ON A FORMULA FOR THE SUM OF A FINITE NUMBER OF TERMS OF THE HYPERGEOMETRIC SERIES WHEN THE FOURTH ELEMENT IS UNITY

(*Second Communication.*)

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Abstract.

IN a paper communicated to the Society and printed in the *Proceedings*, Ser. 2, Vol. 5, pp. 335–341, it was shown that when the real part of $\gamma-\alpha-\beta$ is negative, then, in general, the sum of s terms of the series

$$1+\frac{\alpha\beta}{1\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\;\gamma(\gamma+1)}+\dots$$

was asymptotic with $\dfrac{\Pi(\gamma-1)\,s^{\alpha+\beta-\gamma}}{(\alpha+\beta-\gamma)\,\Pi(\alpha-1)\,\Pi(\beta-1)}$,

l.c., p. 339.

The proof given did not, however, apply to the special case in which $\gamma-\alpha-\beta$ is a negative integer, and it did not apply when $\gamma-\alpha-\beta$ is equal to zero.

The object of the present communication is to show that the formula given above does hold when $\gamma-\alpha-\beta$ is a negative integer, but that, when $\gamma-\alpha-\beta$ is equal to zero, then the sum of s terms is asymptotic with

$$\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\log_e s.$$

This last result was obtained in the first instance by taking the expression given in Art. 4 of the former paper, putting $\gamma=\alpha+\beta+\epsilon$, and equating the terms independent of ϵ. The difficulty of obtaining a thoroughly satisfactory proof by this method led me to build up an independent proof.

The method adopted has a point of interest.

Calling the terms of the original series $T_1+T_2+\dots+T_s$, certain factors

$U_1, U_2, \ldots, U_s$ are obtained, such that $U_n T_n$ can be put into the form $V_n - V_{n-1}$. From this it follows that

$$U_1 T_1 + U_2 T_2 + \ldots + U_s T_s = V_s - V_0.$$

The factors $U_1, U_2, \ldots, U_s$ depend upon an integer r, in such a way that when r is increased to infinity, these factors all tend to unity.

To sum the series $T_1 + \ldots + T_s$, all that remains is to make r infinitely great in $V_{s+1} - V_0$, and then determine the simplest expression with which $V_{s+1} - V_0$ is asymptotic. The result is as given above.

Thus the only discontinuity in the formula takes place at the value of γ which separates those series which are convergent from those which are divergent.

1. With the notation used in the former paper, and also the following,

$$t_{r,s} = \frac{1}{a+r-1} + \frac{1}{\beta+r-1} - \frac{1}{r} - \frac{1}{a+\beta+s+r},$$

$$u_{r,s} = 1 + \frac{a\beta}{a+\beta+s+1} + \frac{a_2\beta_2}{2!\,(a+\beta+s+1)_2} + \ldots + \frac{a_{r-1}\beta_{r-1}}{(r-1)!\,(a+\beta+s+1)_{r-1}},$$

$$w_{r,s} = t_{1,s}u_{1,s} + t_{2,s}u_{2,s} + \ldots + t_{r,s}u_{r,s},$$

it will be proved that

$$\frac{a_s\beta_s}{s!\,(a+\beta)_s}\left(1 - \frac{a_r\beta_r}{r!\,(a+\beta+s)_r} - \frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{a+\beta+s+r}\right)$$

$$= \frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}} w_{r,s} - \frac{a_s\beta_s}{(s-1)!\,(a+\beta)_s} w_{r,s-1}. \quad \text{(I.)}$$

Dividing out by $\dfrac{a_s\beta_s}{(s-1)!\,(a+\beta)_s}$, it is necessary to prove

$$\frac{1}{s}\left(1 - \frac{a_r\beta_r}{r!\,(a+\beta+s)_r} - \frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{a+\beta+s+r}\right)$$

$$= \frac{(a+s)(\beta+s)}{s\,(a+\beta+s)} w_{r,s} - w_{r,s-1}$$

$$= \frac{(a+s)(\beta+s)}{s\,(a+\beta+s)} \sum_{v=1}^{r} t_{v,s}u_{v,s} - \sum_{v=1}^{r} t_{v,s-1}u_{v,s-1}$$

$$= \sum_{v=1}^{r} t_{v,s}\left(\frac{(a+s)(\beta+s)}{s\,(a+\beta+s)} u_{v,s} - u_{v,s-1}\right) + \sum_{v=1}^{r} u_{v,s-1}\,(t_{v,s} - t_{v,s-1}).$$

