

FREE AND FORCED LONGITUDINAL TIDAL MOTION IN A LAKE

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1. The lake considered in this paper is such that the size and shape of its transverse section vary only slowly. The variations may be in any manner, subject only to certain very general conditions which are satisfied in many actual lakes.

The determination of the ordinary longitudinal seiches in such a lake was reduced by Prof. Chrystal* to the solution of a linear differential equation of the second order in the normal form, with certain boundary conditions.

The determination of the "temperature seiche," discovered by E. R. Watson and E. M. Wedderburn, has been shown by the latter of these authors† to be similarly reducible.

Again, the discussion of the vibrations of a string of any law of density forms a problem which is reducible to the same equations and conditions. For the special case in which the string is of uniform density the original method of John and Daniel Bernoulli was to replace it by one in which the mass was concentrated at a finite number of equidistant points.‡ This, when the masses at the points are no longer equal, is the method used in the present paper to suggest the form of the general solution. It involves the regarding of the differential equation as the limiting form of a difference equation, a method which has been very much used.

In the first place the determination of the forced motion in the lake due to a periodic change in atmospheric pressure§ which varies along the

* "Hydrodynamical Theory of Seiches," *Trans. Roy. Soc. Edin.*, Vol. xli, p. 599 (1905).

† "The Temperature Seiche," *ibid.*, Vol. xlvii, p. 619 (1910); "Temperature Observations . . .," *ibid.*, Vol. xlviii, p. 629 (1912).

‡ The analysis is given by Lord Rayleigh, *Theory of Sound*, Vol. i, p. 172.

§ The effect of a number of types of pressure disturbances on a special lake has been considered by Prof. Chrystal, *Trans. Roy. Soc. Edin.*, Vol. xlvi, p. 499 (1908).

lake, as well as to a periodic bodily disturbing force which is directed everywhere along the length of the lake, is reduced to the solution, with the same boundary conditions, of a differential equation which is an extension of that of Prof. Chrystal. Of course we are merely concerned with a very simple linear boundary problem, of which Prof. Chrystal's is the "homogeneous" case.

A general equation for the free periods, and a general expression, in terms of the period, for the motion in any free mode, are obtained, as well as a complete expression for the periodic forced motion.

General Equations.

2. Following Prof. Chrystal, let x denote the area of the surface of the lake from one end up to a transverse section which is at a distance s , measured along the length of the lake, from some fixed transverse section. Then, if a denotes the total area of the surface of the lake, x will range from 0 to a . Let the transverse section corresponding to x have an area $A(x)$ and be of breadth $b(x)$ at the free surface, so that

$$\frac{dx}{ds} = b(x).$$

Now let V denote the total volume of water which has passed the section at x up to time t ; then V will vanish at $x = 0$ and $x = a$. Let ξ denote the forward displacement of a particle in the section at x , and ζ the elevation of the free surface at this section. It is supposed that ξ and ζ are functions only of x and t . We shall then have

$$\xi = \frac{V}{A(x)}, \quad \zeta = -\frac{\partial V}{\partial x},$$

the latter being the equation of continuity.

Let Π denote the atmospheric pressure, and S the bodily disturbing force per unit mass, supposed to act everywhere along the length of the lake. Π and S are supposed to be functions only of x and t .

When the motion is "tidal" in character, the pressure intensity in the water will be

$$\Pi + g\rho (\zeta + \text{depth below mean surface}),$$

where ρ is the density, supposed uniform, of the water, and g is the acceleration due to gravity. The dynamical equation then gives, on

neglecting squares and products of displacements,

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho} \frac{\partial \Pi}{\partial s} - g \frac{\partial \xi}{\partial s} + S, \quad (1)$$

which, on substituting from the above, gives

$$\frac{\partial^2 V}{\partial t^2} = -\frac{b(x) A(x)}{\rho} \frac{\partial \Pi}{\partial x} + g b(x) A(x) \frac{\partial^2 V}{\partial x^2} + A(x) S. \quad (2)$$

If we write

$$p(x) = b(x) A(x), \quad F = \frac{1}{g\rho} \frac{\partial \Pi}{\partial x} - \frac{1}{g b(x)} S, \quad (3)$$

$$(2) \text{ may be written } \frac{\partial^2 V}{\partial x^2} - \frac{1}{g p(x)} \frac{\partial^2 V}{\partial t^2} = F. \quad (4)$$

