1)$.

Now m terms of the series

$$\begin{aligned}
 & (n-1)(n-1) + (n+1)(n-3) + (n+3)(n-5) + \&c. \\
 & = n^2 - 2n + 1 + n^2 - 2n - 1 \cdot 3 + n^2 - 2n - 3 \cdot 5 + \&c. \\
 & \quad + n^2 - 2n - (2m-3)(2m-1) \\
 & = m(n^2 - 2n) + 1 - (1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots \\
 & \quad + (2m-3)(2m-1)) \\
 & = m(n^2 - 2n) + 1 - \frac{(2m-3)(2m-1)(2m+1) + 3}{6}.
 \end{aligned}$$

Put $m = \frac{n}{2}$, then the numerator of the chance-fraction

$$\begin{aligned}
 & = 4 \left\{ \frac{n}{2} (n^2 - 2n) + 1 - \frac{1}{6} (n-3)(n-1)(n+1) - \frac{1}{2} \right\} \\
 & = \frac{2n}{3} (n-1)(2n-1).
 \end{aligned}$$

Put $m = \frac{n-1}{2}$, then the numerator of the chance-fraction

$$= 4 \left\{ \frac{n-1}{2} (n^2 - 2n) + 1 - \frac{(n-4)(n-2)n}{6} - \frac{1}{2} \right\} + 2(n-1) \\ = \frac{2n}{3} (n-1)(2n-1).$$

The chance, therefore, for both odd and even values of n

$$= \frac{2 \cdot \frac{n}{3} (n-1)(2n-1)}{n^2(n^2-1)} = \frac{2}{3} \cdot \frac{2n-1}{n(n+1)}.$$

If $n=8$,

$$\text{the chance} = \frac{5}{36}.$$

If the bishop be restricted to squares of one and the same colour, white or black, then if n be even the numerator and the denominator of the chance-fraction are both halved, and the chance is the same as before. If n be odd, we have two cases to consider. If we take the squares of the same colour as the centre square, the numerator of the chance-fraction is twice $\frac{n-1}{2}$ terms of the series

$$(n-1)(n-1) + (n+1)(n-3) + (n+3)(n-5) + \&c. \\ \text{increased by } 2(n-1), \\ = \frac{n-1}{3} (2n^2 - n + 3).$$

If we take the squares of the other colour, the numerator of the chance-fraction is twice the $\frac{n-1}{2}$ terms of the above series

$$= \frac{(n-1)(n+1)(2n-3)}{3}.$$

Now, when a bishop occupies one square, the king can be placed on any one of $n^2 - 1$ squares. Therefore the chance of check when the bishop is restricted to squares of the same colour as the middle one

$$= \frac{\frac{n-1}{3} (2n^2 - n + 3)}{\frac{n^2+1}{2} \cdot (n^2-1)} = \frac{2}{3} \cdot \frac{2n^2 - n + 3}{(n+1)(n^2+1)}.$$

When the bishop is restricted to squares of the other colour, the chance of check

$$= \frac{\frac{(n-1)(n+1)(2n-3)}{3}}{\frac{n^2-1}{2} \cdot (n^2-1)} = \frac{2}{3} \cdot \frac{2n-3}{n^2-1}.$$

Safe check with one bishop.

This is best found by finding the number of unsafe checks. It will be seen that if the bishop be on a corner square, there is one unsafe check; for any other outside square there are two unsafe checks; and for any other square there are four. The whole number therefore of unsafe checks

$$= 4 + 4(n-2) \cdot 2 + (n-2)^2 \cdot 4 = 4(n^2 - 4n + 4 + 2n - 4 + 1) \\ = 4(n-1)^2.$$

Therefore the numerator of the chance-fraction

$$= \frac{2n}{3}(n-1)(2n-1) - 4(n-1)^2 = \frac{2}{3}(n-1)(n-2)(2n-3),$$

and the chance of safe check

$$= \frac{\frac{2}{3}(n-1)(n-2)(2n-3)}{n^2(n^2-1)} = \frac{2}{3} \cdot \frac{(n-2)(2n-3)}{n^2(n+1)}.$$

If $n=8$,

$$\text{the chance} = \frac{1}{144}.$$

These results will be the same for a bishop restricted to squares of the same colour on a board of an even number of squares.