2. It may be immediately verified for $v = 1, 2, \ldots$, and then, by induction for all values of v, that

$$\frac{(a+s)(\beta+s)}{s(a+\beta+s)} u_{v,s} - u_{v,s-1} = \frac{a_v \beta_v}{s.(v-1)!(a+\beta+s)_v}.$$

3. Hence the equation to be proved is

$$\frac{1}{s}\left(1 - \frac{a_r\beta_r}{r!(a+\beta+s)_r} - \frac{a_r\beta_r}{(r-1)!(a+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{a+\beta+s+r}\right)$$

$$= \sum_{v=1}^{r} \frac{a_v\beta_v t_{v,s}}{s.(v-1)!(a+\beta+s)_v}$$

$$+ \sum_{v=1}^{r} u_{v,s-1}\left(\frac{1}{a+\beta+s+v-1} - \frac{1}{a+\beta+s+v}\right).$$

Now $$\sum_{v=1}^{r} u_{v,s-1}\left(\frac{1}{a+\beta+s+v-1} - \frac{1}{a+\beta+s+v}\right)$$

$$= \frac{u_{1,s-1}}{a+\beta+s} + \frac{u_{2,s-1} - u_{1,s-1}}{a+\beta+s+1}$$

$$+ \frac{u_{3,s-1} - u_{2,s-1}}{a+\beta+s+2} + \ldots + \frac{u_{r,s-1} - u_{r-1,s-1}}{a+\beta+s+r-1} - \frac{u_{r,s-1}}{a+\beta+s+r}$$

$$= \frac{1}{a+\beta+s} + \frac{a\beta}{(a+\beta+s)_2} + \frac{a_2\beta_2}{2!(a+\beta+s)_3} + \ldots + \frac{a_{r-1}\beta_{r-1}}{(r-1)!(a+\beta+s)_r}$$

$$- \frac{u_{r,s-1}}{a+\beta+s+r}.$$

4. Hence the equation to be proved is

$$\frac{1}{s}\left(1 - \frac{a_r\beta_r}{r!(a+\beta+s)_r} - \frac{a_r\beta_r}{(r-1)!(a+\beta+s)_{r+1}} - \frac{s u_{r,s-1}}{a+\beta+s+r}\right)$$

$$= \sum_{v=1}^{r}\left(t_{v,s}\frac{a_v\beta_v}{s.(v-1)!(a+\beta+s)_v} + \frac{a_{v-1}\beta_{v-1}}{(v-1)!(a+\beta+s)_v}\right) - \frac{u_{r,s-1}}{a+\beta+s+r}.$$

Now $$t_{v,s}\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}\left[\frac{a+\beta+2v-2}{(a+v-1)(\beta+v-1)}-\frac{a+\beta+s+2v}{v(a+\beta+s+v)}\right]$$

$$+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_{v-1}\beta_{v-1}}{s.(v-1)!\,(a+\beta+s)_v}\left[(a+\beta+s+v-1)+(v-1)-s\right]$$

$$-\frac{a_v\beta_v}{s.v!\,(a+\beta+s)_{v+1}}\left[(a+\beta+s+v)+v\right]+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}$$

$$=\frac{a_{v-1}\beta_{v-1}}{s.(v-1)!\,(a+\beta+s)_{v-1}}+\frac{a_{v-1}\beta_{v-1}}{s.(v-2)!\,(a+\beta+s)_v}$$

$$-\frac{a_v\beta_v}{s.v!\,(a+\beta+s)_v}-\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_{v+1}}.$$

The third and fourth of the above terms are obtainable from the first and second by increasing v by unity.

When $v=1$, the formula is

$$t_{1,s}\frac{a\beta}{s(a+\beta+s)}+\frac{1}{a+\beta+s}=\frac{1}{s}-\frac{a\beta}{s(a+\beta+s)}-\frac{a\beta}{s(a+\beta+s)_2};$$

therefore

$$\sum_{v=1}^{r}\left(t_{v,s}\frac{a_v\beta_v}{s.(v-1)!\,(a+\beta+s)_v}+\frac{a_{v-1}\beta_{v-1}}{(v-1)!\,(a+\beta+s)_v}\right)$$

$$=\frac{1}{s}-\frac{a_r\beta_r}{s.r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{s.(r-1)!\,(a+\beta+s)_{r+1}}.$$

Hence the equation to be proved is

$$\frac{1}{s}\left(1-\frac{a_r\beta_r}{r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}-\frac{s\,u_{r,s-1}}{a+\beta+s+r}\right)$$

$$=\frac{1}{s}-\frac{a_r\beta_r}{s.r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{s.(r-1)!\,(a+\beta+s)_{r+1}}-\frac{u_{r,s-1}}{a+\beta+s+r},$$

which is correct.

