223. Higher Trigonometry
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Source: *The Mathematical Gazette*, Vol. 4, No. 61 (Jan., 1907), pp. 13-14
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3604265
Accessed: 24-06-2016 13:13 UTC

MATHEMATICAL NOTES.

223. [D. 6. b.] *Higher Trigonometry.*

Since I wrote my rather fragmentary 'Notes' on Higher Trigonometry I have had occasion to work out the theory of the elementary transcendental functions in a rather more systematic way. This attempt has led me to modify my views in some respects. The net result is that I disagree with Mr. Picken more decidedly than I should have done if I could have seen his method of developing the theory six months ago.

I have no time to discuss the question at length. My chief difference with Mr. Picken is about the use to be made of the theorem

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n=\exp x.$$

He makes it fundamental: in fact he takes it as his definition of the exponential function. I adhere to my statement that this is 'logically quite wrong.' Of course I do not mean by this that it is impossible to base a rigorous theory of $\exp x$ and $\log x$ upon this theorem: many writers have done so. What I mean is that to do so is to disturb the proper perspective of the subject. Lewis Carroll based a theory of parallels on the proposition, 'In every Circle, the inscribed equilateral Tetragon is greater than any one of the Segments which lie outside it.' He would have been the first to admit that this was, 'although possible, logically quite wrong.'

Moreover, the result is not encouraging. It is 'not for the immature schoolboy mind.' I am sanguine enough to believe, on the other hand, that it is quite possible for a clever schoolboy to master a good deal of the theory of these functions. But we must look about for methods other than Mr. Picken's. On the whole I incline to the integral definition of $\log x$ as the best starting point. Mr. Picken, I notice, in one place seems to presuppose this definition. If so, why not define the exponential as the inverse of the logarithm? From the equations

$$y=\int_1^x\frac{dt}{t},\quad x=\exp y$$

the greater part of the theory follows with perfect rigour and extreme simplicity. In particular the theorem

$$\lim_{x=\infty}\left(1+\frac{1}{x}\right)^x=e$$

(which Mr. Picken, to judge by his remarks at the top of p. 360, seems to find rather a stumbling block) follows in two or three lines.

There is, of course, really no difference between starting from these definitions and starting from

$$\frac{dx}{dy}=x,\quad x_{y=0}=1$$

as the definition of $x=\exp y$. In this case the logarithm is defined as an inverse function.

Otherwise it seems to me, in spite of what Mr. Picken says, that we must start from the exponential series. The only serious difficulty is the proof of the functional equation by multiplication of series, or the proof of the equation $\frac{dx}{dy}=x$ by differentiation of an infinite series. It is not necessary to face both difficulties: either may be used to avoid the other. Mr. Picken's criticisms (pp. 332-3) I cannot altogether follow.

May I make two other remarks? (1) How does Mr. Picken's investigation of the factors of $\sin x$ affect my statement that the factor theorem is 'really difficult'? Apart from details his proof is that which I was taught years ago—due, I believe, to Tannery (if not much older). And does Mr. Picken mean to imply that his proof is not 'really difficult.'?

(2) Mr Picken seems to me to follow Prof. Chrystal in a certain vagueness as to the distinction between a value for $x=1$ and a limit for $x=1$. The function $\frac{x^2-1}{x-1}$ has *no* value for $x=1$; for $x=1$, $\frac{x^2-1}{x-1}$ is strictly and absolutely meaningless. The fact that its limit for $x=1$ is 2 is entirely irrelevant. The functions $\frac{x^2-1}{x-1}$ and $x+1$ are different functions. They are equal when x is *not* equal to 1. Similarly the function $y=\frac{x}{x}$ is $=1$ when $x \neq 0$ and undefined for $x=0$. To calculate $f(x)$ for $x=0$ we must put $x=0$ in the expression of $f(x)$ and perform the arithmetical operations which the form of the function prescribes, and this we cannot do in this case.

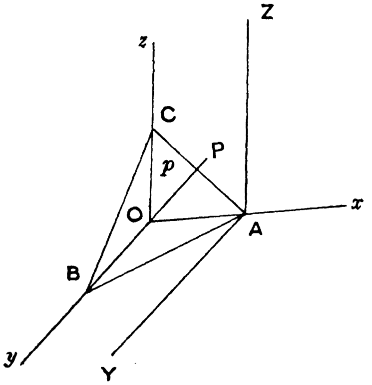
Whether Mr. Picken agrees with me here I cannot say. I lay stress on the point because his language is not quite clear. Thus he says (p. 330) that 'a function of x may have a value for a given value a of the argument, although the expression $f(x)$ fails to provide a value when a is substituted for it'—and I might quote other sentences which I cannot regard as entirely satisfactory. G. H. HARDY.

224. [M[1]. 8. g.] *A curious imaginary curve.*

The curve
$$(x+iy)^2=\lambda(x-iy)$$
is (i) a parabola, (ii) a rectangular hyperbola, and (iii) an equiangular spiral. The first two statements are evidently true. The polar equation is
$$r=\lambda e^{-3i\theta},$$
the equation of an equiangular spiral. The intrinsic equation is easily found to be $\rho=3is$.

It is instructive (i) to show that the equation of any curve which is both a parabola and a rectangular hyperbola can be put in the form given above, or in the form
$$(x+iy)^2=x \text{ (or } y),$$
and (ii) to determine the intrinsic equation directly from one of the latter forms of the Cartesian equation. G. H. HARDY.

225. [L[1]. 1. a.] *The line at infinity, etc.*



Can anyone tell me of an English book which contains a clear and intelligible account of the 'line at infinity'? Such accounts as are contained in the ordinary books on Conics, or in Miss Scott's *Modern Analytical Geometry*, appear to me confusing in the highest degree.

Most undergraduates seem to believe that there really are points at infinity, and that they really do lie on a line, and that if you could get there you would find that $1=0$. The fault lies in the books, which persist in treating conventions as if they were sober statements of fact.

I have found the following construction useful (see figure). Project p in the plane $x+y+z=1$ into P in