The function $p(x)$ can never be negative, and can only vanish at the ends of the lake. For the possibility of the type of motion we are considering $p(x)$ must be a continuous function of x , and we shall further require that dp/dx shall not vanish at an end point for which p vanishes.*

Suppose now the motion to be periodic with speed σ , and take

$$\lambda = \frac{\sigma^2}{g}.$$

Then V is determined by the equation

$$\frac{d^2 V}{dx^2} + \frac{\lambda}{p(x)} V = F, \quad (5)$$

with the condition that V shall vanish at $x = 0$ and $x = a$.

For a free mode we have $F = 0$; let V, λ be then denoted by V_n, λ_n respectively, so that we have

$$\frac{d^2 V_n}{dx^2} + \frac{\lambda_n}{p(x)} V_n = 0. \quad (6)$$

It is to be remarked that, owing to the approximations made in deriving the equations, only the smaller values of λ will admit of interpretation.

When we speak of a solution of (5) or (6), we shall always suppose it understood that the solution vanishes at $x = 0$ and $x = a$.

* Chrystal and Wedderburn have calculated the function $p(x)$ for Lochs Earn and Treig, and in each case this condition is satisfied: *Trans. Roy. Soc. Edin.*, Vol. xli, p. 823 (1905).

The determination of the possible values of λ , for which (6) possesses a solution, together with that of the corresponding solution, has been considered by Picard,* and, in fact, he has given direct processes by which they may be calculated. The processes of the present paper are different.

Solution of a Difference Equation.

3. In accordance with the method indicated in § 1, let us consider the determination of the $m-1$ constants

$$v(1), v(2), \dots, v(m-1),$$

which are subject to the relations

$$v(r-1) + \left\{ \frac{a^2}{m^2} \frac{\lambda}{p(r)} - 2 \right\} v(r) + v(r+1) = \frac{a^2}{m^2} f(r), \quad (7)$$

for $r = 1, 2, \dots, m-1$,

with $v(0) = v(m) = 0$.

The difference equation (7) is suggested by the differential equation (5) which may be regarded as a limiting form of it as $m \rightarrow \infty$.

For our purpose we shall require the determinant

$$\begin{vmatrix} 2 - \frac{a^2}{m^2} \frac{\lambda}{p(\mu+1)}, & -1, & 0, & \dots, & 0 \\ -1, & 2 - \frac{a^2}{m^2} \frac{\lambda}{p(\mu+2)}, & -1, & \dots, & 0 \\ 0, & -1, & 2 - \frac{a^2}{m^2} \frac{\lambda}{p(\mu+3)}, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & 0 \\ 0, & 0, & 0, & \dots, & -1 \\ 0, & 0, & 0, & \dots, & 2 - \frac{a^2}{m^2} \frac{\lambda}{p(\nu-1)} \end{vmatrix},$$

which we shall call $\Delta(\mu, \nu, \lambda)$.

$\Delta(\mu, \nu, 0)$ is easily evaluated and found to be $\nu - \mu$, and then, on expanding $\Delta(\mu, \nu, \lambda)$ in powers of λ , it is easily seen to be given by

$$\Delta(\mu, \nu, \lambda) = (\nu - \mu) + \sum_{n=1}^{\nu-\mu-1} (-\lambda)^n S_n(\mu, \nu), \quad (8)$$

* *Traité d'Analyse*, t. III, ch. vi.

where $S_n(\mu, \nu)$ is given by either of the two equal multiple sums

$$\frac{a^{2n}}{m^{2n}} \sum_{s_n=\mu+n}^{\nu-1} \sum_{s_{n-1}=\mu+n-1}^{s_n-1} \dots \sum_{s_2=\mu+2}^{s_3-1} \sum_{s_1=\mu+1}^{s_2-1} \frac{(\nu-s_n)(s_n-s_{n-1})\dots(s_1-\mu)}{p(s_1)p(s_2)\dots p(s_n)}, \quad (9)$$

$$\frac{a^{2n}}{m^{2n}} \sum_{s_n=\mu+1}^{\nu-n} \sum_{s_{n-1}=s_n+1}^{\nu-n+1} \dots \sum_{s_2=s_3+1}^{\nu-2} \sum_{s_1=s_2+1}^{\nu-1} \frac{(\nu-s_1)(s_1-s_2)\dots(s_n-\mu)}{p(s_1)p(s_2)\dots p(s_n)}. \quad (10)$$