5. There is no difficulty in applying the demonstration of equation (I.) to the case $s=1$.

But for the first term $s=0$ a separate proof is required.

In this case the terms $u_{r,s-1}$ and $w_{r,s-1}$ disappear, and it might be expected that the identity to be proved reduces to

$$1-\frac{\alpha_r\beta_r}{r!\,(\alpha+\beta)_r}-\frac{\alpha_r\beta_r}{(r-1)!\,(\alpha+\beta)_{r+1}}=\frac{\alpha\beta}{\alpha+\beta}w_{r,0}. \qquad \text{(II.)}$$

To prove that this is so

$$w_{r,0}=t_{1,0}u_{1,0}+t_{2,0}u_{2,0}+\ldots+t_{r,0}u_{r,0}.$$

Now $u_{r,0}=1+\dfrac{\alpha\beta}{\alpha+\beta+1}+\dfrac{\alpha_2\beta_2}{2!\,(\alpha+\beta+1)_2}+\ldots+\dfrac{\alpha_{r-1}\beta_{r-1}}{(\alpha+\beta+1)_{r-1}(r-1)!}$

$$=\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(\alpha+\beta+1)_{r-1}},$$

by the result in Art. 5, (a), of the former paper. Also

$$\begin{aligned}t_{r,0}u_{r,0}= &\ \frac{\alpha+\beta+2(r-1)}{(\alpha+r-1)(\beta+r-1)}\,\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(\alpha+\beta+1)_{r-1}}\\ &-\frac{\alpha+\beta+2r}{r(\alpha+\beta+r)}\,\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(\alpha+\beta+1)_{r-1}}\\ = &\ [(r-1)+(\alpha+\beta+r-1)]\frac{(\alpha+1)_{r-2}(\beta+1)_{r-2}}{(r-1)!\,(\alpha+\beta+1)_{r-1}}\\ &-[r+(\alpha+\beta+r)]\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{r!\,(\alpha+\beta+1)_r}\\ = &\ \frac{(\alpha+1)_{r-2}(\beta+1)_{r-2}}{(r-2)!\,(\alpha+\beta+1)_{r-1}}+\frac{(\alpha+1)_{r-2}(\beta+1)_{r-2}}{(r-1)!\,(\alpha+\beta+1)_{r-2}}\\ &-\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(\alpha+\beta+1)_r}-\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{r!\,(\alpha+\beta+1)_{r-1}}.\end{aligned}$$

The third and fourth of the above terms come from the first and second by increasing r by unity. Also

$$t_{1,0}u_{1,0}=\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{\alpha+\beta+1};$$

therefore

$$w_{r,0} = t_{1,0}u_{1,0}+t_{2,0}u_{2,0}+\dots+t_{r,0}u_{r,0}$$

$$= \frac{1}{a}+\frac{1}{\beta}-\frac{(a+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\,(a+\beta+1)_r}-\frac{(a+1)_{r-1}(\beta+1)_{r-1}}{r!\,(a+\beta+1)_{r-1}};$$

therefore $$\frac{a\beta}{a+\beta}w_{r,0} = 1-\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}}-\frac{a_r\beta_r}{r!\,(a+\beta)_r},$$

which is the equation (II.) to be proved.

6. Hence, by means of equations (I.) and (II.) together, it follows that

$$1-\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}}-\frac{a_r\beta_r}{r!\,(a+\beta)_r}$$

$$+\sum_{s=1}^{s}\frac{a_s\beta_s}{s!\,(a+\beta)_s}\left(1-\frac{a_r\beta_r}{r!\,(a+\beta+s)_r}-\frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}-\frac{s\,u_{r,s-1}}{a+\beta+s+r}\right)$$

$$=\frac{a\beta}{a+\beta}w_{r,0}+\sum_{s=1}^{s}\left(\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}}w_{r,s}-\frac{a_s\beta_s}{(s-1)!\,(a+\beta)_s}w_{r,s-1}\right)$$

$$=\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}}w_{r,s}. \tag{III.}$$

In this result it is necessary to make r infinite.

Now $$\frac{a_r\beta_r}{r!\,(a+\beta)_r}=\frac{\Pi(r,\,a+\beta-1)}{r.\Pi(r,\,a-1)\,\Pi(r,\,\beta-1)},$$

and therefore tends to zero as r tends to infinity. Similarly

$$\frac{a_r\beta_r}{(r-1)!\,(a+\beta)_{r+1}},\quad \frac{a_r\beta_r}{r!\,(a+\beta+s)_r},\quad \frac{a_r\beta_r}{(r-1)!\,(a+\beta+s)_{r+1}}$$

all tend to zero as r tends to infinity.

