the ratio a_1/a_2, and by suitably assigning the value of this ratio the surface condition can be satisfied whatever p_1 and p_2 may be.

Hence, in this case, p_1 and p_2 may be *any* two negative quantities satisfying the relation

$$p_1+p_2 = -m',$$

and the velocity of propagation is given by

$$V^2 = \frac{gm'}{p_1 p_2 + m^2}.$$

For a given value of m, the greatest of these values of V^2 is gm'/m^2, so that the irrotational mode would seem to stand by itself, and not to occur as one limiting case of the rotational modes.

It might have been expected, *à priori*, that in the case of infinite depth, the infinity of possible motions would have been of a higher order than in the case of finite depth; and this is seen to be so, the former case being comparable with the infinity of possible positions of points on a line, and the latter with the infinite series of roots of a transcendental equation.

On some Rings of Circles connected with a Triangle, and the Circles which cut them at Equal Angles. By W. W. Taylor.

[*Read June* 13*th*, 1889.]

If any three circles be placed in contact, the lines joining their points of contact A, B, C form a triangle. Hence it would appear that such three circles must play an important part in the geometry of the triangle. They may be defined, with reference to the triangle ABC, as the circles* that touch two of the radii of the circle ABC at the angular points of the triangle. We will proceed to find their equations, and discuss their properties, and those of certain associated circles and triangles.

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* These circles have been called the ex-cosine circles of the triangle ABC. (W. E. Johnson's "Trigonometry," § 194.)

[The centres of the same set of circles form a second triangle, and the circles may be defined, with reference to that triangle, as the circles which are orthogonal to the inscribed circle, and have their centres at the angular points of the triangle. The properties obtained for the first set may be transformed so as to suit the second set of circles by means of three formulæ like

$$\frac{\delta}{d} = \frac{R}{rs}\left(-\alpha\cos^2\frac{A}{2} + \beta\cos^2\frac{B}{2} + \gamma\cos^2\frac{C}{2}\right),$$

where α, β, γ are the trilinear coordinates of a point referred to the first triangle ABC, and δ, ϵ, ξ new coordinates referred to the sides d, e, f, *i.e.*, EF, FD, DE, where D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides.]

If the equation

$$\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C$$
$$-(l\alpha + m\beta + n\gamma)(\alpha\sin A + \beta\sin B + \gamma\sin C) = 0 \quad\text{.............(1)}$$

represent a circle orthogonal to the circle ABC, the straight line whose equation is $l\alpha + m\beta + n\gamma = 0$ must be the polar of O the centre of the circle ABC, *i.e.*, of the point $R\cos A$, $R\cos B$, $R\cos C$. The condition for this is

$$1 - l\cos A - m\cos B - n\cos C = 0 \quad\text{..................(2)},$$

and the equation of the circle becomes

$$(l\cos A + m\cos B + n\cos C)(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C)$$
$$-(l\alpha + m\beta + n\gamma)(\alpha\sin A + \beta\sin B + \gamma\sin C) = 0,$$

which can be written in the form

$$l\sin A(\alpha^2 - \beta\gamma\cos A + \gamma\alpha\cos B + \alpha\beta\cos C)$$
$$+ m\sin B(\beta^2 + \beta\gamma\cos A - \gamma\alpha\cos B + \alpha\beta\cos C)$$
$$+ n\sin C(\gamma^2 + \beta\gamma\cos A + \gamma\alpha\cos B - \alpha\beta\cos C) = 0.$$

If the common chord be the line $\alpha = 0$, then $l = +\sec A$, and the equation of the circle orthogonal to the circle ABC is

$$\alpha^2 - \beta\gamma\cos A + \gamma\alpha\cos B + \alpha\beta\cos C = 0.$$

We will denote the left-hand side of this by the letter A_1; the equations of the three circles that touch OB, OC; OC, OA; OA, OB at the angular points will be

$$\left.\begin{aligned} A_1 &\equiv \alpha^2 - \beta\gamma\cos A + \gamma\alpha\cos B + \alpha\beta\cos C = 0 \\ B_1 &\equiv \beta^2 + \beta\gamma\cos A - \gamma\alpha\cos B + \alpha\beta\cos C = 0 \\ C_1 &\equiv \gamma^2 + \beta\gamma\cos A + \gamma\alpha\cos B - \alpha\beta\cos C = 0 \end{aligned}\right\} \quad\text{............ (3).}$$

The equation of any other circle orthogonal to the circle ABC will be

$$l\sin A \,.\, A_1 + m\sin B\,.\,B_1 + n\sin C\,.\,C_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots(4),$$

where $l\alpha+m\beta+n\gamma=0$ is the equation of their common chord.

It will be observed that, as each of the three circles A_1, B_1, C_1 touches the other two, and as there must be a pair of circles that each touches, these three circles, A_1, B_1, C_1, must form a ring in the sense of Mr. H. M. Taylor's paper on "The Porism of the Ring of Circles touching Two Circles," *Messenger of Mathematics*, Vol. VII., 1878.)*

We shall accordingly refer to them as the 3-ring circles A_1, B_1, C_1.

The centre of the circle A_1 is most easily found by taking the tangents to the circumscribed circle at B, C, and finding their intersection.

It is obviously $$\frac{\alpha}{-a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{2\Delta}{-a^2+b^2+c^2}$$

Hence we see that the centres of these circles are the "associates of the Lemoine point."

To find the length of the intercept cut off on the side AC of the triangle by the circle A_1, we have the equations

$$\beta = 0, \quad \alpha = -\gamma\cos B \text{ by (3), and } a\alpha + c\gamma = 2\Delta,$$

whence $$\alpha = -a\sin C\cos B\sec A,$$

and the intercept

$$CP = \pm\, a\operatorname{cosec} C = \pm\, a\cos B\sec A;$$

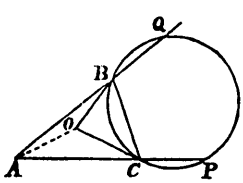
similarly the intercept

$$BQ = \pm\, a\cos C\sec A,$$

and $$BC : CP : BQ = \cos A : \pm\cos B : \pm\cos C.$$

It is worthy of notice that the six points corresponding to P, Q lie on the conic

$$\Sigma\alpha^2 + \Sigma\beta\gamma\,(\cos A + \sec A) = 0,$$



and also that

$$A_1B_1 + B_1C_1 + C_1A_1 \equiv \left\{\Sigma\,(\beta\gamma\sin A)\right\}^2$$

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* Compare also a paper in the same volume "On the Ring of Circles touching Two Circles, and kindred Porisms."