Simple check with one queen.

The queen on any square of a chessboard checks all the squares that a bishop on the same square would, as well as all the squares that a rook on the same square would. The numerator of the chance-fraction, therefore, will be the sum of the numerators for the cases of the rook and the unrestricted bishop. The chance, therefore, for the queen is the sum of the chances for the bishop and rook

$$= \frac{2}{3} \cdot \frac{2n-1}{n(n+1)} + \frac{2}{n+1} = \frac{2}{n+1} \cdot \frac{(2n-1)+3n}{3n} = \frac{2}{3} \cdot \frac{5n-1}{n(n+1)}.$$

If $n=8$,

$$\text{the chance} = \frac{1}{36}.$$

Safe check with one queen.

The number of unsafe checks with one queen is seen thus: if the queen be on a corner square, there are 3 unsafe checks, if on any other outside square there are 5, and if on any other square there are 8 unsafe checks. The total number of unsafe checks, therefore, is

$$4 \cdot 3 + 4(n-2) \cdot 5 + (n-2)^2 \cdot 8 = 4(n-1)(2n-1).$$

The numerator of the chance-fraction, therefore, is

$$\frac{2}{3}n(n-1)(2n-1) + n^2 \cdot 2(n-1) - 4(2n-1)(n-1) \\ = \frac{2(n-1)(5n-3)(n-2)}{3}.$$

Therefore the chance

$$= \frac{\frac{2}{3}(n-2)(n-1)(5n-3)}{n^2(n^2-1)} = \frac{2}{3} \frac{(n-2)(5n-3)}{n^2(n+1)}.$$

If $n=8$,

the chance = $\frac{37}{144}$.

Simple check with two bishops.

We will assume that n is an even number, and that one bishop (A) is restricted to white squares, and the other (B) to black squares. The number of squares checked by the two bishops for all different positions of B when A is on a particular square is obtained from the following scheme :—

a	a'	a	a'
a'	b	b'	b
a	b'	c	c'
a'	b	c'	c

A on a square on the $\left(\frac{n}{2}-r\right)$ th row from the outside.

Number of such squares.	B on	Number of such positions of B.	Number of squares checked by A and B.
$2(2r-1)$	a	$2(n-1)$	$3n-2r-2$
	b	$2(n-3)$	$3n-2r$
	c	$2(n-5)$	$3n-2r+2$
	d	$2(n-7)$	$3n-2r+4$

The total number of checks for all positions of B while A remains fixed

$$= 2(n-1)(3n-2r-2) + 2(n-3)(3n-2r) + 2(n-5)(3n-2r+2) + 2(n-7)(3n-2r+4) + \&c. \text{ to } \frac{n}{2} \text{ terms,}$$

$$= 2(n-1)((2n-2r-3)+n+1) + 2(n-3)((2n-2r-3)+n+3) + 2(n-5)((2n-2r-3)+n+5) + \&c. \text{ to } \frac{n}{2} \text{ terms,}$$

$$= 2 \cdot \frac{n^2}{4} \cdot (2n-2r-3) + 2 \left(\frac{n}{2} \cdot n^2 - \frac{n(n^2-1)}{6} \right)$$

$$= -rn^2 + \frac{n}{6}(10n^2-9n+2).$$

This result must be multiplied by $2(2r-1)$, the number of the positions of A on the given row, and the product summed for values of r between 1 and $\frac{n}{2}$. The numerator of the chance-fraction, therefore,

$$\begin{aligned}
 &= \sum_{r=1}^{\frac{n}{2}} (4r-2) \left(-rn^2 + \frac{n(10n^2-9n+2)}{6} \right) \\
 &= -4n^2 \cdot \sum r^2 + \left\{ \frac{2n}{3} (10n^2-9n+2) + 2n^2 \right\} \sum r - \frac{n}{2} \cdot \frac{n}{3} (10n^2-9n+2) \\
 &= -4n^2 \cdot \frac{\frac{n}{2} \left(\frac{n}{2} + 1 \right) (n+1)}{6} + \frac{\frac{n}{2} \left(\frac{n}{2} + 1 \right)}{2} \cdot \left(\frac{2n}{3} (10n^2-9n+2) + 2n^2 \right) \\
 &\quad - \frac{n^2}{6} (10n^2-9n+2) \\
 &= \frac{n^3}{6} (4n^2-6n+2).
 \end{aligned}$$