Also $u_{r,s-1}$ is a convergent series and therefore finite, and therefore $\frac{s\,u_{r,s-1}}{a+\beta+s+r}$ tends to zero as r tends to infinity. Hence, when r tends to infinity, the left-hand side of equation (III.) tends to

$$1+\sum_{s=1}^{s}\frac{a_s\beta_s}{s!\,(a+\beta)_s}.$$

Next $$\frac{\alpha_{s+1}\beta_{s+1}}{s!\,(\alpha+\beta)_{s+1}} = \frac{\Pi(s+1,\alpha+\beta-1)}{\Pi(s+1,\alpha-1)\,\Pi(s+1,\beta-1)},$$

and hence when s is large it will be asymptotic with

$$\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}.$$

It remains to examine $w_{r,s}$.

$$w_{r,s} = t_{1,s}u_{1,s}+t_{2,s}u_{2,s}+\ldots+t_{r,s}u_{r,s}$$

$$= t_{1,s}+t_{2,s}+\ldots+t_{r,s}+t_{1,s}(u_{1,s}-1)+t_{2,s}(u_{2,s}-1)+\ldots+t_{r,s}(u_{r,s}-1).$$

Now

$$u_{r,s}-1 = \frac{\alpha\beta}{\alpha+\beta+s+1}+\frac{\alpha_2\beta_2}{2!\,(\alpha+\beta+s+1)_2}+\ldots+\frac{\alpha_{r-1}\beta_{r-1}}{(r-1)!\,(\alpha+\beta+s+1)_{r-1}};$$

therefore

$$\frac{u_{r,s}-1}{\left(\dfrac{\alpha\beta}{\alpha+\beta+s+1}\right)} = 1+\frac{(\alpha+1)(\beta+1)}{2!\,(\alpha+\beta+s+2)}+\ldots+\frac{(\alpha+1)_{r-2}(\beta+1)_{r-2}}{(r-1)!\,(\alpha+\beta+s+2)_{r-2}}. \quad \text{(IV.)}$$

Now when s is large the right-hand side of (IV.) tends to the limit 1.

Suppose that the right-hand side of (IV.) lies between $1-\epsilon$ and $1+\eta$, where ϵ, η are two functions of s which tend to zero as s increases.

It follows that

$$t_{1,s}(u_{1,s}-1)+t_{2,s}(u_{2,s}-1)+\ldots+t_{r,s}(u_{r,s}-1)$$

lies between $$\frac{\alpha\beta}{\alpha+\beta+s+1}(t_{1,s}+t_{2,s}+\ldots+t_{r,s})(1-\epsilon)$$

and $$\frac{\alpha\beta}{\alpha+\beta+s+1}(t_{1,s}+t_{2,s}+\ldots+t_{r,s})(1+\eta).$$

Now $$t_{1,s}+t_{2,s}+\ldots+t_{r,s}$$

$$= \left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{\alpha+\beta+s+1}\right)$$

$$+\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}-\frac{1}{2}-\frac{1}{\alpha+\beta+s+2}\right)$$

$$+\ldots$$

$$+\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+s+r}\right)$$

$$= \left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{1}-\frac{1}{\alpha+\beta-1}\right)+\frac{1}{\alpha+\beta-1}-\frac{1}{\alpha+\beta+s+1}$$

$$+\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}-\frac{1}{2}-\frac{1}{\alpha+\beta}\right)+\frac{1}{\alpha+\beta}-\frac{1}{\alpha+\beta+s+2}$$

$$+\dots$$

$$+\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)+\frac{1}{\alpha+\beta+r-2}-\frac{1}{\alpha+\beta+s+r}$$

$$= \sum_{r=1}^{r}\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)$$

$$+\left(\frac{1}{\alpha+\beta-1}+\frac{1}{\alpha+\beta}+\dots+\frac{1}{\alpha+\beta+s}\right)$$

$$-\left(\frac{1}{\alpha+\beta+r-1}+\frac{1}{\alpha+\beta+r}+\dots+\frac{1}{\alpha+\beta+s+r}\right).$$

Now
$$\sum_{r=1}^{r}\left(\frac{1}{\alpha+r-1}+\frac{1}{\beta+r-1}-\frac{1}{r}-\frac{1}{\alpha+\beta+r-2}\right)$$

$$= -\sum_{r=1}^{r}\frac{(\alpha-1)(\beta-1)[\alpha+\beta+2(r-1)]}{r(\alpha+r-1)(\beta+r-1)(\alpha+\beta+r-2)},$$

and is therefore a convergent series with a finite sum, when r is infinite.