Lemma I.

If α, β, γ be the trilinear coordinates of a point, the equation of a circle being expressed in the form

$$\phi(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma \sin A - (l\alpha + m\beta + n\gamma)\,\Sigma\alpha \sin A = 0,$$

or in the form

$$(l, m, n) \equiv \Sigma a\beta\gamma - (l\alpha + m\beta + n\gamma)\,\Sigma a\alpha = 0,$$

the coordinates of its centre are $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, where

$$\bar{\alpha} = R\,(-l + m\cos C + n\cos B + \cos A),$$

$$\bar{\beta} = R\,(-m + n\cos A + l\cos C + \cos B),$$

$$\bar{\gamma} = R\,(-n + l\cos B + m\cos A + \cos C),$$

and its radius is ρ, where

$$\rho^2 = R^2\,(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$
$$-2l\cos A - 2m\cos B - 2n\cos C + 1).$$

The condition that the centre of the circle (l, m, n) is the pole of the line at infinity gives the following equations to find its coordinates:—

$$\{c\beta + b\gamma - a\,(l\alpha + m\beta + n\gamma) - l\,(a\alpha + b\beta + c\gamma)\}/a$$

$$= \{a\gamma + c\alpha - b\,(l\alpha + m\beta + n\gamma) - m\,(a\alpha + b\beta + c\gamma)\}/b$$

$$= \{b\alpha + a\beta - c\,(l\alpha + m\beta + n\gamma) - n\,(a\alpha + b\beta + c\gamma)\}/c.$$

Let each of these $= X - (l\alpha + m\beta + n\gamma)$. Then we have the equations

$$c\beta + b\gamma - l\,(a\alpha + b\beta + c\gamma) = aX,$$

$$a\gamma + c\alpha - m\,(a\alpha + b\beta + c\gamma) = bX,$$

$$b\alpha + a\beta - n\,(a\alpha + b\beta + c\gamma) = cX.$$

Then, eliminating β, γ, we obtain the equation

$$\alpha \begin{vmatrix} -al & c-bl & b-cl \\ c-am & -bm & a-cm \\ b-an & a-bn & -cn \end{vmatrix} = X \begin{vmatrix} a & c-bl & b-cl \\ b & -bm & a-cm \\ c & a-bn & -cn \end{vmatrix}$$

The coefficient of X becomes, on expansion,

$$2abc\,(\cos A - l + m\cos C + n\cos B)$$

Therefore we have

$$\alpha : \beta : \gamma : a\alpha + b\beta + c\gamma$$

$$= \cos A - l + m\cos C + n\cos B : \cos B - m + n\cos A + l\cos C$$

$$: \cos C - n + l\cos B + m\cos A : a\cos A + b\cos B + c\cos C;$$

but $$a\alpha + b\beta + c\gamma = aR\cos A + bR\cos B + cR\cos C,$$

where R is the radius of the circle ABC;

therefore $$\left.\begin{aligned} \alpha &= R\,(\cos A - l + m\cos C + n\cos B) \\ \beta &= R\,(\cos B - m + n\cos A + l\cos C) \\ \gamma &= R\,(\cos C - n + l\cos B + m\cos A) \end{aligned}\right\} \quad \ldots\ldots\;\ldots\ldots\ldots \quad (5);$$

and, if ρ is the radius of the circle $\phi = 0$,

$$\rho^2\phi\,(-1, \cos C, \cos B) = -\phi\,(\alpha_0, \beta_0, \gamma_0);$$

whence, by substituting the above values, we obtain

$$\rho^2 = R^2\,(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$
$$-2l\cos A - 2m\cos B - 2n\cos C + 1) \;\ldots\ldots(6).$$

Lemma II.

We will now proceed to find the cosine of the angle θ, at which two circles (l_1, m_1, n_1), (l_2, m_2, n_2) whose equations are expressed in the form (1) cut one another.

If the centres of these circles be O_1, O_2, and their radii R_1, R_2,

$$\cos\theta = \frac{O_1O_2^2 - R_1^2 - R_2^2}{2R_1R_2},$$

and $$R_1 = \frac{8R^3}{abc}\,\phi_1,$$

where ϕ_1 denotes the value of the expression

$$\Sigma\beta\gamma\sin A - (\Sigma l\alpha)(\Sigma\alpha\sin A),$$

when for α, β, γ we substitute the coordinates of the centre.

Let the equations of the circles whose radii are R_1, R_2, and centres O_1, O_2, be

$$\phi_1\,(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma\sin A - (\Sigma l_1\alpha)(\Sigma\alpha\sin A),$$

$$\phi_2\,(\alpha, \beta, \gamma) \equiv \Sigma\beta\gamma\sin A - (\Sigma l_2\alpha)(\Sigma\alpha\sin A).$$

Then the equation of a circle whose centre is O_1 and radius O_1O_2 will be

$$\phi_1(\alpha, \beta, \gamma) + h(\Sigma\alpha \sin A)^2 = 0;$$

and, as $O_2(\alpha_2, \beta_2, \gamma_2)$ is on this circle,

$$\phi_1(\alpha_2, \beta_2, \gamma_2) + h(\Sigma\alpha \sin A)^2 = 0;$$

and
$$O_1O_2^2 = \frac{8R^3}{abc}\{\phi_1(\alpha_1, \beta_1, \gamma_1) + h(\Sigma\alpha \sin A)^2\}$$

$$= \frac{8R^3}{abc}\{\phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_1(\alpha_2, \beta_2, \gamma_2)\},$$

and

$$\cos\theta = \frac{\phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_1(\alpha_2, \beta_2, \gamma_2) - \phi_1(\alpha_1, \beta_1, \gamma_1) - \phi_2(\alpha_2, \beta_2, \gamma_2)}{\sqrt{\{\phi_1(\alpha_1, \beta_1, \gamma_1)\,\phi_2(\alpha_2, \beta_2, \gamma_2)\}}}$$

