
X. *On the Progress of Heat when communicated to Spherical Bodies from their Centres.* By JOHN PLAYFAIR, F. R. S. LOND. Sec. R. S. EDIN. and Professor of Natural Philosophy in the University of Edinburgh.

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1. **A**N argument against the hypothesis of central heat has been stated by an ingenious author as carrying with it the evidence of demonstration.

“ THE essential and characteristic property of the power
“ producing heat, is its tendency to exist every where in a state
“ of equilibrium, and it cannot hence be preserved without loss
“ or without diffusion, in an accumulated state. In the theory
“ of HUTTON, the existence of an intense local heat, acting for
“ a long period of time, is assumed. But it is impossible to pro-
“ cure caloric in an insulated state. Waving every objection
“ to its production, and supposing it to be generated to any ex-
“ tent, it cannot be continued, but must be propagated to the
“ contiguous matter. If a heat, therefore, existed in the cen-
“ tral region of the earth, it must be diffused over the whole
“ mass; nor can any arrangement effectually counteract this
“ diffusion. It may take place slowly, but it must always con-
“ tinue progressive, and must be utterly subversive of that sy-
“ stem of indefinitely renewed operations which is repre-
VOL. VI. P. II. Y y “ fented

“fented as the grand excellence of the Huttonian Theory *.”
 “Again, he observes, in giving what he says appears to him a demonstration of the fallacy of the first principles of the Huttonian System, “it will not be disputed, that the tendency of
 “caloric is to diffuse itself over matter, till a common temperature is established. Nor will it probably be denied, that a
 “power constantly diffusing itself from the centre of any mass
 “of matter, cannot remain for an indefinite time locally accumulated in that mass, but must at length become equal or
 “nearly so over the whole †.”

2. I must confess, notwithstanding the respect I entertain for the acuteness and accuracy of the author of this reasoning, that it does not appear to me to possess the force which he ascribes to it; nor to be consistent with many facts that fall every day under our observation. A fire soon heats a room to a certain degree, and though kept up ever so long, if its intensity, and all other circumstances remain the same, the heat continues very unequally distributed through the room; but the temperature of every part continues invariable. If a bar of iron has one end of it thrust into the fire, the other end will not in any length of time become red-hot; but the whole bar will quickly come into such a state, that every point will have a fixed temperature, lower as it is farther from the fire, but remaining invariable while the condition of the fire, and of the medium that surrounds the bar, continues the same. The reason indeed is plain: the equilibrium of heat is not so much a primary law in the distribution of that fluid, as the limitation of another law which is general and ultimate, consisting in the tendency of heat to pass with a greater or a less velocity, according to circumstances, from bodies where the temperature
 is

* MURRAY'S *System of Chemistry*, vol. iii. Appendix, p. 49.

† Page 51.

is higher, to those where it is lower, or from those which contain more heat, according to the indication of the thermometer, to those which contain less. It is of this general tendency, that the equilibrium or uniform distribution of heat is a consequence,—but a consequence only contingent, requiring the presence of another condition, which may be wanting, and actually is wanting, in many instances. This condition is no other, than that the quantity of heat in the system should be given, and should not admit of continual increase from one quarter, nor diminution from another. When such increase and diminution take place, what is usually called “the equilibrium of heat” no longer exists. Thus, if we expose a thermometer to the sun’s rays, it immediately rises, and continues to stand above the temperature of the surrounding air. The way in which this happens is perfectly understood: the mercury in the thermometer receives more heat from the solar rays than the air does; it begins therefore to rise as soon as those rays fall on it; at the same time, it gives out a portion of its heat to the air, and always the more, the higher it rises. It continues to rise, therefore, till the heat which it gives out every instant to the air, be equal to that which it receives every instant from the solar rays. When this happens, its temperature becomes stationary; the momentary increment and decrement of the heat are the same, and the total, of course, continues constant. The thermometer, therefore, in such circumstances, never acquires the temperature of the surrounding air; and the only equilibrium of the heat, is that which subsists between the increments and the decrements just mentioned: these indeed are, strictly speaking, *in equilibrio*, as they accurately balance one another. This species of equilibrium, however, is quite different from what is implied in the uniform diffusion of heat.