Next
$$\frac{1}{\alpha+\beta+r-1}+\frac{1}{\alpha+\beta+r}+\dots+\frac{1}{\alpha+\beta+s+r}<\frac{s+2}{\alpha+\beta+r-1},$$

and therefore tends to zero as r tends to infinity.

Next
$$\frac{1}{\alpha+\beta-1}+\frac{1}{\alpha+\beta}+\dots+\frac{1}{\alpha+\beta+s}$$

is asymptotic with $\log_e s$. Hence

$$t_{1,s}+t_{2,s}+\dots+t_{r,s}$$

is asymptotic with $\log_e s$, whilst

$$\frac{\alpha\beta}{\alpha+\beta+s+1}(t_{1,s}+\dots+t_{r,s})$$

tends to the value
$$\frac{\alpha\beta}{\alpha+\beta+s+1}\log_e s;$$

and is therefore very small when s is large; therefore

$$t_{1,s}(u_{1,s}-1)+\ldots+t_{r,s}(u_{r,s}-1)$$

is very small when r is infinite and s is large; therefore $w_{r,s}$ is asymptotic with

$$t_{1,s}+t_{2,s}+\ldots+t_{r,s};$$

and therefore with $\log_e s$. Hence

$$\frac{a_{s+1}\beta_{s+1}}{s!\,(a+\beta)_{s+1}}\,w_{r,s}$$

is asymptotic with $$\frac{\Pi(a+\beta-1)}{\Pi(a-1)\,\Pi(\beta-1)}\log_e s.$$

Hence $$1+\sum_{s=1}^{s}\frac{a_s\beta_s}{s!\,(a+\beta)_s}$$

is asymptotic with $$\frac{\Pi(a+\beta-1)}{\Pi(a-1)\,\Pi(\beta-1)}\log_e s.$$

7. Consider now the case where $\gamma=a+\beta-n$.

Putting $\gamma=a+\beta-n, \quad t=n-1,$

in equations (V.) and (VI.), p. 337, of the former paper, it follows that

$$G(a,\beta,a+\beta-n,s)=\frac{(a-n)_n(\beta-n)_n}{(-n)_n(a+\beta-n)_n}G(a,\beta,a+\beta,s)$$
$$+\frac{a_{s+1}\beta_{s+1}}{n.s!\,(a+\beta-n)_{s+1}}f(a,\beta,a+\beta-n,s,n-1)$$

where $f(a,\beta,a+\beta-n,s,n-1)$

$$=1+\frac{(a-n)(\beta-n)}{(-n+1)(a+\beta-n+s+1)}+\frac{(a-n)_2(\beta-n)_2}{(-n+1)_2(a+\beta-n+s+1)_2}+\ldots$$
$$+\frac{(a-n)_{n-1}(\beta-n)_{n-1}}{(-n+1)_{n-1}(a+\beta-n+s+1)_{n-1}}.$$

Now $$\frac{a_{s+1}\beta_{s+1}}{n.s!\,(a+\beta-n)_{s+1}}=\frac{\Pi(s+1,a+\beta-n-1)}{\Pi(s+1,a-1)\,\Pi(s+1,\beta-1)}\,\frac{(s+1)^n}{n};$$

and is therefore, for a large s, asymptotic with

$$\frac{\Pi(a+\beta-n-1)}{\Pi(a-1)\,\Pi(\beta-1)}\,\frac{(s+1)^n}{n}.$$

Also $f(a,\beta,a+\beta-n,s,n-1)$ tends to unity as s increases.

Further, $G(\alpha, \beta, \alpha+\beta, s)$ has been shown to be asymptotic with a finite multiple of $\log_e s$, which is very small compared with $(s+1)^n$, when n is a positive integer and s a large positive integer.

Consequently $G(\alpha, \beta, \alpha+\beta-n, s)$ is asymptotic with

$$\frac{\Pi(\alpha+\beta-n-1)}{\Pi(\alpha-1)\,\Pi(\beta-1)}\,\frac{(s+1)^n}{n},$$

which is the value assumed by the expression

$$\frac{\Pi(\gamma-1)(s+1)^{\alpha+\beta-\gamma}}{(\alpha+\beta-\gamma)\,\Pi(\alpha-1)\,\Pi(\beta-1)}$$

given in Art. 4, on p. 339 of the former paper, when $\gamma=\alpha+\beta-n$.