$$= \frac{-2\Sigma\beta_2\gamma_2 \sin A + \{\Sigma(l_1 + l_2)\,\alpha_2\}\,\Sigma\alpha_2 \sin A}{\sqrt{\{\phi_1(\alpha_1, \beta_1, \gamma_1\}\{\phi_2(\alpha_2, \beta_2, \gamma_2)\}}};$$

whence, by substituting the values for $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, we obtain

$$\cos\theta = \frac{-1 - \Sigma l_1 l_2 + \Sigma\cos A\,(l_1 + l_2 + m_1 n_2 + m_2 n_1)}{\sqrt{\{[1 + \Sigma l_1^2 - 2\Sigma\cos A\,(l_1 + m_1 n_1)][1 + \Sigma l_2^2 - 2\Sigma\cos A\,(l_2 + m_2 n_2)]\}}} \qquad \text{............(7)}$$

Hence, if the circle (l, m, n) make the same angle θ with each of the three circles

$$(\sec A, 0, 0), \quad (0, \sec B, 0), \quad (0, 0, \sec C),$$

we must have, by (7),

$$\cos\theta\,\sqrt{[1 + \Sigma l^2 - 2\Sigma\cos A\,(l + mn)]}$$

$$= \{-1 - l\sec A + \cos A\,(l + \sec A) + \cos B\,(m + n\sec A)$$
$$+ \cos C\,(n + m\sec A)\}/\sqrt{[1 + \sec^2 A - 2]}$$

$$= \sec A\{-l(1 - \cos^2 A) + m(\cos C + \cos A\cos B)$$
$$+ n(\cos B + \cos A\cos C)\}/\tan A$$

$$= \sec A\{-l\sin^2 A + m\sin A\sin B + n\sin A\sin C\}/\tan A$$

$= -l\sin A + m\sin B + n\sin C$, and in like manner

$$= +l\sin A - m\sin B + n\sin C$$

$$= +l\sin A + m\sin B - n\sin C;$$

therefore
$$l\sin A = m\sin B = n\sin C.$$

Let each of these $= p$. Then, substituting for l, m, n their values in terms of p,

$$\cos\theta\,\sqrt{\left[1+p^2\Sigma\operatorname{cosec}^2 A-2p\Sigma\cot A-2p^2\Sigma\cos A\operatorname{cosec} B\operatorname{cosec} C\right]}=p;$$

or, putting $\qquad \cot A+\cot B+\cot C=\cot\omega,$

$$\cos\theta\sqrt{\left[1+p^2\operatorname{cosec}^2\omega-2p\cot\omega-4p^2\right]}=p,$$

$$\cos\theta=\frac{p}{\sqrt{\{(1-p\cot\omega)^2-3p^2\}}}=\frac{1}{\sqrt{(\lambda^2-3)}},$$

where $$\lambda=\frac{1}{p}-\cot\omega.$$

Making these substitutions, the equation of the circle $(l,\ m,\ n)$ reduces to the form

$$\Sigma\beta\gamma\sin A-p\,(\Sigma\alpha\operatorname{cosec} A)\,(\Sigma\alpha\sin A)=0,$$

or $$-p\Sigma\alpha^2+\Sigma\beta\gamma\sin A-p\Sigma\beta\gamma\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 B+\sin^2 C)=0,$$

or $$p\Sigma\alpha^2-\Sigma\beta\gamma\sin A\,(1-p\cot\omega)+p\Sigma\beta\gamma\cos A=0,$$

$$\Big[\text{since}\quad \tfrac{1}{2}\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 A+\sin^2 B+\sin^2 C)$$
$$+\tfrac{1}{2}\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C\,(\sin^2 B+\sin^2 C-\sin^2 A)$$
$$=\cot\omega+\operatorname{cosec} A\cos A\Big],$$

and substituting λ for $(1-p\cot\omega)/p$, we obtain for the equation of our circle

$$\Sigma\alpha^2+\Sigma\beta\gamma\cos A-\lambda\Sigma\beta\gamma\sin A=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and, if this touch the circles A_1, B_1, C_1,

$$\sqrt{(\lambda^2-3)}=\sec\theta=1,$$

$$\lambda=\pm 2.$$

The formulæ for the centre of a circle become in this case

$$\left.\begin{aligned}\alpha&=R\,(\sin A+\lambda\cos A)/(\lambda+\cot\omega)\\ \beta&=R\,(\sin B+\lambda\cos B)/(\lambda+\cot\omega)\\ \gamma&=R\,(\sin C+\lambda\cos C)/(\lambda+\cot\omega)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

With the same substitutions

$$\rho^2=R^2\,(\Sigma l^2-2\Sigma mn\cos A-2\Sigma l\cos A+1)$$
$$=R^2\,(\Sigma p^2\operatorname{cosec}^2 A-2\Sigma p^2\cos A\operatorname{cosec} B\operatorname{cosec} C-2\Sigma p\cot A+1)$$
$$=R^2\,(p^2\operatorname{cosec}^2\omega-2p^2\Sigma\,(1-\cot A\cot B)-2p\cot\omega+1)$$

$$= R^2\,(p^2 \operatorname{cosec}^2 \omega - 4p^2 - 2p\cot\omega + 1)$$
$$= R^2\,(p\cot\omega - 1)^2 - 3p^2$$
$$= R^2\,(\lambda^2 - 3)\,p^2$$
$$= R^2\,(\lambda^2 - 3)\,/\,(\lambda + \cot\omega)$$
$$\rho = \pm R\,\sqrt{(\lambda^2 - 3)}\,/\,(\lambda + \cot\omega) \quad \ldots\ldots\ldots\ldots (10).$$