The cofactor of the element in the r -th row and s -th column of the determinant $\Delta(0, m, \lambda)$ is, for the respective cases in which $r <, =, > s$,

$$\Delta(0, r, \lambda) \Delta(s, m, \lambda), \quad \Delta(0, r, \lambda) \Delta(r, m, \lambda), \quad \Delta(0, s, \lambda) \Delta(r, m, \lambda), \quad (11)$$

where

$$\Delta(0, 1, \lambda) = \Delta(m-1, m, \lambda) = 1.$$

Now, when $f(r) = 0$ for all values of r concerned, the condition for the existence of a solution of the problem in question is

$$\Delta(0, m, \lambda) = 0. \quad (12)$$

This is an algebraic equation of the $(m-1)$ -th degree in λ ; let its roots be denoted by

$$\lambda_1, \lambda_2, \dots, \lambda_{m-1}.$$

Corresponding to λ_n we have, by the ordinary theory,

$$\frac{v(1)}{\Delta(0, 1, \lambda_n)} = \frac{v(2)}{\Delta(0, 2, \lambda_n)} = \dots = \frac{v(r)}{\Delta(0, r, \lambda_n)} = \dots = \frac{v(m-1)}{\Delta(0, m-1, \lambda_n)}, \quad (13)$$

and

$$\frac{v(1)}{\Delta(1, m, \lambda_n)} = \frac{v(2)}{\Delta(2, m, \lambda_n)} = \dots = \frac{v(r)}{\Delta(r, m, \lambda_n)} = \dots = \frac{v(m-1)}{\Delta(m-1, m, \lambda_n)}, \quad (14)$$

and the determination is unique. We notice that when $\Delta(0, m, \lambda) = 0$, $\Delta(0, r, \lambda)/\Delta(r, m, \lambda)$ must be independent of r .

When $f(r)$ is not zero for all values of r concerned, and λ is prescribed, we shall have, provided

$$\Delta(0, m, \lambda) \neq 0,$$

$$v(r) = -\frac{1}{\Delta(0, m, \lambda)} \left\{ \Delta(r, m, \lambda) \sum_{s=1}^{r-1} \Delta(0, s, \lambda) f(s) + \Delta(0, r, \lambda) \Delta(r, m, \lambda) f(r) + \Delta(0, r, \lambda) \sum_{s=r+1}^{m-1} \Delta(s, m, \lambda) f(s) \right\}, \quad (15)$$

for

$$r = 1, 2, \dots, m-1.$$

Solution of the Differential Equation.

4. We now proceed to solve the original problem on the lines suggested by the preceding section, giving, however, an independent justification.

From the expressions (9) and (10) for $S_n(\mu, \nu)$ we are led to consider the two multiple integrals

$$\int_{s_n=\xi}^{\eta} \int_{s_{n-1}=\xi}^{s_n} \dots \int_{s_2=\xi}^{s_3} \int_{s_1=\xi}^{s_2} \frac{(\eta-s_n)(s_n-s_{n-1}) \dots (s_2-s_1)(s_1-\xi)}{p(s_1)p(s_2) \dots p(s_n)} ds_1 ds_2 \dots ds_n, \quad (16)$$

$$\int_{s_n=\xi}^{\eta} \int_{s_{n-1}=s_n}^{\eta} \dots \int_{s_2=s_3}^{\eta} \int_{s_1=s_2}^{\eta} \frac{(\eta-s_1)(s_1-s_2) \dots (s_{n-1}-s_n)(s_n-\xi)}{p(s_1)p(s_2) \dots p(s_n)} ds_1 ds_2 \dots ds_n, \quad (17)$$

for $n > 0$, where $0 \leq \xi \leq \eta \leq a$. The integrands of these are always positive, so that if the integrals exist we may change the order of integration. Doing this for (16), we obtain