The chance, therefore,

$$\begin{aligned}
 &= \frac{\frac{n^3}{6} (4n^2-6n+2)}{\frac{n^2}{2} \cdot \frac{n^2}{2} \cdot (n^2-2)} \\
 &= \frac{4}{3} \cdot \frac{(n-1)(2n-1)}{n(n^2-2)}.
 \end{aligned}$$

If $n=8$,

$$\text{the chance} = 1\frac{35}{24}.$$

Simple check from two rooks.

We will call the rooks A and B. A can be placed on n^2 squares, B on n^2-1 squares for each position of A, and the king on n^2-2 squares for each position of A and B. If A and B defend each other, they check altogether $3n-4$ squares, and there are $2n-2$ squares on which B will defend A on a given square. If B does not defend A, they check $4n-6$ squares, and there are $(n-1)^2$ squares on which B will not defend A on a given square. The chance of the king being in check

$$= \frac{n^2(3n-4)(2n-2) + n^2(4n-6)(n-1)^2}{n^2(n^2-1)(n^2-2)} = \frac{2(2n^2-2n-1)}{(n+1)(n^2-2)}.$$

If $n=8$,

$$\text{the chance} = \frac{87}{37}.$$

Chance of the king being in check.

	For chessboards of n^2 squares (n even).		For ordinary chessboard of 64 squares.	
	Simple.	Safe.	Simple.	Safe.
Knight	$\frac{8(n-2)}{n^2(n+1)}$	$\frac{8(n-2)}{n^2(n+1)}$	$\frac{1}{12}$	$\frac{1}{12}$
Bishop	$\frac{2}{3} \cdot \frac{2n-1}{n(n+1)}$	$\frac{2}{3} \cdot \frac{(n-2)(2n-3)}{n^2(n+1)}$	$\frac{5}{36}$	$\frac{13}{144}$
Rook	$\frac{2}{n+1}$	$\frac{2(n-2)}{n(n+1)}$	$\frac{2}{9}$	$\frac{1}{6}$
Queen	$\frac{2}{3} \cdot \frac{5n-1}{n(n+1)}$	$\frac{2}{3} \cdot \frac{(n-2)(5n-3)}{n^2(n+1)}$	$\frac{13}{36}$	$\frac{37}{144}$
Two bishops.	$\frac{4}{3} \cdot \frac{(n-1)(2n-1)}{n(n^2-2)}$		$\frac{35}{124}$	
Two rooks...	$\frac{2(2n^2-2n-1)}{(n+1)(n^2-2)}$		$\frac{37}{93}$	

We may conclude by remarking that the relative values of the knight, bishop, rook, and queen are, according as we measure them by the chance of simple check or of safe check, on the ordinary chessboard in the ratio of 3, 5, 8, 13, or 12, 13, 24, 37 respectively; while the values of the pieces in the same order, as given by Staunton in the 'Chess-Player's Handbook,' are 3.05, 3.50, 5.48, and 9.94, the value of the pawn being taken as unity.

N.B.—The value of a pawn depends so much on the fact that it is possible to convert it into a queen, that this method does not appear applicable to it.

XXVIII. On a new Form of Wave-apparatus suitable for the Lecture-room. By C. J. WOODWARD, B.Sc.*

THE apparatus about to be described illustrates the motion of the air-particles when a sound-wave is propagated, and also the motion of the æther molecules in a wave of plane-polarized light.

Wave motion consists in the repetition by a number of particles of some prescribed motion which is given to the first of the particles and taken up successively by the others, a certain

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