The cosine of the angle between this circle (8) and a ring circle (l, m, n) for which $l\cos A + m\cos B + n\cos C = 1$, is

$$\frac{-1 - \Sigma lp \operatorname{cosec} A + \Sigma\{l\cos A + p\cot A + pl(\cos B\operatorname{cosec} C + \cos C\operatorname{cosec} B)\}}{\sqrt{\{[(l^2+m^2+n^2) - (l\cos A + m\cos B + n\cos C)^2 - 2\Sigma mn\cos A](\lambda^2-3)p^2\}}}$$
$$= \frac{-1 + \Sigma lp\,(\sin A - \cos A\cot\omega) + 1 + p\cot\omega}{p\sqrt{\{(\Sigma l^2\sin^2 A - 2\Sigma mn\sin B\sin C)(\lambda^2-3)\}}}$$
$$= \frac{\Sigma l\sin A}{\sqrt{\{(\Sigma l^2\sin^2 A - 2\Sigma mn\sin B\sin C)(\lambda^2-3)\}}} \quad \ldots\ldots\ldots\ldots (11).$$

The series of circles included in the equation

$$\Sigma\alpha^2 + \Sigma\beta\gamma\cos A - \lambda\Sigma\beta\gamma\sin A = 0$$

has been discussed by Professor P. H. Schoute, in Vol. III., Series 3, of the *Verslagen en Mededeelingen* of the *Koninklijke Akademie van Wetenschappen*, Amsterdam, from an entirely different point of view. He shows that, when a point P moves so that, if D, E, F be the feet of the perpendiculars from it on the sides of the triangle, the Brocard angle of the triangle DEF is constant, the locus of P is a circle of the above series, and λ is the cotangent of the said Brocard angle.

He has shown that this series of circles includes, as particular cases: the Brocard circle ($\lambda = \cot\omega$); the imaginary circle whose equation is

$$\Sigma\alpha^2 + \Sigma\beta\gamma\cos A = 0, \quad (\lambda = 0);$$

the circle ABC ($\lambda = \infty$); the Lemoine line ($\lambda = -\cot\omega$), and the isodynamic points ($\lambda = \pm\sqrt{3}$).

Of these six particular results the first three are obvious by a comparison of the equation No. 8 with the equations of the other circles, and the last three can be obtained by making the centre of the circle lie on the locus.

The condition for this is

$$\begin{vmatrix} 2 & \cos C - \lambda\sin C & \cos B - \lambda\sin B \\ \cos C - \lambda\sin C & 2 & \cos A - \lambda\sin A \\ \cos B - \lambda\sin B & \cos A - \lambda\sin A & 2 \end{vmatrix} = 0,$$

which reduces to the form

$$(\lambda^2-3)(\lambda \sin A \sin B \sin C+1+\cos A \cos B \cos C) = 0.$$

The coordinates of the point circle, for which $\lambda = \sqrt{3}$, must be given by substituting this value in the equations of the centre of a circle (9).

Then
$$\alpha = R(\sin A+\sqrt{3}\cos A)/(\sqrt{3}+\cot\omega)$$
$$= 2R\cos(A-60^\circ)/(\sqrt{3}+\cot\omega),$$
$$\beta = 2R\cos(B-60^\circ)/(\sqrt{3}+\cot\omega),$$
$$\gamma = 2R(\cos C-60^\circ)/(\sqrt{3}+\cot\omega),$$

and the remaining value

$$\lambda \sin A \sin B \sin C+1+\cos A \cos B \cos C = 0,$$

or
$$\lambda = -\cot\omega,$$

gives all the coordinates of the centre infinite. This shows that the circle is a straight line which is at once found to be

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c} = 0$$

The centre of the circle

$$\Sigma(\beta\cos B-\gamma\cos C)^2+\Sigma(\beta\sin B+\gamma\sin C)^2 \equiv \Sigma\alpha^2+\Sigma\beta\gamma\cos A = 0$$

is Lemoine's point K, and its radius is

$$R\tan\omega\sqrt{(-3)}.$$

The equation of the real circle corresponding to this (centre K radius $= R\tan\omega\sqrt{3}$) is

$$4\Delta\{\Sigma\alpha^2+\Sigma\beta\gamma\cos A\}-3\tan\omega\{\Sigma a\alpha\}^2 = 0.$$

We will now for a time desert analysis and employ inversion, using inversion in the sense that we take a fixed point $O_{,,}$, and find a point Q corresponding to any other point P, such that $O_{,,}$, P, Q are in a straight line, and the rectangle $O_{,,}P$, $O_{,,}Q$ is equal to a constant (the square of the radius of inversion).

If we take the centre and radius of the circle ABC as the centre and radius of inversion, the three-ring circles invert into themselves, and a circle cutting them at the angle θ must invert into a circle cutting them at the angle θ; we see, therefore, that the $+\lambda$ circle of Schoute's series inverts into the $-\lambda$ circle of his series, and if V, W be the isodynamic points, $OV.OW = R^2$. It also follows that O

is the external centre of similitude of the $\pm\lambda$ pair of Schoute's circles.

If, again, we take O_n the centre of inversion on Lemoine's line—the radical axis of the system—and the tangent from O_n to the circle ABC as the radius of inversion, each circle of Schoute's system will invert into itself; but the three-ring circles will assume a different position for each position of O_n, and will always possess Schoute's system of circles *each for each* as before. Their points of contact will accordingly form new triangles, each of which possesses the same system of Schoute's circles. That these are the co-Brocardal triangles of ABC can be proved by finding the envelope of a side as O_n moves along Lemoine's line, or thus:—The three-ring circles of the co-Brocardal triangles must touch a pair of circles, which are coaxal with the Brocard circle and the circle ABC, and must be orthogonal to circle ABC; and the only rings of three circles that satisfy these conditions are the rings of circles obtained by our inversion.

This can also be proved thus. All our three-ring circles touch or cut all Schoute's circles at the same angle. So, taking (l, m, n), $(\sec A, 0, 0)$ as two specimens of three-ring circles, we have, by (11),

$$\Sigma l \sin A = \surd\{\Sigma l^2 \sin^2 A - 2\Sigma mn \sin B \sin C\},$$

or

$$\Sigma mn \sin B \sin C = 0$$

and any ring-circle has an equation of the form

$$l \sin A \,.\, A_1 + m \sin B \,.\, B_1 + n \sin C \,.\, C_1 = 0.$$

This gives, as the form of the general equation of a three-ring circle,

$$A_2 \equiv A_1 - B_1 (1+\mu) - C_1 \left(1 + \frac{1}{\mu}\right) = 0.$$

The common chord of this circle and the circle ABC is

$$\mu \frac{\alpha}{a} - \mu (1+\mu) \frac{\beta}{b} - (1+\mu) \frac{\gamma}{c} = 0,$$

and, as this is a quadratic equation in μ, the envelope of all such lines is given by the equation

$$\left(\frac{\alpha}{a} - \frac{\beta}{b} - \frac{\gamma}{c}\right)^2 = 4 \frac{\beta\gamma}{bc},$$

which is the equation of Brocard's ellipse.