$$\int_{s_1=\xi}^{\eta} \int_{s_2=s_1}^{\eta} \dots \int_{s_{n-1}=s_{n-2}}^{\eta} \int_{s_n=s_{n-1}}^{\eta} \frac{(\eta-s_n)(s_n-s_{n-1}) \dots (s_2-s_1)(s_1-\xi)}{p(s_1)p(s_2) \dots p(s_n)} ds_1 ds_2 \dots ds_n,$$

which only differs by a change in notation from (17). The integrals obviously exist for $0 < \xi \leq \eta < a$, since the integrands are then continuous; let their common value be denoted by

$$I_n(\xi, \eta),$$

and take
$$I_0(\xi, \eta) = \eta - \xi. \quad (18)$$

The equality of (16) and (17) was to be expected from that of (9) and (10).

From (16) we have

$$I_n(\xi, \eta) = \int_{\xi}^{\eta} \frac{\eta-s}{p(s)} I_{n-1}(\xi, s) ds, \quad (19)$$

and this may be written as the double integral

$$\int_{s=\xi}^{\eta} \int_{s'=s}^{\eta} \frac{I_{n-1}(\xi, s)}{p(s)} ds ds', \quad (20)$$

which, by interchanging the order of integration, may be written

$$\int_{s'=\xi}^{\eta} \int_{s=\xi}^{s'} \frac{I_{n-1}(\xi, s)}{p(s)} ds ds'. \quad (21)$$

Similarly, from (17), we obtain

$$I_n(\xi, \eta) = \int_{\xi}^{\eta} \frac{s - \xi}{p(s)} I_{n-1}(s, \eta) ds \quad (22)$$

$$= \int_{s=\xi}^{\eta} \int_{s'=\xi}^s \frac{I_{n-1}(s, \eta)}{p(s)} ds ds' \quad (23)$$

$$= \int_{s'=\xi}^{\eta} \int_{s=s'}^{\eta} \frac{I_{n-1}(s, \eta)}{p(s)} ds ds'. \quad (24)$$

These relations are true for $n > 0$.

For a particular value of ξ , $I_n(\xi, \eta)$ increases as η increases, while for a particular value of η , $I_n(\xi, \eta)$ decreases as ξ increases.

We now proceed to show that $I_n(0, a)$ will exist, provided that

$$p(x) > kx(a-x), \quad (25)$$

where k is any positive constant. It will obviously be sufficient to show the existence for the case in which $p(x) = x(a-x)$. Now

$$\int_0^x \frac{x-s}{a-s} \left(\frac{s}{a}\right)^n ds = \sum_{r=2}^{\infty} \frac{1}{(n+r-1)(n+r)} \frac{x^{n+r}}{a^{n+r-1}} \leq \frac{1}{n+1} \frac{x^{n+2}}{a^{n+1}},$$

for $0 \leq x \leq a$, so that we have in succession, on using (19),

$$\begin{aligned} I_0(0, x) &= x, \\ I_1(0, x) &= \int_0^x \frac{x-s}{a-s} ds \leq \frac{x^2}{a}, \\ I_2(0, x) &\leq \int_0^x \frac{x-s}{a-s} \frac{s}{a} ds \leq \frac{1}{2} \frac{x^3}{a^2}, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ I_n(0, x) &\leq \frac{1}{(n-1)!} \int_0^x \frac{x-s}{a-s} \left(\frac{s}{a}\right)^{n-1} ds \leq \frac{1}{n!} \frac{x^{n+1}}{a^n}. \end{aligned}$$

We thus see that when (25) is satisfied, $I_n(0, a)$ will exist and have a value which is $< a/n! k^n$. The only restriction imposed by (25), further than those already required, is that dp/dx shall not vanish at an end for which p vanishes.

From (24), (21) we obtain respectively

$$\frac{\partial}{\partial \xi} I_n(\xi, \eta) = - \int_{\xi}^{\eta} \frac{I_{n-1}(s, \eta)}{p(s)} ds, \quad \frac{\partial}{\partial \eta} I_n(\xi, \eta) = \int_{\xi}^{\eta} \frac{I_{n-1}(\xi, s)}{p(s)} ds, \quad (26)$$

for $0 < \xi \leq \eta \leq a$ and $0 \leq \xi \leq \eta < a$ respectively, while from (18) we obtain

$$\frac{\partial}{\partial \xi} I_0(\xi, \eta) = -1, \quad \frac{\partial}{\partial \eta} I_0(\xi, \eta) = 1. \quad (27)$$