3. IN order to state the argument more generally, let A, B, C, D, &c. be a series of contiguous bodies; or let them be parts of the same body; and let us suppose that A receives, from some cause, into the nature of which we are not here to inquire, a constant and uniform supply of heat. It is plain, that heat will flow continually from A to B, from B to C, &c.; and in order that this may take place, A must be hotter than B, B than C, and so on; so that no uniform distribution of heat can ever take place. The state, however, to which the system will tend, and at which, after a certain time, it must arrive, is one in which the momentary increase of the heat of each body is just equal to its momentary decrease; so that the temperature of each individual body becomes fixed, all these temperatures together forming a series decreasing from A downwards. To be convinced that this is the state which the system must assume, suppose any body D, by some means or other, to get more heat than that which is required to make the portion of heat which it receives every moment from C, just equal to that which it gives out every moment to E; as its excess of temperature above E is increased, it will give out more heat to E, and as the excess of the temperature of C above that of D is diminished, D will receive less heat from C; therefore, for both reasons, D must become colder, and there will be no stop to the reduction of its temperature, till the increments and decrements become equal as before.

4. If, therefore, heat be communicated to a solid mass, like the earth, from some source or reservoir in its interior, it must go off from the centre on all sides, toward the circumference. On arriving at the circumference, if it were hindered from proceeding farther, and if space or vacuity presented to heat an impenetrable barrier, then an accumulation of it at the surface, and at last a uniform distribution of it through the whole mass, would inevitably be the consequence. But if
heat

heat may be lost and dissipated in the boundless fields of vacuity, or of ether, which surround the earth, no such equilibrium can be established. The temperature of the earth will then continue to augment only, till the heat which issues from it every moment into the surrounding medium, become equal to the increase which it receives every moment from the supposed central reservoir. When this happens, the temperature at the superficies can undergo no farther change, and a similar effect must take place with respect to every one of the spherical and concentric strata into which we may conceive the solid mass of the globe to be divided. Each of these must in time come to a temperature, at which it will give out as much heat to the contiguous stratum on the outside, as it receives from the contiguous stratum on the inside; and, when this happens, its temperature will remain invariable.

5. THAT we may trace this progress with more accuracy, let us suppose a spherical body to be heated from a source of heat at its centre; and let b, b', b'' , be the temperatures at the surfaces of two contiguous and concentric strata, the distances from the centre being x, x', x'' ; and let it also be supposed, that the thickness of each of the strata, to wit, $x' - x$, and $x'' - x'$, is very small.

THEN supposing the body to be homogeneous, the quantity of heat that flows from the inner stratum into the outward, in a given time, will be proportional to the excess of its temperature above that of the outward stratum multiplied into its quantity of matter, that is, to $(b - b') (x'^3 - x^3)$.

6. IN the same manner, the heat which goes off from the second stratum in the same time, is proportional to $(b' - b'')$ $(x''^3 - x'^3)$; and these two quantities, when the temperature of the second stratum becomes constant, must be equal to one another, or $(b - b') (x'^3 - x^3) = (b' - b'') (x''^3 - x'^3)$.

BUT because $b - b'$, and $x' - x$ are indefinitely small, $b - b' = \dot{b}$, and $x'^3 - x^3 = 3x^2 \dot{x}$; therefore $\dot{b} \times 3x^2 \dot{x} =$ a given quantity; which quantity, since \dot{x} is given, we may represent by $a^2 \dot{x}^2$; so that $\dot{b} = \frac{a^2 \dot{x}^2}{3x^2 \dot{x}} = \frac{a^2 \dot{x}}{3x^2}$, or, because \dot{b} is negative in respect of \dot{x} , being a decrement, while the latter is an increment, $\dot{b} = -\frac{a^2 \dot{x}}{3x^2}$, and therefore $b = C + \frac{a^2}{3x}$.