The circle A_2 meets the circle ABC where

$$A_1 B_1 + B_1 C_1 + C_1 A_1 \equiv \{\Sigma \beta\gamma \sin A\}^2 = 0,$$

and $$A_1 - B_1(1+\mu) - C_1\left(1+\frac{1}{\mu}\right) = 0;$$

that is, where $$\mu A_1 = -(\mu+1)\,B_1 = C_1,$$

and where $$A_1 = \mu B_1 = -(\mu+1)\,C_1.$$

The third point of this co-Brocardal triangle* must be

$$-(\mu+1)\,A_1 = B_1 = \mu C_1,$$

and the equation to the other circles of the same ring will be

$$B_2 \equiv -A_1\left(1+\frac{1}{\mu}\right) + B_1 - C_1(1+\mu) = 0,$$

$$C_2 \equiv -A_1(1+\mu) - B_1\left(1+\frac{1}{\mu}\right) + C_1 = 0.$$

Here A_2, B_2, C_2 satisfy the relations

$$A_2 + B_2 + C_2 \equiv -\left(\mu+1+\frac{1}{\mu}\right)(A_1+B_1+C_1),$$

$$B_2C_2 + C_2A_2 + A_2B_2 \equiv \left(\mu+1+\frac{1}{\mu}\right)^2 (B_1C_1 + C_1A_1 + A_1B_1).$$

Again, we will invert with regard to one of the isodynamic points V, W as our centre of inversion, and the tangent to the circle ABC as our radius of inversion. Since

$$OV\,.\,OW = R^2,$$

$$WV\,.\,WO = WO^2 - R^2 = WT^2;$$

therefore, inverting with respect to W, as above, V inverts into the centre of the circle ABC, and the circle ABC inverts into itself; and two circles of the *coaxal* system having become concentric by inversion, the rest must have done so also, and the whole system of Schoute's circles becomes a concentric system, and, in consequence of these circles having become *concentric*, the three-ring circles become necessarily equal circles, and their points of contact form an equi-

* The coordinates of the angular points of this triangle are given by the equations

$$\frac{\alpha}{a} = \mu\frac{\beta}{b} = -(\mu+1)\frac{\gamma}{c},$$

$$-(\mu+1)\frac{\alpha}{a} = \frac{\beta}{b} = \mu\frac{\gamma}{c},$$

$$\mu\frac{\alpha}{a} = -(\mu+1)\frac{\beta}{b} = \frac{\gamma}{c}.$$

lateral triangle. This applies not only to our original triangle, but to any of the co-Brocardal system of triangles; and as at the same time the other isodynamic point inverts into the centre of the circle ABC, the circles of Apollonius, which passed through both points V, W and the angular points of the triangle, become straight lines passing through O and the angular points of the equilateral triangles, that is to say, become diameters of the circle ABC through the angular points of the equilateral triangles. Therefore in any triangle the circles of Apollonius cut one another at an angle of 60°.

The circle of inversion in this case is at the isodynamic point—in other words, is concentric with the circle

$$\alpha^2+\beta^2+\gamma^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)=0.$$

[The $-$ sign gives the inner point (V), the $+$ sign the outer point (W)]. A circle concentric with this must have an equation of the form

$$\Sigma\alpha^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)+h\,(\Sigma\alpha\sin A)^2=0.$$

If this be also of the form (4),

$$l\sin A=1+h\sin^2 A,$$

$$m\sin B=1+h\sin^2 B,$$

$$n\sin C=1+h\sin^2 C,$$

$$-l\sin A\cos A+m\sin B\cos A+n\sin C\cos A$$

$$=\cos A\pm\sqrt{3}\sin A+2h\sin B\sin C,$$

and also from above $\quad=\cos A\left[1+h\,(-\sin^2 A+\sin^2 B+\sin^2 C)\right];$

whence

$$h\left[\cos A\,(\sin^2 A-\sin^2 B-\sin^2 C)+2\sin B\sin C\right]=\mp\sqrt{3}\sin A,$$

or
$$h=\frac{\mp\sqrt{3}}{2\sin A\sin B\sin C},$$

and the equation of the required circle becomes

$$\Sigma\alpha^2+\Sigma\beta\gamma\,(\cos A\pm\sqrt{3}\sin A)\mp\frac{\sqrt{3}}{2\sin A\sin B\sin C}(\Sigma\alpha\sin A)^2=0.$$

The upper signs will give an imaginary and the lower a real circle, showing that the inversion is in the first case across the point, in the second away from it.

The centre of the circle

$$\alpha^2+\beta^2+\gamma^2+\Sigma\beta\gamma\cos A = 0$$

was found to be the symmedian point K_1, and its radius

$$\rho = R\tan\omega\sqrt{(-3)}.$$

Again, if we draw the chord AKA' of the circle ABC, the product $AK.KA'$ is equal to

$$R^2-OK^2 = R^2-4\rho'^2$$

(where ρ' is the radius of the Brocard circle, *i.e.*, of the Schoute circle for which $\lambda = \cot\omega$) $= R^2-R^2(1-3\tan^2\omega)$, by (10),

$$= 3R^2\tan^2\omega.$$

This proves that, if we invert with K as centre of inversion, and the radius of this impossible circle as the radius of inversion, the circle ABC will invert into itself; consequently, all circles orthogonal to it will invert into circles orthogonal to it, and the three-ring circles of the triangle ABC will invert into the three-ring circles of the co-symmedian triangle; as this is a co-Brocardal triangle, the system of Schoute's circles must invert into one another as circles cut one another at the same angles as their inverses. It also follows that K is the internal centre of similitude of each pair $(\pm\lambda)$ of Schoute's circles.