We next obtain

$$\frac{\partial^2}{\partial \xi^2} I_n(\xi, \eta) = \frac{1}{p(\xi)} I_{n-1}(\xi, \eta), \quad \frac{\partial^2}{\partial \eta^2} I_n(\xi, \eta) = \frac{1}{p(\eta)} I_{n-1}(\xi, \eta), \quad (28)$$

for $n > 0$ and the same ranges, while

$$\frac{\partial^2}{\partial \xi^2} I_0(\xi, \eta) = 0, \quad \frac{\partial^2}{\partial \eta^2} I_0(\xi, \eta) = 0. \quad (29)$$

The conditions necessary for the validity of the operations involved in the derivation of (26) and (28) are satisfied.

The above relations may be used for the determination of $I_n(\xi, \eta)$ instead of (16) and (17). For instance, $I_n(\xi, \eta)$, regarded as a function of the parameter ξ and the variable η , may be determined step by step from the differential equations in (28) and (29), together with the conditions at the initial point $\eta = \xi$,

$$I_0(\xi, \eta) = 0, \quad \frac{\partial}{\partial \eta} I_0(\xi, \eta) = 1,$$

$$I_n(\xi, \eta) = 0, \quad \frac{\partial}{\partial \eta} I_n(\xi, \eta) = 0 \quad (n > 0),$$

which are derived from (26) and (27).

5. Now take

$$R(\xi, \eta, \lambda) = \sum_{n=0}^{\infty} (-\lambda)^n I_n(\xi, \eta). \quad (30)$$

This is suggested by (8); we shall expect the condition for a solution of (6) to be

$$R(0, a, \lambda_n) = 0,$$

and the solution itself to be given either by

$$V_n = R(0, x, \lambda_n) \quad \text{or} \quad V_n = R(x, a, \lambda_n).$$

When (25) holds, we see from the upper limit obtained for $I_n(0, a)$, that (30) will be absolutely convergent, as well as uniformly convergent with respect to ξ, η , for all values of λ .

We shall then have

$$\begin{aligned}\frac{\partial}{\partial \xi} R(\xi, \eta, \lambda) &= \sum_{n=0}^{\infty} (-\lambda)^n \frac{\partial}{\partial \xi} I_n(\xi, \eta) \\ &= -1 + \lambda \sum_{n=0}^{\infty} (-\lambda)^n \int_{\xi}^{\eta} \frac{I_n(s, \eta)}{p(s)} ds \\ &= -1 + \lambda \int_{\xi}^{\eta} \frac{R(s, \eta, \lambda)}{p(s)} ds,\end{aligned}\quad (81)$$

and

$$\begin{aligned}\frac{\partial}{\partial \eta} R(\xi, \eta, \lambda) &= \sum_{n=0}^{\infty} (-\lambda)^n \frac{\partial}{\partial \eta} I_n(\xi, \eta) \\ &= 1 - \lambda \sum_{n=0}^{\infty} (-\lambda)^n \int_{\xi}^{\eta} \frac{I_n(\xi, s)}{p(s)} ds \\ &= 1 - \lambda \int_{\xi}^{\eta} \frac{R(\xi, s, \lambda)}{p(s)} ds.\end{aligned}\quad (82)$$

We have here used the relations (26) and (27).

We next obtain

$$\frac{\partial^2}{\partial \xi^2} R(\xi, \eta, \lambda) = -\frac{\lambda}{p(\xi)} R(\xi, \eta, \lambda), \quad \frac{\partial^2}{\partial \eta^2} R(\xi, \eta, \lambda) = -\frac{\lambda}{p(\eta)} R(\xi, \eta, \lambda). \quad (83)$$

We now see that $R(\xi, x, \lambda)$ and $R(x, \eta, \lambda)$ are each solutions of the differential equation

$$\frac{d^2 V}{dx^2} + \frac{\lambda}{p(x)} V = 0, \quad (84)$$

for $0 \leq \xi \leq x < a$, and $0 < x \leq \eta \leq a$ respectively, and that

$$R(\xi, x, \lambda) = 0, \quad \frac{\partial}{\partial x} R(\xi, x, \lambda) = 1,$$

$$\text{for } x = \xi, \text{ while } R(x, \eta, \lambda) = 0, \quad \frac{\partial}{\partial x} R(x, \eta, \lambda) = -1,$$

for $x = \eta$.