7. To determine the constant quantity C, let us suppose that the temperature at the surface of the internal *nucleus* of ignited matter is $= H$, and $r =$ radius of that nucleus. Then, in the particular case, when $x = r$ and $b = H$, the preceding equation gives $H = C + \frac{a^2}{3r}$; so that $C = H - \frac{a^2}{3r}$, and consequently $b = H - \frac{a^2}{3r} + \frac{a^2}{3x}$; or $b = H + \frac{a^2}{3} \left(\frac{1}{x} - \frac{1}{r} \right)$.

8. IT is evident, from this formula, that for every value of x there is a determinate value of b , or that for every distance from the centre there is a fixed temperature, which, after a certain time, must be acquired, and will remain invariable as long

long as the intensity and magnitude of the central fire continue the same.

9. It remains for us to determine the value of a^2 , which, though constant, is not yet given, or known from observation.

At the surface of the globe we may suppose the mean temperature to be known: let T be that temperature, and let $R =$ the radius of the globe. Then, when $x = R$, $b = T$, and by

substituting in the general formula, we have $T = H + \frac{a^2}{3} \left(\frac{1}{R} - \frac{1}{r} \right)$,

$$\text{and } a^2 = \frac{3(T - H)}{\frac{1}{R} - \frac{1}{r}} = \frac{3Rr(H - T)}{R - r}.$$

$$\text{THUS } b = H + \frac{Rr(H - T)}{R - r} \left(\frac{1}{x} - \frac{1}{r} \right)$$

$$= H - \frac{Rr(H - T)}{R - r} \left(\frac{1}{r} - \frac{1}{x} \right).$$

HENCE also by reduction

$$b = \frac{RT - rH}{R - r} + \frac{Rr(H - T)}{x(R - r)},$$

$$\text{or } b = \frac{1}{R - r} (RT - rH) + \frac{Rr(H - T)}{x}.$$

FROM this equation, it is evident, that $b = \frac{RT - rH}{R - r}$, or

the excess of the temperature at any distance x from the centre, above a certain given temperature, is inversely as x . But the construction of the hyperbola which is the locus of the
preceding

preceding equation, will exhibit the relation between the temperature and the distance, in the way of all others least subject to misapprehension.

LET the circle (Plate X. fig. 3.) described with the radius AB , represent the globe of the earth; and the circle described with the radius AH an ignited mass at the centre. Let HK , perpendicular to AB , be the temperature at H , the surface of the ignited mass; and let FD be the temperature at any point whatever, in the interior of the earth, BM representing that at the surface. Then AB being $= R$ in the preceding equation, $AH = r$, $HK = H$, $BM = T$; $AF = x$, and $FD = b$, these two last being variable quantities; since

$$\left(b - \frac{RT - rH}{R - r}\right)x = \frac{Rr(H - T)}{R - r} \text{ we have, (taking } AE = \frac{RT - rH}{R - r}, \text{ and drawing } EL \text{ parallel to } AB, \text{ meeting } HK$$

$$\text{in } N, \text{ and } FD \text{ in } O,) OD \times OE = \frac{BA \cdot AH (HK - BM)}{BH},$$

which is a given quantity.

THEREFORE D is in a rectangular hyperbola, of which the centre is E , the asymptotes EG and EL , and the rectangle of the co-ordinates, equal to $BA \cdot AH \times \frac{HK - BM}{BH}$, or, which amounts to the same, to $KN \cdot NE$.

IT is evident from this, that if the sphere were indefinitely extended, the temperature at the point B and all other things remaining the same, the temperature at its superficies would not be less than AE , or than the quantity $\frac{RT - rH}{R - r}$.

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THE quantity AE , or $\frac{RT - rH}{R - r}$, is supposed here to be subtracted; if RT be less than rH , it will change its sign, and must be taken on the other side of the centre A .

10. THE results of these deductions may be easily represented numerically, and reduced into tables, for any particular values that may be assigned to the constant quantities. Thus, if the radius of the globe, or $R = 100$, that of the ignited nucleus or $r = 1$; the temperature of the nucleus, or $H = 1000$, and T the temperature at the surface $= 60$, the formula be-

comes $b = 50.505 + \frac{949.494}{x}$.

Values of x	Values of b
10	145°.454
20	98.423
30	82.599
40	74.686
50	69.938
60	66.330
70	63.926
80	62.361
90	61.055
100	60.