We can by means of this result obtain the equations of the six-ring circles which are orthogonal to the circle ABC, and are cut at equal angles by each of the Schoute circles. For, drawing the chords AKA', BKB', CKC' of the circle ABC, A', B', C' must, by our last result, be the points where the six-ring circles at ABC meet the circle ABC again; and as the coordinates of K are $a:b:c$, the equation of $A'K$ is

$$\beta\sin C = \gamma\sin B,$$

and the equations of $A'B$, $A'C$ are found by eliminating β and γ from the equations of $A'K$ and the circle ABC; therefore the equation of $A'B$ is

$$2\alpha\sin C+\gamma\sin A = 0;$$

and therefore, by equation (4), a ring-circle of the six-ring through $A'B$ will have for its equation

$$2A_1+C_1 = 0,$$

and the equations of a complete ring of the six-ring circles will be

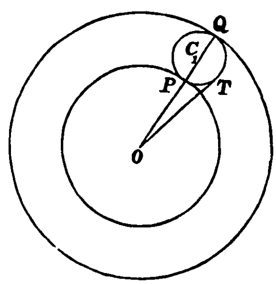
$$B_1+2C_1 = 0,$$
$$A_1+2C_1 = 0,$$
$$C_1+2A_1 = 0,$$
$$B_1+2A_1 = 0,$$
$$A_1+2B_1 = 0,$$
$$C_1+2B_1 = 0.$$

The value of λ for the Schoute circles that touch the six-ring circles is

$$\lambda = \pm 2\sqrt{3}.$$

The general equation of a member of the six-ring circles can be found by taking an arbitrary point O_{11} on Lemoine's line, drawing the lines $O_{11}A$, $O_{11}B'$, and finding where these meet the circle ABC again, and forming the equation of that chord of the circle ABC; whereupon we at once know the equation of the corresponding ring-circle, and proceeding in the same way we can find the equations of all the circles of any six-ring system.

We will briefly indicate how the same can be done for any other ring. The chords common to the circles of the ring and the circumscribed circle form a harmonic polygon (Casey's "Sequel to Euclid," 199-206), and always touch an ellipse of a family of which Brocard's is the best known example. Their centres lie on another ellipse whose foci are the centres of a $\pm\lambda$ pair of Schoute's circles.



To determine the value of λ for the tangent circle to any ring, formula (11) shows us that the angle at which any of Schoute's circles cuts the three-ring circles depends only on the value of λ, and not on the angles A, B, C; so we can determine the value of λ for an equilateral triangle.

Now, for any ring of r circles, we must have in the above figure—where O_1 is the centre of PTQ one of the ring-circles, and P, T, Q are the points where the ring-circle meets the λ-Schoute circle, the circle ABC, and the $-\lambda$-Schoute circle—

$$\tan\frac{\pi}{r} = \tan O_1OT = O_1T/\sqrt{(OP.\,OQ)} = \tfrac{1}{2}(OQ-OP)/\sqrt{(OP.OQ)},$$

and therefore, taking the values of OP, OQ from equation (10) with due regard to sign, and remembering that $\lambda < \sqrt{3}$ numerically, and

$$\cot\omega = \sqrt{3},$$

we have $$\tan\frac{\pi}{r} = \frac{1}{2}\left(\frac{1}{\lambda-\cot\omega}-\frac{1}{\lambda+\cot\omega}\right)\Big/\sqrt{\left(\frac{1}{\lambda^2-\cot^2\omega}\right)}$$

$$= \frac{\cot\omega}{\sqrt{(\lambda^2-\cot^2\omega)}} = \frac{\sqrt{3}}{\sqrt{(\lambda^2-3)}},$$

$$\lambda^2 = 3\,\mathrm{cosec}^2\frac{\pi}{r},$$

$$\lambda = \pm\sqrt{3}\,\mathrm{cosec}\,\frac{\pi}{r},$$

and for the six-ring circle

$$\lambda = \pm 2\sqrt{3}.$$

The general relation between l, m, n, when a circle belongs to the ring of r circles, is found by substituting the values

$$\cos\theta = 1 \quad\text{and}\quad \lambda = \pm\sqrt{3}\,\mathrm{cosec}\,\frac{\pi}{r}$$

from above in formula (11), and can be written

$$(\Sigma l\sin A)\tan\frac{\pi}{r} = \sqrt{3}\,.\,\sqrt{(\Sigma l^2\sin^2 A - 2\Sigma mn\sin B\sin C)} \quad \ldots(12)$$

and this is, consequently, also the relation between l, m, n when

$$l\alpha + m\beta + n\gamma = 0$$

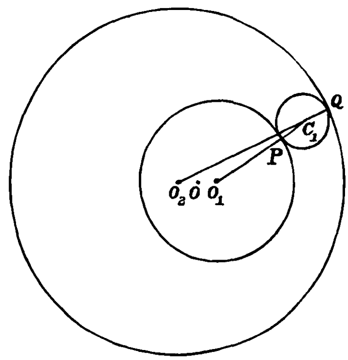
is a side of an harmonic polygon of r sides belonging to the same system.

The locus of the centres of the ring-circles of any series is plainly an ellipse or other conic of which the foci are the centres of the $\pm\lambda$ circles that touch all the members of that ring.

For, if O_1, O_2 be the centres of the $\pm\lambda$ circles that touch at P and

Q the ring-circle whose centre is C_1, then

$$O_1C_1 + C_1O_2 = R_1 - C_1P + R_2 + C_1Q = R_1 + R_2$$
$$= \text{a constant} = 2\lambda R\sqrt{(\lambda^2-3)}/(\lambda^2 - \cot^2\omega).$$



However, we can find the equation of this locus more readily by finding the locus of the pole of

$$l\alpha + m\beta + n\gamma = 0,$$

subject to the condition (12), which may be written in the form

$$(\Sigma l \sin A)^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 6\Sigma l^2 \sin^2 A.$$

Since α, β, γ, a point on the locus, is the pole of

$$l\alpha + m\beta + n\gamma = 0$$

with respect to the circle ABC,

$$\frac{l}{b\gamma + c\beta} = \frac{m}{a\gamma + c\alpha} = \frac{n}{a\beta + b\alpha},$$

and the locus of the pole must therefore be given by the equation

$$\{\Sigma (ab\gamma + ac\beta)\}^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 6\Sigma (ab\gamma + ac\beta)^2,$$

$$(2\Sigma bc\alpha)^2 \left(\tan^2 \frac{\pi}{r} + 3\right) = 12\Sigma (b^2c^2\alpha^2 + a^2bc\beta\gamma),$$

$$\Sigma (b^2c^2\alpha^2 + 2a^2bc\beta\gamma) \left(\tan^2 \frac{\pi}{r} + 3\right) = 3\Sigma (b^2c^2\alpha^2 + a^2bc\beta\gamma),$$

$$\Sigma b^2c^2\alpha^2 \tan^2 \frac{\pi}{r} + \Sigma a^2bc\beta\gamma \left(2 \tan^2 \frac{\pi}{r} + 3\right) = 0.$$

It is plain that the sides also of the harmonic polygons, being the polars of these centres with respect to the circle ABC, must envelope an ellipse.