Again, these conditions, which are "conditions of Cauchy," may be used to determine the functions $R(\xi, x, \lambda)$, $R(x, \eta, \lambda)$, and this determination may be carried out in a manner similar to that of Picard mentioned in § 2, which is based on successive approximations. If we assume, as a solution of (84), for example,

$$V = \sum_{n=0}^{\infty} \lambda^n v_n,$$

with the conditions $V = 0$, $dV/dx = 1$, at the initial point $x = \xi$, we obtain by Picard's method a differential equation and initial conditions for

v_n which, on comparison with those of § 4, show that

$$v_n = (-1)^n I_n(\xi, x).$$

Picard himself does not use these "conditions of Cauchy," but takes his primary solution to have a prescribed value at each of the end points. The functions $R(\xi, x, \lambda)$, $R(x, \eta, \lambda)$ are included in a solution given by Forsyth,* while the function $R(\xi, x, \lambda)$ has been obtained by Liapounoff† for the case in which the function $p(x)$ is periodic.

Since $R(\xi, x, \lambda)$ and $R(x, \eta, \lambda)$ are two solutions of (34), we know that

$$\frac{\partial}{\partial x} R(\xi, x, \lambda) R(x, \eta, \lambda) - \frac{\partial}{\partial x} R(x, \eta, \lambda) R(\xi, x, \lambda),$$

will be independent of x ; we have, in fact,

$$\frac{\partial}{\partial x} R(\xi, x, \lambda) R(x, \eta, \lambda) - \frac{\partial}{\partial x} R(x, \eta, \lambda) R(\xi, x, \lambda) = R(\xi, \eta, \lambda), \quad (35)$$

on putting either $x = \xi$ or $x = \eta$. When $R(\xi, \eta, \lambda) = 0$, we see further from (35) that $R(\xi, x, \lambda)/R(x, \eta, \lambda)$ will be independent of x . This was to be expected from § 3. We may put $\xi = 0$, $\eta = a$, if $0 < x < a$.

6. With regard to the tidal problem, the solution for the *free modes* is now seen, as was expected, to be given by

$$V_n = R(0, x, \lambda_n) \quad \text{or} \quad V_n = R(x, a, \lambda_n), \quad (36)$$

where λ_n is a root of the equation

$$R(0, a, \lambda) = 0. \quad (37)$$

This is the *period equation*; the left-hand side is an integral function in λ , and we know that the roots are real, positive, infinite in number, and isolated.‡

The ratio of the two solutions in (36) is seen from the end of the preceding section to be independent of x .

For the *forced motion*, we assume, on the suggestion of (15), when

$$R(0, a, \lambda) \neq 0,$$

$$V(x) = -\frac{1}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) ds + R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) ds \right\}. \quad (38)$$

* *A Treatise on Differential Equations*, 3rd ed., p. 120, Ex.

† *Ann. Fac. Sci. Toulouse*, (2), t. ix, p. 403.

‡ See, for instance, Picard, *l.c.*

This vanishes at $x = 0$ and $x = a$. We have further

$$\frac{d}{dx} V(x) = -\frac{1}{R(0, a, \lambda)} \left\{ \frac{\partial}{\partial x} R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) ds \right. \\ \left. + \frac{\partial}{\partial x} R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) ds \right\}. \quad (39)$$

and, consequently,

$$\frac{d^2}{dx^2} V(x) = \frac{\lambda}{p(x)} \frac{1}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) ds \right. \\ \left. + R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) ds \right\} \\ - \frac{1}{R(0, a, \lambda)} \left\{ \frac{\partial}{\partial x} R(x, a, \lambda) R(0, x, \lambda) \right. \\ \left. - \frac{\partial}{\partial x} R(0, x, \lambda) R(x, a, \lambda) \right\} F(x). \quad (40)$$

Equations (39) and (40) are valid for $0 \leq x \leq a$. Therefore

$$\frac{d^2}{dx^2} V(x) + \frac{\lambda}{p(x)} V(x) = F(x),$$

on using (35), so that (38) provides the solution of (5) required.