11. OTHER things remaining as before, if we now make

$$r = 10, \text{ then } b = -44.444 + \frac{10444}{x}$$

x	b
20	477°.556
30	303.556
40	226.556
50	164.556
60	145.556
70	104.556
80	85.056
90	64.556
100	60.000

12. If $R = 10$, $r = 1$, $H = 10000$, and $T = 60$,

$$b = -1044.44 + \frac{11044.44}{x}$$

Values of x	Values of b
1	10000°.00
2	4477.70
3	2637.04
4	1716.67
5	1164.44
6	796.30
7	533.33
8	346.16
9	182.72
10	60.00

13. THE

13. 1. THE general conclusions which result from all this are, that when we suppose an ignited nucleus of a given magnitude, and a given intensity of heat, there is in the sphere to which it communicates heat a fixed temperature for each particular stratum, or for each spherical shell, at a given distance from the centre; and that a great intensity of heat in the interior, is compatible with a very moderate temperature at the surface.

2. HOWEVER great the sphere may be, the heat at its surface cannot be less than a given quantity; R, r, H and T remaining the same. It must be observed, that though R is put for the radius of the globe; it signifies in fact nothing, but the distance at which the temperature is T , as r does the distance at which the temperature is H .

THEREFORE were the sphere indefinitely extended, the temperature at its superficies would not be less than the quantity

$$\frac{RT - rH}{R - r}, \text{ that is, not less than } 50.5 \text{ in the first of the preceding examples, than } -44.4 \text{ in the second, or } -1044.4 \text{ in the third.}$$

14. IN all this the sphere is supposed homogeneous; but if it be otherwise, and vary in density, in the capacity of the parts for heat, or in their power to conduct heat, providing it do so as any function of the distance from the centre, the calculus may be instituted as above. For example, let the density

be supposed to vary as $\frac{b}{b+x}$, then we have as before

$$(b-b')(x'^3 - x^3) \frac{b}{b+x} \text{ for the momentary increment of}$$

heat in a stratum placed at the distance x from the centre,

Z z 2

or

or $\dot{b} \times 3x^2 \dot{x} \times \frac{b}{b+x} =$ to a given quantity, or to $a^2 \dot{x}^2$, and

therefore $\dot{b} = -\frac{a^2(b+x)\dot{x}}{3bx^2} = -\frac{a^2\dot{x}}{3x^2} - \frac{a^2\dot{x}}{bx}$. Hence $b =$

$C + \frac{a^2}{3x} - \frac{a^2}{b} \text{Log } x$. Suppose that when $x=r$, the radius of the heated nucleus, $b=H$; then $H =$

$$C + \frac{a^2}{3r} - \frac{a^2}{b} \text{Log } r, \text{ and } C =$$

$$H - \frac{a^2}{3r} + \frac{a^2}{b} \text{Log } r; \text{ therefore } b =$$

$$H - \frac{a^2}{3r} + \frac{a^2}{3x} + \frac{a^2}{b} \text{Log } \frac{r}{x}.$$

In this expression a^2 will be determined, if the temperature at any other distance R from the centre is known. Let this be T ; then by substitution we have

$$T = H - \frac{a^2}{3r} + \frac{a^2}{3R} + \frac{a^2}{b} \text{Log } \frac{r}{R},$$

$$\text{and } a^2 = \frac{T-H}{\frac{1}{3R} - \frac{1}{3r} + \frac{1}{b} \text{Log } \frac{r}{R}}.$$

$$\text{Hence } b = H + \left(\frac{T-H}{\frac{1}{3R} - \frac{1}{3r} + \frac{1}{b} \text{Log } \frac{r}{R}} \right) \times$$

$$\left(-\frac{1}{3r} + \frac{1}{3x} + \frac{1}{b} \text{Log } \frac{r}{x} \right).$$

15. **THIS** is given merely as an example of the method of conducting the calculus when the variation of the density is taken into account, and not because there is reason to believe that the law which that variation actually follows, is the same that has now been hypothetically assumed.