Its equation, being the envelope of

$$l\alpha+m\beta+n\gamma=0$$

subject to the condition (12), is

$$\Sigma b^2c^2\alpha^2\tan^2\frac{\pi}{r}-\left(\tan^2\frac{\pi}{r}+3\right)\Sigma a^2bc\beta\gamma=0.$$

Both series of ellipses belong to the same family

$$\Sigma b^2c^2\alpha^2+\mu\Sigma a^2bc\beta\gamma=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

and the values of μ for a pair of them are connected by the relation

$$\mu_1+\mu_2=1.$$

The usual formulæ for the foci of the conic

$$A\alpha^2+B\beta^2+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta=0$$

are $\alpha^2(A'a^2+B'b^2+C'c^2+2D'bc+2E'ca+2F'ab)$

$$-4\Delta\alpha\,(bF'+cE'+aA')+4\Delta^2A',$$

= two like expressions in β and γ, where A', B', &c. are the minors of A, B, &c. in the determinant

$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} \quad \text{(Whitworth, p. 269).}$$

In the present case these equations reduce to

$$\alpha^2\left\{\mu\left(-\Sigma a^4+2\Sigma b^2c^2\right)-2\Sigma a^4\right\}$$
$$-4\Delta a\alpha\left\{(b^2+c^2-a^2)\,\mu-2a^2\right\}-4\Delta^2a^2(\mu+2)=\&\text{c}.$$

Putting each of these $=K$, and rearranging,

$$\alpha^2\left\{\mu\,(\Sigma a^2)^2-2\,(\mu+1)\,\Sigma a^4\right\}-4\Delta a\alpha\left\{\mu\Sigma a^2-2\,(\mu+1)\,a^2\right\}$$
$$+4\Delta^2a^2\left\{\mu-2\,(\mu+1)\right\}=K,$$

and two like equations.

Eliminating from these the ratios

$$\mu:-2\,(\mu+1):K,$$

we obtain, as the equation of the locus of the foci,

$$\begin{vmatrix} (\alpha\Sigma a^2-2\Delta a)^2, & \alpha^2\Sigma a^4-4\Delta a^3\alpha+4\Delta^2 a^2, & 1 \\ (\beta\Sigma a^2-2\Delta b)^2, & \beta^2\Sigma a^4-4\Delta b^3\beta+4\Delta^2 b^2, & 1 \\ (\gamma\Sigma a^2-2\Delta c)^2, & \gamma^2\Sigma a^4-4\Delta c^3\gamma+4\Delta^2 c^2, & 1 \end{vmatrix} = 0,$$

which, on expansion, yields the two factors

$$\Sigma bc\,(b^2-c^2)\,\alpha \quad\text{and}\quad abc\Sigma\alpha^2-\Sigma a^3\beta\gamma.$$

For the locus the real foci are on the straight line KO,

$$\Sigma bc\,(b^2-c^2)\,\alpha = 0,$$

and the imaginary ones on the Brocard circle

$$abc\Sigma\alpha^2 = \Sigma a^3\beta\gamma.$$

Since the equation (13) can be arranged in the form

$$(\Sigma bc\alpha)^2+\overline{\mu-2}\,\Sigma a^2bc\beta\gamma = 0,$$

the family of ellipses (13) have imaginary double contact with one another and the circle ABC where they meet the line

$$\Sigma bc\alpha = 0.$$

There is one parabola belonging to the series, which is the locus of centres of circles that touch Brocard's circle and Lemoine's line, and then

$$\mu = \tan^2\omega-3.$$

(Lemoine's line)^2 is given twice over by making the discriminant vanish, and we also obtain Lemoine's point as a particular case of these conics.

To find the general equation of a circle of Apollonius, we know that it is orthogonal to the circle ABC, and therefore its equation is of the form (4),

$$l\sin A\,.\,A_1+m\sin B\,.\,B_1+n\sin C\,.\,C_1 = 0.$$

To be a circle of Apollonius it must also pass through the point

$$\sin(A+60^\circ) : \sin(B+60^\circ) : \sin(C+60^\circ).$$

Making these substitutions for α, β, γ in A_1, we obtain, for

$$\alpha^2-\beta\gamma\cos A+\gamma\alpha\cos B+\alpha\beta\cos C,$$

$$\tfrac{1}{4}\{\sin^2 A+3\cos^2 A-\cos A\,(\sin B\sin C+3\cos B\cos C)$$
$$+\cos B\,(\sin A\sin C+3\cos A\cos C)$$
$$+\cos C\,(\sin A\sin B+3\cos A\cos B)\}$$
$$+\tfrac{1}{4}\sqrt{3}\,\{2\sin A\cos A-\sin A\cos A+\sin B\cos B+\sin C\cos C\}$$
$$= \tfrac{1}{2}\{1+\cos A\cos B\cos C+\sqrt{3}\sin A\sin B\sin C\};$$

therefore, in this case, $A_1 = B_1 = C_1$,

and the necessary condition for the above equation (4) to represent a circle of Apollonius is

$$l \sin A + m \sin B + n \sin C = 0.$$

The particular one through the angular point A of the original triangle must satisfy (4), when we put

$$\alpha : \beta : \gamma = 1 : 0 : 0;$$

A_1 becomes 1, $B_1 = C_1 = 0$,

therefore $l = 0$,

and the equation reduces to $B_1 = C_1$.

Hence $B_1 = C_1, \quad C_1 = A_1, \quad A_1 = B_1$

are the three primary circles of Apollonius. Any other can be found by making (4) pass through some other point on the circle ABC. The circle on VW as diameter is plainly the smallest of all these circles, and its equation can be found by making its centre lie on OK whose equation is

$$\Sigma a (b^2 - c^2)\, \alpha = 0,$$

or, more simply, by making the common chord with the circle ABC,

$$l\alpha + m\beta + n\gamma = 0,$$

parallel to Lemoine's line. The condition for this is

$$\begin{vmatrix} l, & m, & n \\ \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c} \\ a, & b, & c \end{vmatrix} = 0,$$

or $$la (b^2 - c^2) + mb (c^2 - a^2) + nc (a^2 - b^2) = 0.$$

And the equation to this circle may be obtained by eliminating l, m, n from the three equations

$$l \sin A \,.\, A_1 + m \sin B \,.\, B_1 + n \sin C \,.\, C_1 = 0,$$

$$l \sin A \quad + m \sin B \quad + n \sin C \quad = 0,$$

$$l \sin A (\sin^2 B - \sin^2 C) + m \sin B (\sin^2 C - \sin^2 A)$$
$$+ n \sin C (\sin^2 A - \sin^2 B) = 0.$$

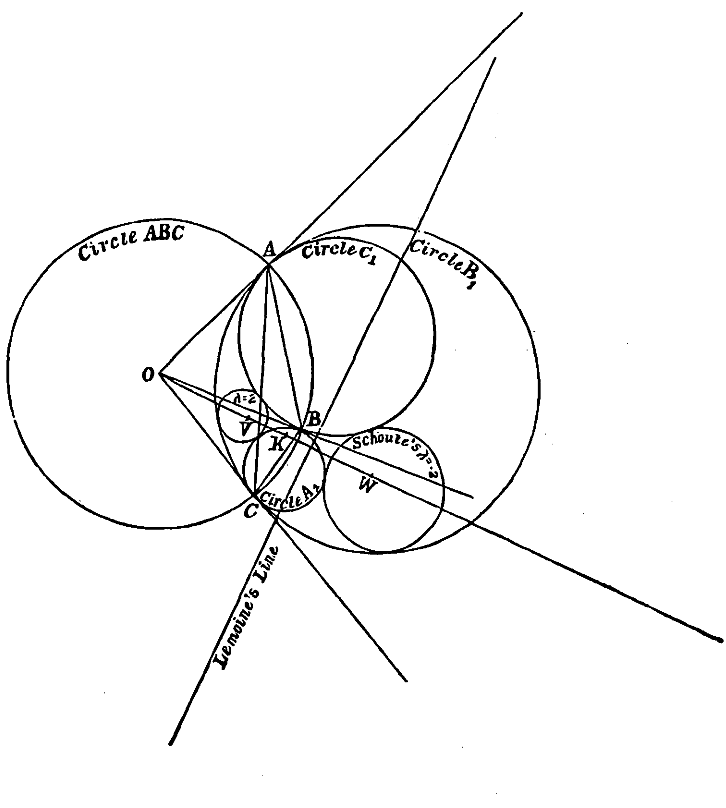
Its equation, therefore, is

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ 1, & 1, & 1 \\ \sin^2 B - \sin^2 C, & \sin^2 C - \sin^2 A, & \sin^2 A - \sin^2 B \end{vmatrix} = 0,$$

or $$A_1\,(2a^2-b^2-c^2)+B_1\,(2b^2-a^2-c^2)+C_1\,(2c^2-a^2-b^2)=0.$$

We have previously omitted to remark that, if we take any other point anywhere, and invert with the tangent to the circle ABC as the radius of inversion, we shall obtain a new set of harmonic polygons, ring-circles and Schoute's circles; as is evident, since all curves cut at the same angle as their inverses.

It is worthy of notice that the circles ABC, BCW, CAW, and ABW, being the inverses of the circle ABC, and of the sides of an equilateral triangle, intersect at angles of 60°, and the circles of Apollonius round VAW, VBW, VCW, being the inverses of the bisectors of the angles of the equilateral triangle, intersect at 60° and bisect the angles between the circles BCW, CAW, ABW.



Presents received during the Recess:—

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Pamphlets by M. Maurice d'Ocagne:—

"Sur certaines Courbes qu'on peut adjoindre aux Courbes Planes pour l'étude de leurs Propriétés Infinitésimales." (*American Jour. of Math.*, Vol. XI., No. 1.)

"Calcul direct des Termes d'une réduite de rang quelconque d'une Fraction Continue Périodique."

"Détermination du Rayon de Courbure de la Courbe Intégrale." (*Nouvelles Annales.*)

"Quelques propriétés de l'Ellipse; Deviation, Ecart normal." (*Nouvelles Annales.*)

"Sur les Systèmes de Péninvariants principaux d'une Forme Binaire. (*Bulletin de la S. Math. de France.*)

"Formules nouvelles pour résoudre le problème de la Carte au moyen de données particulières." (*Revue Maritime et Coloniale*, Fev., 1889.)

APPENDIX.

Mr. Basset points out that the following corrections should be supplied in his investigation of the stability of Maclaurin's Spheroid (Vol. XIX., pp. 52–54):—"The correct result, which was first obtained by Riemann, is that for an ellipsoidal displacement, the spheroid becomes unstable, when the excentricity exceeds the root of the equation

$$e\,(1-e^2)^{\frac{1}{2}}\,(3+4e^2) = (3+2e^2-4e^4)\sin^{-1}e,$$

which gives e equal to about ·95." See his work on "Hydrodynamics," Vol. II., p. 124.

The following recently published papers bear upon the subject:—

Love, *Phil. Mag.*, Vol. XXVII., p. 254.

Bryan, *Phil. Trans.*, 1889, p. 187, and *Proc. Camb. Phil. Soc.*, Vol. VI., p. 248.

There is a paper by H. Weber in the *Math. Annalen*, Band XXXIII., Heft. 3, p. 391, "Zur complexen Multiplication elliptischen Funktionen." (See Prof. Greenhill's paper on "Complex Multiplication Moduli of Elliptic Functions," Vol. XIX., p. 362.)