16. **THE** principle on which we have proceeded, applies not only to solids, such as we suppose the interior of the earth, but it applies also to fluids like the atmosphere, provided they are supposed to have reached a steady temperature. The propagation of heat through fluids is indeed carried on by a law very different from that which takes place with respect to solids; it is not by the motion of heat, but by the motion of the parts of the fluid itself. Yet, when we are seeking only the mean result, we may suppose the heat to be so diffused, that it does not accumulate in any particular stratum, but is limited by the equality of the momentary increments and decrements of temperature which that stratum receives. This is conformable to experience; for we know that a constancy, not of temperature, but of difference between the temperature of each point in the atmosphere and on the surface, actually takes place. Thus, near the surface, an elevation of 280 feet produces, in this country, a diminution of one degree. The strata of our atmosphere, however, differ in their capacity of heat, or in the quantity of heat contained in a given space, at a given temperature. Concerning the law which the change of capacity follows, we have no certain information to guide us; and we have no resource, therefore, but to assume a hypothetical law, agreeing with such facts as are known, and, after deducing the results of this law, to compare them with the observations made on the temperature of the air, at different heights above the surface of the earth.

17. **LET**

17. LET us then suppose, that the strata of the atmosphere have a capacity for heat, which increafes as the air becomes rarer, fo as to be proportional to $m b^{-\frac{r}{x}}$, x denoting, as before, the distance from the centre of the earth, r the radius of the earth, m and b determinate, but unknown quantities, fuch that $m b^{-1}$ or $\frac{m}{b}$, expreffes the capacity of air for heat, when of its ordinary density, at the furface of the earth. The formula thus affumed, agrees with the extreme cafes; for, when $x = r$, the capacity of heat $= \frac{m}{b}$, a finite quantity; when x increafes, $\frac{r}{x}$ diminifhes, and fo alfo does $b^{\frac{r}{x}}$, if b is greater than unity, and therefore $\frac{m}{b^{\frac{r}{x}}}$ increafes continually. It does not, however, increafe beyond a certain limit, for when x is infinite $\frac{m}{b^{\frac{r}{x}}}$ becomes $\frac{m}{1}$, or m .

18. HENCE,

18. HENCE, by reasoning as in § 6. the momentary increment of the temperature, or sensible heat, of any stratum, is as $-\frac{a^2 \dot{x}}{3x^2}$ directly, and its capacity for heat, or $mb^{-\frac{r}{x}}$ inversely,

$$\text{that is, } \dot{b} = -\frac{a^2 \dot{x}}{3x^2} \times \frac{b^{\frac{r}{x}}}{m} = -\frac{a^2 b^{\frac{r}{x}} \dot{x}}{3m x^2}.$$

LET $\frac{r}{x} = y$, then $-\frac{r \dot{x}}{x^2} = \dot{y}$, so that $-\frac{a^2 \dot{x}}{3m x^2} = \frac{a^2 \dot{y}}{3m r}$, and

$$\text{therefore } \dot{b} = \frac{a^2 b^y}{3m r} \dot{y}. \quad \text{Hence } b = C + \frac{a^2}{3mr \text{Log } b} b^y =$$

$$C + \frac{a^2}{3mr \text{Log } b} b^{\frac{r}{x}}.$$

19. To determine C, if T be the temperature of the air at the surface, when $x=r$, $T = C + \frac{a^2 b}{3mr \text{Log } b}$, and $C =$

$$T - \frac{a^2 b}{3mr \text{Log } b}.$$

$$\text{HENCE } b = T - \frac{a^2 b}{3mr \text{Log } b} + \frac{a^2 b^{\frac{r}{x}}}{3mr \text{Log } b} =$$

$$T - \frac{a^2 (b - b^{\frac{r}{x}})}{3m r \text{Log } b}.$$

THIS formula, when $x=r$ gives $b=T$, and when x is infinite, it gives $b = T - \frac{a^2 (b-1)}{3m r \text{Log } b}$. In all intermediate cases,

cases, as x is greater than r , $b^{\frac{r}{x}}$ is less than b , (b being a number greater than 1) and therefore $b - b^{\frac{r}{x}}$ is positive, so that b is less than T , as it ought to be.

20. WE may obtain an approximate value of this formula, without exponential quantities, that will apply to all the cases in which x and r differ but little in respect of r , that is, in all the cases to which our observations on the atmosphere can possibly extend.

IF, in the term $b^{\frac{r}{x}}$ we write $r + z$ for x , z being the height of any stratum of air above the surface of the earth, we have $b^{\frac{r}{x}} = b^{\frac{r}{r+z}}$.

$$\begin{aligned} 21. \text{ BUT, from the nature of exponentials, we know } b^{\frac{r}{x}} = \\ 1 + \frac{r}{x} \text{ Log } b + \frac{r^2 (\text{Log } b)^2}{2 x^2} + \frac{r^3 (\text{Log } b)^3}{2.3 x^3}, \text{ \&c.} = \\ 1 + \frac{r}{r+z} \text{ Log } b + \frac{r^2 (\text{Log } b)^2}{2 (r+z)^2} +, \text{ \&c.} \end{aligned}$$

Now $\frac{r}{r+z} = 1 - \frac{z}{r} + \frac{z^2}{r^2} -, \text{ \&c.}$ And if we leave out the higher powers of z , we have nearly

$$\frac{r}{r+z}$$

$$\begin{aligned}\frac{r}{r+z} &= 1 - \frac{z}{r} \\ \frac{r^2}{(r+z)^2} &= 1 - \frac{2z}{r} \\ \frac{r^3}{(r+z)^3} &= 1 - \frac{3z}{r}, \text{ \&c.}\end{aligned}$$

THEREFORE, by substitution, we have $b^{\frac{r}{r+z}} =$

$$1 + \left(1 - \frac{z}{r}\right) \text{Log } b + \left(1 - \frac{2z}{r}\right) \frac{(\text{Log } b)^2}{2} +, \text{ \&c.} =$$

$$\left\{ \begin{aligned} &1 + \text{Log } b + \frac{(\text{Log } b)^2}{2} + \frac{(\text{Log } b)^3}{2 \cdot 3} +, \text{ \&c.} \\ &-\frac{z}{r} \text{Log } b - \frac{z}{r} (\text{Log } b)^2 - \frac{z}{r} \frac{(\text{Log } b)^3}{2}, \text{ \&c.} \end{aligned} \right\}$$

Now, from the nature of exponentials,

$$b = 1 + \text{Log } b + \frac{\text{Log } b^2}{2} + \frac{\text{Log } b^3}{2 \cdot 3} +, \text{ \&c.}$$

$$\text{And } \frac{z}{r} \text{Log } b + \frac{z}{r} (\text{Log } b)^2 + \frac{z}{r} \frac{(\text{Log } b)^3}{2}, \text{ \&c.}$$

$$= \frac{z}{r} \text{Log } b \left(1 + \text{Log } b + \frac{(\text{Log } b)^2}{2} +, \text{ \&c.} \right) =$$

$$\frac{z}{r} b \text{Log } b; \text{ therefore when } z \text{ is very small, } b^{\frac{r}{r+z}} =$$

$$b - \frac{z}{r} \text{Log } b, \text{ and therefore } (\S 19.), a^2 \frac{(b - b^{\frac{r}{r+z}})}{3 m r \text{Log } b} =$$

$$a^2 \frac{\left(b - b + \frac{b z \text{Log } b}{r}\right)}{3 m r \text{Log } b} = \frac{a^2 b z}{3 m r^2}; \text{ hence when } z \text{ is very}$$

small, $b = T - \frac{a^2 b z}{3 m r^2}.$

22. THEREFORE when z , or the height above the surface is small, b diminishes in the same proportion that the height increases, which is conformable to experience.

IN our climate, when $z = 280$ feet, $\frac{a^2 b \times 280}{3 m r^2} = 1^\circ$; so that the co-efficient $\frac{a^2 b}{3 m r^2} = \frac{1}{280}$, and therefore

$$b = T - \frac{z}{280}.$$

WHEN the constant quantities are thus determined, the formula agrees nearly with observation. In the rule for barometrical measurements, it is implied, that the heat of the atmosphere decreases uniformly; but the rate for each particular case is determined by actual observation, or by thermometers observed at the top and bottom of the height to be measured.